Comparison of Analog and Digital Relay Methods with Network Coding for Wireless Multicast

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Abstract—We study wireless multicasting from two sources to two destinations with the assistance of a single half-duplex relay. The objective is to evaluate the throughput and error performance of different analog and digital relay schemes with linear network coding at the relay. The analog relay node forwards either a scaled version of the received signal to the destinations, or alternatively, first filters the received signals to generate a linear Minimum Mean Squared Error (MMSE) estimate, which is subsequently forwarded. The digital relay scheme first detects the source transmissions, combines the packets with a network code, and forwards the resulting symbols to the destinations. For all schemes the destinations recover the source and relay signals by first applying linear MMSE filters, followed by decoding of the source bits. The performances of the schemes are compared in terms of normalized throughput (bits per channel use accounting for the delay due to the relay) and uncoded error probability, given a normalized power constraint. Both narrowband and wideband transmission schemes are considered. Our results show that the analog relay schemes outperform the digital network coding scheme with respect to both throughput and error probability because of error propagation through the relay. Numerical results are presented, which illustrate throughput-reliability trade-offs for all schemes considered.

I. INTRODUCTION

Network coding extends store-and-forward-based routing solutions and allows intermediate relay nodes to code over the incoming packets [1]. For multicast communication from a single source to multiple destinations, linear network coding is known to achieve the Max-Flow Min-Cut bounds in wired networks [2]. However, optimal multi-source network coding is in general an open problem [3].

Network coding has been extended to wireless network models, which account for omnidirectional transmissions, half-duplex nodes, and interference effects [4]–[6]. Since the received signals at the relay nodes are a superposition of transmitted signals, the simple amplify-and-forward (AF) scheme is a form of network coding at the signal level [7], [8]. Conventional digital network coding at the packet level has been compared to analog network coding for bi-directional relaying between source pairs [9], [10]. Relay-assisted communication with network coding has been also studied in terms of information-theoretic bounds on achievable rates [11]. Those bounds, of course, assume optimal coding over long time intervals without regard to the accumulation of delay across the relay node. In addition, the digital relay scheme in [11] assumes perfect decoding at the relay without error propagation, and the associated decoding delay is not taken into account.

In this paper, we assume the simple “butterfly” network in Fig. 1 with two sources broadcasting packets to two destinations with the assistance of a half-duplex relay node. Wireless network coding is studied from the perspective of filtering and detection at packet and signal levels. A common framework is established to compare digital network coding with two analog relay schemes in which the relay node forwards either scaled or filtered versions of the received signal. Linear filters are used to detect the source symbols at the relay and destinations. The destinations then map the signal estimates to packets. In the digital relay scheme the relay node decodes each incoming packet, applies a network code across the decoded packets, and forwards the result to the destinations. The destinations then filter the received samples and use a Maximum Likelihood (ML) detector to recover the source bits.

We compare the relay schemes with respect to achievable rate and probability of decoding error, where because of the multicast assumption, the worst-case value is taken over the two destinations. We assume QPSK source symbols, and compute the achievable rate by modeling the paths from source to destinations as binary symmetric channels. First, we consider narrowband (NB) analog and digital relay schemes subject to a power constraint, which is normalized by the total number of channel uses. Then we extend the models to include multiple degrees of freedom, e.g., using wideband Code-Division-Multiple-Access (CDMA), and compare the normalized achievable rate in bits per channel uses (accounting for the relay).

Our results show that the analog relay schemes outperform the digital relay with network coding in terms of both throughput and reliability. Digital relaying is limited by errors caused by linear filtering with hard quantization, which are propagated to the destinations. In contrast, in the analog schemes the destinations filter the source and relayed signals directly, which improves robustness.

II. SYSTEM MODEL

Fig. 1 shows two source nodes S = {1, 2}, which use a half-duplex, causal relay node R = {3} to transmit their packets to each of two destination nodes D = {4, 5}. We assume
complex baseband channels that are linear, time-invariant and frequency-flat. The complex channel gain between transmitter node $i \in S \cup R$ and receiver node $j \in R \cup D$, where $i \neq j$, is $h_{ij}$. We assume additive complex Gaussian noise $n_j(t) \sim \mathcal{CN}(0, \sigma_j^2)$ at receiver node $j \in R \cup D$ at time $t$. We consider omnidirectional transmissions in a synchronous slotted system.

Each source $s \in S$ transmits two bits, represented by $b_s = [b_{s,1} b_{s,2}]$. These are mapped to a Quadrature Phase-Shift Keying (QPSK) symbol $x_s$ via Gray mapping\(^1\), which is transmitted with power $P_s$. Since all nodes are half-duplex, relay node 3 is scheduled as a transmitter or receiver over different time slots. If the relay is not transmitting in time slot $t$, it receives the signal $y_s(t)$ from the sources. After all sources have transmitted their symbols, the relay transmits the symbol $x_3$ (which depends on previous received symbols) in a subsequent time slot $t' > t$ with power $P_3$. The destinations and the relay node use linear filters to estimate the signals followed by a standard (minimum distance) quantizer for QPSK. In the next two sections we describe the specific relay methods, which will be compared.

III. ANALOG RELAY SCHEMES

All relay schemes, including analog and digital, are illustrated in Fig. 2.

A. Direct Transmission (DT)

This is the reference scheme, which does not use the relay. Source node 1 transmits in time slot 1 and source node 2 transmits in time slot 2. The vector of received samples at destination $d \in D$ is given by

$$
\begin{bmatrix}
y_d(1) \\
y_d(2) \\
y_d(3)
\end{bmatrix}
= \begin{bmatrix}
h_{1d} & 0 & P_3 & 0 & \frac{1}{2} [x_1] \\
0 & h_{2d} & 0 & P_2 & [x_2] \\
0 & 0 & h_{3d} g_{h_{13}} & h_{3d} g_{h_{23}} & [x_1] + [x_2]
\end{bmatrix} + \begin{bmatrix}
n_d(1) \\
n_d(2) \\
n_d(3)
\end{bmatrix}.
$$

B. Amplify and Forward (AF)

In AF, both sources transmit in the first time slot, and then the relay amplifies and transmits the received signal in the second time slot (see Fig. 2). The relay therefore receives

$$
y_3(1) = \begin{bmatrix} h_{13} & h_{23} \end{bmatrix} \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \frac{1}{2} [x_1] + n_3(1).
$$

The signal transmitted by the relay in slot 2 is scaled as $x_3 = g y_3(1)$, where the constant $g = \frac{\sqrt{P_3}}{\sqrt{P_1 h_{13}^2 + P_2 h_{23}^2 + \sigma_3^2}}$ is chosen such that the relay transmits with power $P_3$, i.e., $E[|x_3|^2] = P_3$. Hence the received vector of samples at destination $d \in D$ is given by

$$
\begin{bmatrix}
y_d(1) \\
y_d(2)
\end{bmatrix}
= \begin{bmatrix}
h_{1d} & h_{2d} \\
h_{3d} g_{h_{13}} & h_{3d} g_{h_{23}}
\end{bmatrix} \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \frac{1}{2} [x_1] + \begin{bmatrix} n_d(1) \\ n_d(2)
\end{bmatrix} + \begin{bmatrix}
0 \\
h_{3d} g n_3(1)
\end{bmatrix}.
$$

C. Filter and Forward (FF)

In FF, source node 1 transmits in the first time slot and source node 2 transmits in the second time slot. The relay filters the signals received over the two time slots and transmits a weighted sum in the third time slot (see Fig. 2). Specifically, the relay receives $y_3(1)$ and $y_3(2)$, which are given by (1) with $d = 3$. In the third time slot the relay transmits

$$
x_3 = \begin{bmatrix} g_1 & g_2 \end{bmatrix} \begin{bmatrix} y_3(1) \\ y_3(2)
\end{bmatrix},
$$

where $g_s = \alpha_s \frac{\sqrt{P_s}}{\sigma_s^2}$, $s \in S$, is the scaled minimum mean square error (MMSE) weight for $x_s$.\(^2\) The scaling constant $\alpha^2 = P_3 \sum_s P_s |h_{s,3}|^2 (\sigma_s^2 + P_s |h_{s,3}|^2)^{-1}$ is again chosen such that $E[|x_3|^2] = P_3$. The vector of received samples for FF at destination $d \in D$ is given by

$$
\begin{bmatrix}
y_d(1) \\
y_d(2) \\
y_d(3)
\end{bmatrix}
= \begin{bmatrix}
h_{1d} & 0 & P_3 & 0 & \frac{1}{2} [x_1] \\
0 & h_{2d} & 0 & P_2 & [x_2] \\
h_{3d} g_{h_{13}} & h_{3d} g_{h_{23}} & 0 & 0 & [x_1] + [x_2] + [n_d(1) + n_d(2)]
\end{bmatrix}.
$$

D. Receive Filters for DT, AF and FF

Denoting the vector of received samples over time at destination $d \in D$ as $y_d$, for the analog schemes considered we can rewrite (1), (2), and (4) as

$$
y_d = H_d P_3^{\frac{1}{2}} x + \tilde{n}_d + n_d,
$$

where $H_d$ is the channel matrix, which depends on the relay scheme, $n_d$ is the noise at the destination, and $\tilde{n}_d$ is the noise

\(^1\)It is sufficient to consider only one symbol per block of bits, since there is no inter-symbol interference. The analysis can easily be extended to arbitrary QAM modulation schemes.

\(^2\)In FF, the relay samples the received signals and transmits the digitally filtered version over the next time slot. This is different from AF, unless the relay can store the analog signal.
forwarded through the relay ($\hat{n}_d = 0$ for DT).\(^3\) The source symbol vector $\hat{x}$ and power matrix $P$ are the same for all relaying schemes.

Each destination estimates the source symbols as $\hat{x}_{s,d} = [G_d y_d]_s$ where $G_d$ is the linear MMSE filter $G_d = \arg \min_G \mathbb{E}[\|G y_d - x\|^2]$

$$= P^{\frac{1}{2}} \mathbf{H}^H_d (R_{n_d} + R_{\hat{n}_d} + H_d P \mathbf{H}^H_d)^{-1},$$

where the expectation is over the signal and noise, $R_{n_d}$ and $R_{\hat{n}_d}$ are the covariance matrices of $n_d$ and $\hat{n}_d$, respectively, and $H_d^H$ is the conjugate transpose of $H_d$. The minimum-distance quantizer detects the transmitted source bits as $b_{s,d}$ from the MMSE estimates. The multicast mean square error (MSE) corresponding to source $s$ is then given by

$$\varepsilon_s = \max_{d \in D} \mathbb{E}[\|\hat{x}_{s,d} - x_s\|^2]$$

$$= \max_{d \in D} [(I + P^{\frac{1}{2}} \mathbf{H}^H_d (R_{n_d} + R_{\hat{n}_d} + H_d P \mathbf{H}^H_d)^{-1})]_{s,s},$$

where $\lfloor . \rfloor_{s,s}$ is the $s$th diagonal element. The maximum over the destinations reflects the multicast requirement that both destinations receive the source message.

With MMSE filters the multicast Signal-to-Interference-and-Noise Ratio (SNR) for $s \in S$ is given by $\gamma_s = \frac{1}{\varepsilon_s} - 1$. For DT there is no interference, hence the SNR for that scheme is the Signal-to-Noise Ratio (SNR). The multicast bit error probability $\pi_s = \max_{d \in D} P(b_{s,d} \neq b_s)$ is then given by $\pi_s = 1 - \Phi(\sqrt{\gamma_s})$, where $\Phi$ is the Gaussian cumulative distribution function. For AF and FF, this approximates the bit error probability.

The relation between the transmitted bits $b_s$ corresponding to $x_s$ and the estimate $\hat{b}_{s,d}$ at the destination $d$ with the maximum bit error rate can be represented as a binary symmetric channel (BSC) with cross-over probability $\pi_s$. Assuming independent coding and decoding of the bit stream from source $s$ with QPSK symbols (i.e., two coded bits per symbol), the maximum achievable multicast rate is

$$R_s = \frac{2}{T} (1 - H_2(\pi_s)), \quad (6)$$

where $T = 2$ time slots for DT and AF, $T = 3$ for FF, and $H_2(p) = -p \log_2 p - (1-p) \log_2 (1-p)$ is the binary entropy function.

IV. DIGITAL RELAY SCHEME

Here the relay demodulates the source message before forwarding. In contrast to the analog relay, which operates symbol by symbol, the digital relay combines the decoded symbols (bits) at the packet level.

A. Decode, Code, and Forward (DF)

In this scheme source node 1 transmits in the first time slot and source node 2 transmits in the second time slot. The relay node filters the received samples as in the FF scheme and demodulates both bit streams from the filter outputs. Then, it combines the two streams using the XOR operation, modulates the coded bits, and transmits the resulting symbol in the third time slot. The relay therefore receives the signal given by (1) for $d = 3$, and transmits in the third time slot the signal $x_3 = \sqrt{P_3} M (Q(g_1 y_3(1)) + Q(g_2 y_3(2)))$, where $g_s$, $s \in S$, are the same as in (3) with $\alpha = 1$, $Q$ is the minimum-distance quantizer with Gray mapping, which maps complex numbers to pairs of bits, and $M$ is the modulation mapping, which maps the pairs of bits to QPSK symbols.

Since $x_3$ is pairwise statistically independent of $x_s$, $s \in S$, linear filtering cannot recover any information about symbol $x_s$ from the relay transmission. Therefore, the vector of received samples at destination $d \in D$ is given by

$$\begin{bmatrix} y_{d(1)} \\ y_{d(2)} \\ y_{d(3)} \end{bmatrix} = \begin{bmatrix} h_{1d} & 0 & 0 \\ 0 & h_{2d} & 0 \\ 0 & 0 & h_{3d} \end{bmatrix} \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n_{d(1)} \\ n_{d(2)} \\ n_{d(3)} \end{bmatrix},$$

The estimated bits for slot $i$ corresponding to the estimated QPSK symbol are then

$$a_d(i) = Q(g_i y_d(i)),$$

where $g_d = \frac{\sqrt{P_d}}{\sigma_d^2 + P_d |h_d|^2}$, $d \in D$. Assuming QPSK with Gray mapping, and that the two bits represented by each symbol are treated independently at the relay, the pairs of bits are independent at the destinations. Hence the bit-level model from source to destination $d \in D$ is

$$\begin{bmatrix} a_{d(1)} \\ a_{d(2)} \\ a_{d(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} z_{d(1)} \\ z_{d(2)} \\ z_{d(3)} \end{bmatrix}, \quad (7)$$

or equivalently,

$$a_d = H b + z_d,$$

where $a_d(t)$ denotes the demodulated bits and $z_d(t)$ denotes binary noise with distribution given by

$$\Pr(z_d(t) = 1) = \begin{cases} \Pr(a_d(t) \neq b_s), & t = s, s \in S \\ \Pr(a_d(t) \neq b_1 + b_2), & t = R = \{3\}. \end{cases}$$

The error probability in the first two time slots depends only on the SNR at the destination in the respective time slot. In the third time slot it depends also on the SNRs at the relay node for the first two time slots.

B. Detection in DF

The destinations have to recover the source bits $b_s$, $s \in S$, from the demodulated bits in (7). This problem is solved by using a Maximum Likelihood (ML) detector at the bit-level, assuming that bits are equiprobable. Given the binary observations $a_d$, the estimate of $b_s$ at destination $d$ is

$$\hat{b}_{d,s} = \arg \max_b \sum_{b_b} P(a_d | b, b_b), \quad s \neq s.$$\(^4\)

This is the optimal decoder in terms of the residual bit error rate (BER) for bits from both sources. The residual multicast BER $\pi_s$ after decoding is given by

$$\pi_s = \max_{d \in D} \Pr(\hat{b}_{d,s} \neq b_s),$$
and the maximum achievable multicast rate $R_s$ with end-to-end channel coding is given by (6).

The transmission powers can be further chosen to minimize $\sum_{s \in S} \varepsilon^T_x$ or $\sum_{s \in S} \pi_s$ in each of the analog or digital relay schemes, subject to maximum power constraints. The optimal power allocation in DT, FF and DF is to use the full available power budget at each node. This also holds for AF, as long as the channel matrix in (2) is full rank. The performance of AF strongly depends on relative channel phases between sources.

V. RELAYING WITH MULTIPLE DEGREES OF FREEDOM

We now extend the framework of Sections III and IV to a system with multiple degrees of freedom, such as CDMA. Network coding in CDMA systems has been considered in [10] for information exchange between two sources over a relay node. In this section, we assume that each relay method uses two time slots and the same amount of power. Each source transmits a QPSK symbols $(x_1)$ and $(x_2)$ in the first time slot by modulating a sequence of spreading sequences (unit column vectors in $\mathbb{C}^2$). The relay receives the signal

$$y_3 = [H_{13} \quad H_{23}] \left[ \begin{array}{c} P_1 \\ 0 \\ P_2 \end{array} \right] \left[ \begin{array}{c} \frac{1}{2} \end{array} x_1 \quad \frac{1}{2} x_2 \right] + n_3,$$

where $H_3 = [H_{13} \quad H_{23}] \in \mathbb{C}^{2 \times 2}$. The channel matrices $H_{ij}$ include the spreading sequence of transmitter node $i$ at receiver node $j$, along with the delay, channel attenuation, and multipath after chip matched filtering. The relay transmits the sequence of symbols $x_3 \in \mathbb{C}^2$ in the second time slot again using a spreading sequence for each symbol.

A. Analog Relay

The relay simply forwards a normalized linear transformation of the received signal, $x_3 = G_3 y_3$, where $G_3 = \alpha I$ for AF, or $G_3 = \alpha P^2 H_3^H (R_3 + H_3 P H_3^H)^{-1}$ for FF, and $\alpha$ is selected so that $E[|x_3|^2] = P_3$. Destination $d$ receives signal

$$y_d = \left[ \begin{array}{c} H_{1d} \\ H_{3d} G_3 H_{13} \\ H_{3d} G_3 H_{23} \end{array} \right] \left[ \begin{array}{c} P^2 x_1 \\ 0 \\ H_{34} G_3 n_3 \end{array} \right] + n_d,$$

which coincides with the model (5). The receiver is the same as in Section III-D. At packet (bit) level the system is again represented as collection of BSCs for source-destination pairs.

B. Digital Relay

The relay first applies an MMSE receiver $G_3 = P^2 H_3^H (R_3 + H_3 P H_3^H)^{-1}$ to $y_3$ and decodes the transmitted signal. Then, it codes the detected bits with a binary network coding matrix $C = [C_1 \quad C_2] \in \{0, 1\}^{2 \times 2}$, which can be chosen either deterministically or randomly. The network coded bits are then mapped to QPSK symbols, given by $x_3$. Hence the relay operations are given by $x_3 = \sqrt{\frac{2}{5}} \mathcal{M}(C Q(G_3 y_3))$. As opposed to [10], we consider individual decoding of all bits at the relay and destinations.

VI. THROUGHPUT AND ERROR PERFORMANCE

The scalar channel $h_{ij}$ from node $i$ to $j$ is represented by a complex, circularly symmetric zero-mean Gaussian random variable with variance $E[|h_{ij}|^2] = \frac{1}{d_{ij}^\alpha}$, where $d_{ij}$ is the distance between the nodes and $\alpha \geq 2$ is the pathloss exponent. Here we assume $\alpha = 4$. The results are averaged over 10000 channel realizations. The noise at each receiver and relay is uncorrelated complex Gaussian with zero mean and unit variance. The sources and relay have the same transmit power constraint ($P_i = P$, $i = 1, 2, 3$).

Refraining to Fig. 1, we consider a symmetric network in which the nodes are located on the vertices of a rectangle with $d_{14} = d_{25} = \frac{1}{2} d_{12}$. The relay is located at the center of the rectangle.

We consider two different scenarios: (a) narrowband (NB) relaying, as presented in Sections III and IV, and (b) wideband (CDMA) relaying, as presented in Section V. For scenario (b) each node chooses random spreading sequences and distributes its power equally across symbols. Based on the DT scheme, the reference SNR is defined as $\text{SNR}_{\text{ref}} = \frac{2}{7} \sum_{s \in S} E[|h_{sd}|^2] P$.

Fig. 3 shows the sum of the maximum achievable coded transmission rates $\sum_{s \in S} R_s$ in bits per time slot or code dimension, versus $\text{SNR}_{\text{ref}}$. For the NB model, AF outperforms the other schemes over all SNR values. FF and DF outperform DT for low to intermediate SNRs, but saturate at 2/3, due to the use of an additional time slot. The rate gain due to relaying over DT increases with $\alpha$. For the CDMA model, AF and FF have nearly the same performance. MMSE filtering at the relay yields only a small gain if the destinations also use MMSE.

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4 The analysis can be extended to multiple antennas (MIMO) and more general models with more than two sources and destinations.
interference at the relay causes additional errors after filtering and quantization. Joint detection (e.g., sphere decoding) would mitigate this effect at the expense of significantly increasing the computational complexity at the relay, particularly if more than two sources are present. Consequently, these results indicate that with computationally inexpensive linear filtering and quantization, analog relaying is superior to digital relaying.

VII. CONCLUSIONS

We have compared the performance of different analog and digital relay methods with network coding for wireless multicast with two sources and two destination nodes. Our approach uses linear filtering at the relay and destinations to detect the transmitted symbols. Additional channel coding provides error protection and optimizes achievable rates at the expense of introducing decoding delay. The CDMA model with network coding can be easily extended to any two-hop network with arbitrarily many sources, relays and destinations.

The use of a relay node significantly improves throughput and reliability. Both FF and AF perform better than DF with network coding with respect to both rate and error objectives. The linear filters for AF can be implemented adaptively (e.g., using training sequences). AF is likely to be less complex than DF, which requires knowledge of the bit error rates for ML decoding, and also uses linear filters at the signal level.

Future work may analyze the performance gains achieved by using more sophisticated detection (e.g., ML) at the relay along with optimizing the resources (power and code dimensions) allocated across the sources and relay. In this context, distributed implementations should be studied. Also, the model could be extended to unicast and multi-group multicast communications.

REFERENCES