ERGODIC CONTROL OF SWITCHING DIFFUSIONS*

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ABSTRACT. We study the ergodic control problem of switching diffusions representing a typical hybrid system that arises in numerous applications such as fault-tolerant control systems, flexible manufacturing systems, etc. Under fairly general conditions, we establish the existence of a stable, nonrandomized Markov policy which almost surely minimizes the pathwise long-run average cost. We then study the corresponding Hamilton-Jacobi-Bellman (HJB) equation and establish the existence of a unique solution in a certain class. Using this, we characterize the optimal policy as a minimizing selector of the Hamiltonian associated with the HJB equations. As an example we apply the results to a failure prone manufacturing system and obtain closed form solutions for the optimal policy.

1. Introduction. We address the problem of controlling switching diffusions by continually monitoring the continuous and discrete component of the state. The objective is to minimize, almost surely, the pathwise long-run average (ergodic) cost over all admissible policies. A controlled switching diffusion is a typical example of a hybrid system which arises in numerous applications of systems with multiple modes or failure modes, such as fault tolerant control systems, multiple target tracking, flexible manufacturing systems etc. [13], [14], [23]. The state of the system at time t is given by a pair $\big(X(t), S(t)\big) \in \mathbb{R}^d \times \mathcal{S}$, $\mathcal{S} = \{1, 2, \ldots, N\}$. The continuous component $X(t)$ is governed by a "controlled diffusion process" with a drift vector which depends on the discrete component $S(t)$. Thus, $X(t)$ switches from one diffusion path to another as the discrete component $S(t)$ jumps from one state to another. On the other hand, the discrete component $S(t)$ is a "controlled Markov chain" with a transition matrix depending on the continuous component. The evolution of the process $\big(X(t), S(t)\big)$ is governed by the following equations:

$$dX(t) = b\big(X(t), S(t), u(t)\big)dt + \sigma\big(X(t), S(t)\big)dW(t), \tag{1.1}$$

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$$(1.2)\quad P\big(S(t+\delta t)=j \mid S(t)=i, X(s), S(s), s\le t\big)=\lambda_{ij}\big(X(t),u(t)\big)\delta t+o(\delta t),\quad i\neq j,$$

for $t\ge 0$, $X(0)=X_0$, $S(0)=S_0$, where b, σ, λ are suitable functions, $\lambda_{ij}\ge 0$ for $i\neq j$, $\sum_{j=1}^{N}\lambda_{ij}=0$, $W(\cdot)$ is a standard Brownian motion and $u(\cdot)$ is a non-anticipative control process (admissible policy). The latter is called a Markov policy if $u(t)=v\big(X(t),S(t)\big)$ for a suitable function v. Our goal is to minimize almost surely (a.s.) over all admissible policies the functional

$$(1.3)\qquad \limsup_{T\to\infty}\frac{1}{T}\int_0^T c\big(X(t),S(t),u(t)\big)\,dt\,,$$

where c is the running-cost function. Note that in (1.3) there is no expectation; we are minimizing the limiting pathwise average cost. Such a criterion is very important in practical applications since we often deal with a single realization. Under certain conditions, we show that there exists a Markov policy v^* and constant ρ^* such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T c\big(X(t),S(t),v^*(X(t),S(t))\big)dt=\rho^*,\quad \text{a.s.}$$

and for any other admissible policy $v(\cdot)$

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T c\big(X(t),S(t),v(t)\big)dt\ge\rho^*,\quad \text{a.s.}$$

This establishes that v^* is optimal in a much stronger sense; viz., the most "pessimistic" average cost under v^* is no worse than the most "optimistic" average cost under any other admissible policy. Also, under the conditions assumed in this paper, the optimal pathwise average cost coincides with the optimal expected average cost. So we do not distinguish between these two criteria.

Our paper is organized as follows. In Section 2 we present and analyze a motivating example, while in Section 3 we introduce a concise mathematical model of the switching diffusion. Section 4 is devoted to the study of recurrence and ergodicity of switching diffusions. The existence of an optimal policy is established in Section 5. The Hamilton-Jacobi-Bellman (HJB) equations are studied in Section 6. Conclusions are in Section 7.

2. A Motivating Example. The failure prone manufacturing system presented in [1], [5], [14], is a very good example of the class of systems studied in this paper. This section is devoted to the analysis of this manufacturing model. Results from subsequent sections will be used in this example and thus the reader will have the opportunity to glimpse at some of the key developments of the paper.

Suppose there is one machine producing a single commodity. We assume that the demand rate is a constant $d>0$. Let the machine state $S(t)$ take values in $\{0,1\}$, $S(t)=0$ or 1, according as the machine is down or functional. We model $S(t)$ as a continuous time Markov chain with generator

$$\begin{bmatrix}-\lambda_0 & \lambda_0\\ \lambda_1 & -\lambda_1\end{bmatrix},$$

where λ_0 and λ_1 are positive constants corresponding to the infinitesimal rates of repair and failure respectively. The inventory $X(t)$ is governed by the Ito equation

$$dX(t) = \big(u(t) - d\big)dt + \sigma dW(t)\,, \tag{2.1}$$

where $\sigma > 0$, $u(t)$ is the production rate and $W(t)$ is a one-dimensional Wiener process independent of $S(t)$. The last term in (2.1) can be interpreted as "sales return", "inventory spoilage", "sudden demand fluctuations", etc. A negative value of $X(t)$ represents backlogged demand. The production rate is constrained by

$$u(t) \in \begin{cases} \{0\}, & \text{if } S(t) = 0 \\ [0, r], & \text{if } S(t) = 1. \end{cases}$$

Let $c : \mathbb{R} \to \mathbb{R}_+$ be the cost function which is assumed to be convex and Lipschitz. Also $c(x) \geq g(|x|)$ for some increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Thus, c satisfies (5.3), a required condition for the results we apply to hold. We show later in this section that a certain hedging-point policy is stable. Therefore, by the results of Section 5 there exists an a.s. optimal nonrandomized Markov policy with respect to the cost criterion

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T c(X(t))dt\,.$$

The HJB equations in this case are

$$\begin{pmatrix} \frac{\sigma^2}{2} V''(x,0) - dV'(x,0) \\ \frac{\sigma^2}{2} V''(x,1) + \displaystyle\min_{u\in[0,r]} \big\{(u-d)V'(x,1)\big\} \end{pmatrix} + \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix} \begin{pmatrix} V(x,0) \\ V(x,1) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} c(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rho\,. \tag{2.2}$$

The results of Section 6 ensure existence of a C^2 solution (V, ρ^*) of (2.2), where ρ^* is the optimal cost. Using the convexity of $c(\cdot)$, it can be shown that $V(\cdot, i)$ is convex for each i. Hence, there exists an x^* such that

$$\begin{aligned} V'(x,1) &\leq 0, &\quad& \text{for } x \leq x^* \\ V'(x,1) &\geq 0, && \text{for } x \geq x^*. \end{aligned} \tag{2.3}$$

It follows, from (2.3), that the value of u which minimizes $(u - d)V'(x, 1)$ is

$$u = \begin{cases} r, & \text{if } x < x^* \\ 0, & \text{if } x > x^*. \end{cases}$$

Since $V'(x^*, 1) = 0$, any $u \in [0, r]$ minimizes $(u - d)V'(x^*, 1)$. Therefore, in view of Theorem 6.2, the action $u \in [0, r]$ can be chosen arbitrarily at $x = x^*$. To be specific, we let

$u(x^*) = d$, i.e., we produce at the level that meets the demand exactly. Thus, the following stable, nonrandomized Markov policy is optimal

$$v^*(x,0) = 0, \qquad v^*(x,1) = \begin{cases} r, & \text{if } x < x^* \\ d, & \text{if } x = x^* \\ 0, & \text{if } x > x^*. \end{cases} \tag{2.4}$$

Note that the stability of the policy (2.4) follows from Theorem 6.3 provided that the set of stable, nonrandomized Markov policies is nonempty. We show next that the zero-inventory policy v given by

$$v(x,0) = 0, \qquad v(x,1) = \begin{cases} r, & \text{if } x \le 0 \\ 0, & \text{if } x > 0 \end{cases} \tag{2.5}$$

is stable if and only if

$$\frac{(r-d)}{\lambda_1} > \frac{d}{\lambda_0}\,. \tag{2.6}$$

The condition (2.6) is in accord with intuition. Note that λ_0^{-1} and λ_1^{-1} are the mean sojourn times of the chain in states 0 and 1 respectively. In state 0 the mean inventory depletes at a rate d while in state 1 it builds up at a rate $(r-d)$. Thus, if (2.6) is satisfied, one would expect the zero-inventory policy to stabilize the system. Our analysis confirms this intuition. We first show that under v the process $\bigl(X(\cdot), S(\cdot)\bigr)$ has an invariant probability measure η_v with a strictly positive density. In view of Lemma 4.1, it then follows from the ergodic theory of Markov processes [25, Chap. 1] that $\bigl(X(\cdot), S(\cdot)\bigr)$ is positive recurrent, or equivalently that v is stable.

By Lemma 5.2, the density φ of the invariant probability measure η_v can be obtained by solving the adjoint system

$$(L^v)^*\varphi(x,i) = 0\,, \tag{2.7}$$

subject to

$$\varphi(x,i) > 0, \qquad \sum_{i\in\{0,1\}} \int_{\mathbb{R}} \varphi(x,i)\,dx = 1\,, \tag{2.8}$$

where L^v is the differential generator defined in (3.6)–(3.8). Define

$$\tilde\lambda_0 := \frac{2\lambda_0}{\sigma^2}, \quad \tilde\lambda_1 := \frac{2\lambda_1}{\sigma^2}, \quad \tilde d := \frac{2d}{\sigma^2} \quad \text{and} \quad \tilde r := \frac{2r}{\sigma^2}\,.$$

Then (2.7) is equivalent to

$$\begin{aligned} &\varphi''(x,0) + \tilde d\varphi'(x,0) - \tilde\lambda_0\varphi(x,0) + \tilde\lambda_1\varphi(x,1) = 0 \\ &\varphi''(x,1) + \tilde d\varphi'(x,1) - \tilde\lambda_1\varphi(x,1) + \tilde\lambda_0\varphi(x,0) = 0 \end{aligned} \qquad \text{for } x > 0, \tag{2.9a}$$

$$\begin{aligned} &\varphi''(x,0) + \tilde d\varphi'(x,0) - \tilde\lambda_0\varphi(x,0) + \tilde\lambda_1\varphi(x,1) = 0 \\ &\varphi''(x,1) - (\tilde r - \tilde d)\varphi'(x,1) - \tilde\lambda_1\varphi(x,1) + \tilde\lambda_0\varphi(x,0) = 0 \end{aligned} \qquad \text{for } x < 0. \tag{2.9b}$$

A solution of (2.9), subject to the constraint (2.8), exists if and only if (2.6) holds and takes the form:

$$\varphi(x)=\begin{pmatrix}\varphi(x,0)\\ \varphi(x,1)\end{pmatrix}=\begin{cases} a_1\begin{pmatrix}\tilde\lambda_1\\ \tilde\lambda_0\end{pmatrix}e^{-s_1x}+a_2\begin{pmatrix}-\tilde\lambda_1\\ \tilde\lambda_1\end{pmatrix}e^{-s_2x}, & \text{for } x\ge 0,\\[2ex] a_3\begin{pmatrix}\tilde\lambda_1\\ -\psi(s_3)\end{pmatrix}e^{s_3x}+a_4\begin{pmatrix}-\tilde\lambda_1\\ \psi(s_4)\end{pmatrix}e^{s_4x}, & \text{for } x< 0,\end{cases} \tag{2.10}$$

where $\psi(s)=s^2+\tilde d s-\tilde\lambda_0$, $s_1=\tilde d$, $s_2=\frac{\tilde d}{2}+\frac12\big[\tilde d^2+4(\tilde\lambda_0+\tilde\lambda_1)\big]^{1/2}$, and s_3, s_4 are the positive roots of the polynomial

$$s^3-(\tilde r-2\tilde d)s^2-\big[(\tilde r-\tilde d)\tilde d+\tilde\lambda_0+\tilde\lambda_1\big]s+\big[(\tilde r-\tilde d)\tilde\lambda_0-\tilde d\tilde\lambda_1)\big]\,,$$

ordered by $0<s_3<s_4$. Also, the coefficients $\{a_1,a_2,a_3,a_4\}$, are given by:

$$\begin{aligned}
a_1&=\frac{1}{\Delta}\Big\{\frac{(s_4-s_3)s_2}{\tilde\lambda_0+\tilde\lambda_1}+\frac{s_4+s_2}{s_3+\tilde d}-\frac{s_3+s_2}{s_4+\tilde d}\Big\}\\
a_2&=\frac{1}{\Delta}\frac{(s_4-s_3)s_2}{\tilde\lambda_0+\tilde\lambda_1}\\
a_3&=\frac{1}{\Delta}\frac{s_4+s_2}{s_3+\tilde d}\\
a_4&=\frac{1}{\Delta}\frac{s_3+s_2}{s_4+\tilde d}\\
\Delta&=\frac{(s_4-s_3)(s_2-\tilde d)}{\tilde d}+\frac{\tilde\lambda_0+\tilde\lambda_1}{\tilde d}\Big\{\frac{s_4+s_2}{s_3}-\frac{s_3+s_2}{s_4}\Big\}
\end{aligned}\tag{2.11}$$

Note that if $\varphi_{x^*}(\cdot)$ denotes the density of the invariant measure corresponding to a hedging-point policy as in (2.4), then

$$\varphi_{x^*}(x)=\varphi(x-x^*)\,.$$

Given a convex cost function, the average cost $\rho(x^*)$ corresponding to such a policy can be readily computed and is a convex function of the threshold value x^*.

In [5], Bielecki and Kumar have studied the mean square stability of the piecewise deterministic system, i.e., (2.1) with $\sigma=0$. They have shown that under (2.6) the policy (2.5) is mean square stable, and have computed the optimal threshold value x^* in (2.4). These results can be easily reproduced here by computing the limiting value of the invariant distribution as $\sigma\to 0$, which we do next. The roots s_2, s_3 and s_4 have the following asymptotic dependence on σ:

$$s_2=\frac{2d}{\sigma^2}+\mathcal{O}(1),\quad s_3=\frac{(r-d)\lambda_0-d\lambda_1}{d(r-d)}+\mathcal{O}(\sigma^2),\quad s_4=\frac{2(r-d)}{\sigma^2}+\mathcal{O}(1)\,. \tag{2.12}$$

Thus, using (2.11), we obtain

$$
\begin{aligned}
a_1, a_2 &= \frac{d\big[(r-d)\lambda_0 - d\lambda_1\big]}{r(\lambda_0+\lambda_1)^2} + \mathcal{O}(\sigma^2) \\
a_3 &= \frac{\sigma^2}{2}\,\frac{\big[(r-d)\lambda_0 - d\lambda_1\big]}{d(r-d)(\lambda_0+\lambda_1)} + \mathcal{O}(\sigma^4) \\
a_4 &= \frac{\sigma^2}{2}\,\frac{d\big[(r-d)\lambda_0 - d\lambda_1\big]}{r^2(r-d)(\lambda_0+\lambda_1)} + \mathcal{O}(\sigma^4)\,.
\end{aligned}
\tag{2.13}
$$

Let

$$
\alpha_0 := \frac{(r-d)\lambda_0 - d\lambda_1}{d(r-d)}
$$

and $\delta_z(x)$ denote the Dirac measure centered at z. Using (2.12) and (2.13), we can show that as $\sigma \to 0$, $\varphi_{x^*}(\cdot)$ converges weakly to a distribution with "density" $\overline{\varphi}_{x^*}(\cdot)$, given by

$$
\overline{\varphi}_{x^*}(x,i) = \begin{cases} \frac{\lambda_1\alpha_0}{\lambda_0+\lambda_1} e^{\alpha_0(x-x^*)}, & \text{for } x \le x^*,\ i=0, \\ \frac{d\alpha_0}{\lambda_0+\lambda_1}\delta_{x^*}(x) + \frac{d\lambda_1\alpha_0}{(r-d)(\lambda_0+\lambda_1)} e^{\alpha_0(x-x^*)}, & \text{for } x \le x^*,\ i=1, \\ 0, & \text{for } x > x^*. \end{cases}
$$

Using a cost as in [5] of the form

$$
c(x) = \frac{c^+ + c^-}{2}|x| + \frac{c^+ - c^-}{2}x\,, \tag{2.14}
$$

with c^+ and c^- positive constants, the average cost corresponding to the policy in (2.4) takes the form:

$$
\begin{aligned}
\rho(x^*) &= \sum_{i=0,1}\int_{-\infty}^{x^*} c(x)\overline{\varphi}_{x^*}(x,i)\,dx \\
&= c^+x^* - \frac{c^+ r\lambda_1}{(r-d)(\lambda_0+\lambda_1)\alpha_0} + \frac{r\lambda_1(c^+ + c^-)}{(r-d)(\lambda_0+\lambda_1)\alpha_0} e^{-\alpha_0 x^*}.
\end{aligned}
$$

In this manner, the results in [5] are reproduced exactly. One advantage of our approach is that the class of admissible policies does not have to be restricted as is done in [5], in order to guarantee the existence of solutions. With our method, optimality is obtained with respect to the class of all non-anticipative policies. Furthermore, our analysis shows that the stability of the zero-inventory policy is retained under additive noise in (2.1). Let us also note that conditions for the optimality of the zero-inventory policy under additive noise can be readily obtained for the cost in (2.14) using the density in (2.10).

3. The Mathematical Model. We first exhibit that the switching diffusion (1.1), (1.2) can be constructed on a given probability space. Our presentation follows [13], [14]; we repeat it here for the sake of clarity and completeness. Let U be a compact metric space, $\mathcal{S} := \{1, 2, \dots, N\}$, and

$$\begin{aligned}
&\bar{b} = \left[\bar{b}_1, \dots, \bar{b}_d\right]' : \mathbb{R}^d \times \mathcal{S} \times U \to \mathbb{R}^d, \\
&\sigma = \left[\sigma_{ij}(\cdot,\cdot)\right] : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^{d\times d}, \\
&\bar{\lambda}_{ij} : \mathbb{R}^d \times U \to \mathbb{R}, \qquad i, j \in \mathcal{S}, \\
&\bar{\lambda}_{ij} \geq 0, \text{ for } i \neq j, \qquad \sum_{j\in\mathcal{S}} \bar{\lambda}_{ij} = 0, \text{ for any } i \in \mathcal{S}.
\end{aligned}$$

We also define the matrix $\widetilde{\Lambda} : \mathbb{R}^d \times U \to \mathbb{R}^{N\times N}$ by

$$\left[\widetilde{\Lambda}(x,u)\right]_{ij} = \begin{cases} \bar{\lambda}_{ij}(x,u), & i \neq j \\ 0, & i = j. \end{cases}$$

We make the following assumptions which will be in effect throughout the paper.

Assumption 3.1.

(i) *The functions $\bar{b}(x,k,u)$, $\sigma_{ij}(x,k)$ and $\bar{\lambda}_{ij}(x,u)$ are continuous and Lipschitz in x, uniformly with respect to u, with a Lipschitz constant γ_0. Let m_0 denote the least upper bound of $\|\bar{b}(0,k,\cdot)\|_\infty$, $|\sigma_{ij}(0,k)|$ and $\|\bar{\lambda}_{ij}(0,\cdot)\|_\infty$.*

(ii) *$\sigma_{ij}(\cdot,\cdot)$ is uniformly elliptic, i.e., there exists a constant $m > 0$ such that $\sigma(\cdot,k)\sigma'(\cdot,k) \geq mI$.*

(iii) *The matrix $\widetilde{\Lambda}(x,u)$ is irreducible for all $(x,u) \in \mathbb{R}^d \times U$.*

For a Polish space Y, $\mathfrak{B}(Y)$ denotes its Borel σ-field and $\boldsymbol{\mathcal{P}}(Y)$ the space of probability measures endowed with the Prohorov topology, i.e., the topology of weak convergence. Let $\mathfrak{M}(Y)$ be the set of all nonnegative, integer-valued, σ-finite measures on $\mathfrak{B}(Y)$. Let $\mathfrak{M}_\sigma(Y)$ be the smallest σ-field on $\mathfrak{M}(Y)$ with respect to which all the maps from $\mathfrak{M}(Y)$ to $\mathbb{N}\bigcup\{\infty\}$ of the form $\mu \mapsto \mu(B)$ with $B \in \mathfrak{B}(Y)$ are measurable. $\mathfrak{M}(Y)$ is assumed to be endowed with this measurability structure. Let $\mathcal{V} = \boldsymbol{\mathcal{P}}(U)$ and $b = \left[b_1, \dots, b_d\right]' : \mathbb{R}^d \times \mathcal{S} \times \mathcal{V} \to \mathbb{R}^d$ be defined by

$$b_i(\cdot,\cdot,v) = \int_U \bar{b}_i(\cdot,\cdot,u)v(du). \tag{3.1}$$

Similarly, for $i, j \in \mathcal{S}$ and $v \in \mathcal{V}$, λ_{ij} is defined as

$$\lambda_{ij}(\cdot,v) = \int_U \bar{\lambda}_{ij}(\cdot,u)v(du). \tag{3.2}$$

For $i, j \in \mathcal{S}$, $x \in \mathbb{R}^d$ and $v \in \mathcal{V}$, let $\Delta_{ij}(x,v)$ be consecutive (with respect to the lexicographic ordering on $\mathcal{S} \times \mathcal{S}$), left closed, right open intervals of the real line, each having length $\lambda_{ij}(x,v)$. Define a function $h : \mathbb{R}^d \times \mathcal{S} \times \mathcal{V} \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$h(x,i,v,z) = \begin{cases} j - i, & \text{if } z \in \Delta_{ij}(x,v) \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Let $\big(X(t), S(t)\big)$ be the $(\mathbb{R}^d \times \mathcal{S})$-valued, controlled, switching diffusion process given by the following stochastic differential equations.

$$
\begin{aligned}
dX(t) &= b\big(X(t), S(t), v(t)\big)\, dt + \sigma\big(X(t), S(t)\big) dW(t) \\
dS(t) &= \int_{\mathbb{R}} h\big(X(t), S(t-), v(t), z\big)\, \mathfrak{p}(dt, dz)\,,
\end{aligned} \tag{3.4}
$$

for $t \geq 0$ with $X(0) = X_0$, $S(0) = S_0$, where

(i) X_0 is a prescribed $\mathbb{R}^d$-valued random variable.
(ii) S_0 is a prescribed $\mathcal{S}$-valued random variable.
(iii) $W(\cdot) = \big[W_1(\cdot), \dots, W_d(\cdot)\big]'$ is a d-dimensional standard Wiener process.
(iv) $\mathfrak{p}(dt, dz)$ is an $\mathfrak{M}(\mathbb{R}_+ \times \mathbb{R})$-valued Poisson random measure with intensity $dt \times m(dz)$, where m is the Lebesgue measure on $\mathbb{R}$.
(v) $\mathfrak{p}(\cdot,\cdot)$, $W(\cdot)$, X_0 and S_0 are independent.
(vi) $v(\cdot)$ is a $\mathcal{V}$-valued process with measurable sample paths satisfying the non-anticipativity property that the σ-fields $\mathfrak{F}^v_t$ and $\mathfrak{F}^{W,\mathfrak{p}}_{[t,\infty)}$ given by

$$
\begin{aligned}
\mathfrak{F}^v_t &= \sigma\{v(s), s \leq t\}, \\
\mathfrak{F}^{W,\mathfrak{p}}_{[t,\infty)} &= \sigma\big\{W(s) - W(t), \mathfrak{p}(A, B) \,:\, A \in \mathfrak{B}\big([s,\infty)\big), B \in \mathfrak{B}(\mathbb{R}), s \geq t\big\}.
\end{aligned}
$$

are independent, for each $t \in \mathbb{R}$.

A process $v(\cdot)$ satisfying (vi) is called an *admissible (control) policy*. If $v(\cdot)$ is a Dirac measure, i.e., $v(\cdot) = \delta_{u(\cdot)}$, where $u(\cdot)$ is U-valued, then it is called an admissible *nonrandomized* policy. An admissible policy is called *feedback* if $v(\cdot)$ is progressively measurable with respect to the natural filtration $\mathfrak{F}_t = \{X(s), S(s),\ s \leq t\}$.

A particular subclass of feedback policies is of special interest. A feedback policy $v(\cdot)$ is called a (homogeneous) Markov policy if $v(t) = \tilde{v}\big(X(t), S(t)\big)$ for a measurable map $\tilde{v} : \mathbb{R}^d \times \mathcal{S} \to \mathcal{V}$. With an abuse in notation the map $\tilde{v}$ itself is called a Markov policy. Let Π, Π_M and Π_{MD} denote the sets of all admissible, Markov and nonrandomized Markov policies respectively.

If $\big(W(\cdot), \mathfrak{p}(\cdot,\cdot), X_0, S_0, v(\cdot)\big)$ satisfying (i)–(vi) above are given on a prescribed probability space $(\Omega, \mathfrak{G}, P)$, then under Assumption 3.1, (3.4) admits an a.s. unique strong solution [17, Chap. 3], and $X(\cdot) \in C(\mathbb{R}_+; \mathbb{R}^d)$, $S(\cdot) \in D(\mathbb{R}_+; \mathcal{S})$, where $D(\mathbb{R}_+; \mathcal{S})$ is the space of right continuous functions on $\mathbb{R}_+$ with left limits taking values in $\mathcal{S}$. However, if $v(\cdot)$ is a feedback policy, then there exists a measurable map

$$
f : \mathbb{R}_+ \times C(\mathbb{R}_+; \mathbb{R}^d) \times D(\mathbb{R}_+; \mathcal{S}) \longrightarrow \mathcal{V}
$$

such that for each $t \geq 0$, $v(t) = f\big(t, X(\cdot), S(\cdot)\big)$ and is progressively measurable with respect to $\{\mathfrak{F}_t\}$. Thus, $v(\cdot)$ cannot be specified a priori in (3.4). Instead, one has to replace $v(t)$ by $f\big(t, X(\cdot), S(\cdot)\big)$, and (3.4) takes the form

$$
\begin{aligned}
dX(t) &= b\big(X(t), S(t), f(t, X(\cdot), S(\cdot))\big) dt + \sigma\big(X(t), S(t)\big) dW(t) \\
dS(t) &= \int_{\mathbb{R}} h\big(X(t), S(t-), f(t, X(\cdot), S(\cdot)), z\big) \mathfrak{p}(dt, dz)\,,
\end{aligned} \tag{3.5}
$$

for $t \geq 0$ with $X(0) = X_0$, $S(0) = S_0$. In general, (3.5) does not even admit a weak solution. However, if the feedback policy is Markov, then the existence of a unique strong solution can be established.

If $\boldsymbol{\mathcal{K}}(\mathbb{R}^d)$ is a vector space of real functions over $\mathbb{R}^d$, we adopt the notation $\boldsymbol{\mathcal{K}}(\mathbb{R}^d \times \mathcal{S})$ to indicate the space $\bigl(\boldsymbol{\mathcal{K}}(\mathbb{R}^d)\bigr)^N$, endowed with the product topology. For example,

$$L^p(\mathbb{R}^d \times \mathcal{S}) := \Bigl\{f : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R} \ :\ f(\cdot, i) \in L^p(\mathbb{R}^d),\ \text{for all } i \in \mathcal{S}\Bigr\}$$

and similarly we define $C^k(\mathbb{R}^d \times \mathcal{S})$, $W^{k,p}(\mathbb{R}^d \times \mathcal{S})$, etc. For $f \in W^{2,p}_{\ell oc}(\mathbb{R}^d \times \mathcal{S})$ and $u \in U$, we write

$$L^u f(x,k) = L^u_k f(x,k) + \sum_{j \in \mathcal{S}} \overline{\lambda}_{kj}(x,u) f(x,j)\,, \tag{3.6}$$

where

$$L^u_k = \frac{1}{2} \sum_{i,j,\ell=1}^{d} \sigma_{i\ell}(x,k)\sigma_{j\ell}(x,k) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} \overline{b}_j(x,k,u) \frac{\partial}{\partial x_j} \tag{3.7}$$

and more generally, for $v \in \mathcal{V}$

$$L^v f(x,k) = \int_U L^u f(x,k) v(du)\,. \tag{3.8}$$

The following result is proved in [14].

Theorem 3.1. *Under a Markov policy v, (3.4) admits an a.s. unique strong solution such that $\bigl(X(\cdot), S(\cdot)\bigr)$ is a strong Feller process with differential generator L^v.*

A Markov policy v is called *stable* if the corresponding process $\bigl(X(\cdot), S(\cdot)\bigr)$ is positive recurrent. In this case, the process has a unique invariant probability measure, denoted by $\eta_v \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d \times \mathcal{S})$. The uniqueness of η_v is guaranteed by Assumption 3.1. We assume that the set of stable Markov policies is nonempty.

The Optimization Problem. Let $\overline{c} : \mathbb{R}^d \times \mathcal{S} \times U \to \mathbb{R}_+$ be the cost function. The following assumption on the cost, $\overline{c}$ will be in effect throughout the paper.

Assumption 3.2. *For each $i \in \mathcal{S}$, $\overline{c}(\cdot, i, \cdot)$ is continuous.*

We define $c : \mathbb{R}^d \times \mathcal{S} \times \mathcal{V} \to \mathbb{R}_+$ by

$$c(x,i,v) = \int_U \overline{c}(x,i,u) v(du)\,. \tag{3.9}$$

Let $v(\cdot)$ be an admissible policy and $\bigl(X(\cdot), S(\cdot)\bigr)$ the corresponding process. The pathwise (long-run) average cost incurred under $v(\cdot)$ is

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T c\bigl(X(t), S(t), v(t)\bigr) dt\,. \tag{3.10}$$

We wish to a.s. minimize (3.10) over all admissible policies. Our goal is to establish the existence of a stable Markov policy which is a.s. optimal. In general, this is not the case, as the following simple counterexample shows [6]. Let $\bar{c}(x,i) = \exp(-\|x\|^2)$. Then for every stable Markov policy the average cost is positive a.s., while we can find an unstable Markov policy for which the average cost is a.s. zero, making it an optimal policy. We want to rule out this possibility, as stability is a very desirable property. We carry out our study under two alternate sets of hypotheses: (a) a condition on the cost which penalizes unstable behavior, (b) a blanket stability condition which implies that all Markov policies are stable. We describe these conditions in Section 6.

4. Recurrence, Ergodicity and Harmonic Functions of Switching Diffusions. Due to the interaction between the continuous and discrete components, the study of recurrence and ergodicity of switching diffusions is quite involved. Let v be a Markov policy which will be fixed throughout this section unless explicitly stated otherwise. Let $P^v : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{S} \to \boldsymbol{\mathcal{P}}(\mathbb{R}^d \times \mathcal{S})$ denote the transition function of the corresponding process $\big(X(\cdot), S(\cdot)\big)$. Also $P^v_{x,i}$ and $E^v_{x,i}$ denote the probability measure and the expectation operator, respectively, on the canonical space, of the process $\big(X(\cdot), S(\cdot)\big)$ starting at $(x,i) \in \mathbb{R}^d \times \mathcal{S}$. The following result plays a crucial role in recurrence.

Lemma 4.1. *For any $(t,x,i) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{S}$, the support of $P^v(t,x,i;\cdot)$ is $\mathbb{R}^d \times \mathcal{S}$.*

Proof. For each $i \in \mathcal{S}$, let τ_i denote the sojourn time of $S(t)$ in state i. Then

$$P^v_{x,i}\big(\tau_i > t\big) = E^v_{x,i}\left[\exp\Big(\int_0^t \lambda_{ii}\big(X(s), v(X(s),S(s))\big)\,ds\Big)\right].$$

Let $\lambda^v_{ij}(s) := \lambda_{ij}\big(X(s), v(X(s),S(s))\big)$, $I_{A,j}(s) := I\{X(s) \in A, S(s) = j\}$, and P^v_i be the transition function of the diffusion corresponding to L^v_i, i.e., the diffusion with no switching and $S(t) \equiv i$. For $A \in \mathfrak{B}(\mathbb{R}^d)$, $i,j \in \mathcal{S}$ and $t > 0$,

$$\begin{aligned}
P^v\big(t,x,i,A\times\{j\}\big) &= E^v_{x,i}\big[I_{A,j}(t) \bigm| \tau_i > t\big]P^v_{x,i}\big(\tau_i > t\big) + E^v_{x,i}\big[I_{A,j}(t) I\{\tau_i < t\}\big] \\
&= E^v_{x,i}\left[\exp\Big(\int_0^t \lambda^v_{ii}(s)\,ds\Big)\right] P^v_i(t,x,A)\delta_{ij} \\
&\quad + E^v_{x,i}\bigg[\int_0^t -\lambda^v_{ii}(s)\exp\Big(\int_0^s \lambda^v_{ii}(s')ds'\Big)ds \\
&\qquad \int_{\mathbb{R}^d} P^v_i(s,x,dy)\sum_{k\neq i}\lambda^v_{ik}(s)P^v\big(t-s,y,k,A\times\{j\}\big)\bigg] \qquad (4.1) \\
&= E^v_{x,i}\left[\exp\Big(\int_0^t \lambda^v_{ii}(s)\,ds\Big)\right] P^v_i(t,x,A)\delta_{ij} \\
&\quad + \sum_{k\neq i}\int_0^t E^v_{x,i}\left[-\lambda^v_{ii}(s)\lambda^v_{ik}(s)\exp\Big(\int_0^s \lambda^v_{ii}(s')ds'\Big)\right] \\
&\qquad \int_{\mathbb{R}^d} P^v_i(s,x,dy)P^v\big(t-s,y,k,A\times\{j\}\big)ds\,.
\end{aligned}$$

Define the transition matrix $\widetilde{\Pi}^v$ by

$$[\widetilde{\Pi}^v(t,x,A)]_{ij} = P^v\big(t,x,i,A\times\{j\}\big)\,.$$

Then we can suitably define the matrix measures

$$\Gamma_1^v, \Gamma_2^v : \mathbb{R}\times\mathbb{R}^d \to \big(\boldsymbol{\mathcal{P}}(\mathbb{R}^d)\big)^{N\times N}$$

with $\Gamma_1^v(t,x,A)$ positive, diagonal and $\Gamma_2^v(t,x,A)$ non-negative, irreducible (by Assumption 3.1 (iii)), for all $(t,x,A)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathfrak{B}(\mathbb{R}^d)$, provided A has positive Lebesgue measure, so as to write (4.1) in the form:

$$\widetilde{\Pi}^v(t,x,A) = \Gamma_1^v(t,x,A) + \int_0^t\int_{\mathbb{R}^d}\Gamma_2^v(s,x,dy)\widetilde{\Pi}^v(t-s,y,A)ds\,. \tag{4.2}$$

The desired result follows from (4.2), using the irreducibility of $\Gamma_2^v(t,x,A)$. $\square$

Let τ_{ii}, τ_j be the stopping times defined as follows:

$$\tau_{ii} = \inf\big\{t>0\,:\,S(t)=i \text{ and } S(t')\neq i,\ \text{for some } 0<t'<t\big\} \tag{4.3}$$

$$\tau_j = \inf\big\{t>0\,:\,S(t)=j\big\}. \tag{4.4}$$

Let $D\subset\mathbb{R}^d$ be a bounded open set and J a subset of $\mathcal{S}$. Define

$$\tau_{D,J} = \inf\big\{t\geq 0\,:\,\big(X(t),S(t)\big)\notin D\times J\big\} \tag{4.5}$$

$$\tau_D = \inf\big\{t\geq 0\,:\,X(t)\notin D\big\}. \tag{4.6}$$

Using (4.2) and well known arguments in Markov processes [12, Vol. I, p. 111] the following results can be proved.

Lemma 4.2. *If τ is a stopping time of the form τ_{ii}, τ_j, $\tau_{D,J}$ or τ_D, as defined in (4.3)–(4.6), then, for each compact set $K\subset\mathbb{R}^d$,*

$$\sup_{v\in\Pi_M,\,x\in K} E_{x,i}^v[\tau]<\infty\,.$$

It is well known that harmonic functions play an important role in the study of recurrence and ergodicity of Markov processes [3]. Therefore, we now turn to the analysis of some properties of the harmonic functions of the process $\big(X(\cdot),S(\cdot)\big)$ under the Markov policy v. The function f is called L^v*-harmonic* in D if it is bounded on compact subsets of D, and for all $x\in D$, $i\in\mathcal{S}$,

$$f(x,i) = E_{x,i}^v f\big(X(\tau_{V,J}),S(\tau_{V,J})\big), \tag{4.7}$$

for every neighborhood V of x having compact closure $\overline{V}$ in D and every subset $J\subset\mathcal{S}$ containing i. It is clear that if f is L^v-harmonic then

$$f(x,i) = E_{x,i}^v f\big(X(\tau_V),S(\tau_V)\big). \tag{4.8}$$

On the other hand, if (4.8) holds then by conditioning on $\mathfrak{F}_{\tau_{V,J}}$ we obtain

$$\begin{aligned} f(x,i) &= E_{x,i}^v\Big[E^v\big[f\big(X(\tau_V),S(\tau_V)\big)\ \big|\ \mathfrak{F}_{\tau_{V,J}}\big]\Big] \\ &= E_{x,i}^v\Big[E_{X_{\tau_{V,J}},S_{\tau_{V,J}}}^v\big[f\big(X(\tau_V-\tau_{V,J}),S(\tau_V-\tau_{V,J})\big)\big]\Big] \\ &= E_{x,i}^v\big[f\big(X(\tau_{V,J}),S(\tau_{V,J})\big)\big]\,, \end{aligned}$$

concluding that (4.7) and (4.8) are actually equivalent.

Lemma 4.3. *Let $D \subset \mathbb{R}^d$ be open. Then,*

(i) *Every L^v-harmonic function in D is continuous in D.*

(ii) *If $L^v f = 0$ in D and $f \in W^{2,p}(D \times \mathcal{S})$, then f is L^v-harmonic. Conversely, if f is L^v-harmonic and $f \in W^{2,p}_{\ell oc}(D \times \mathcal{S})$, then $L^v f = 0$ in D.*

(iii) *(Maximum Principle) Let D be connected and $f \geq 0$ and L^v-harmonic in D. Then f is either strictly positive in $D \times \mathcal{S}$ or identically zero.*

Proof. The proof of (i) is standard [3], [12, Vol. II, Chap. 13] and (ii) can easily be proved using generalized Ito's formula [18]. Let $x_0 \in D$, $i_0 \in \mathcal{S}$ and $r > 0$ be such that $f(x_0, i_0) = 0$ and $\overline{B}(x_0, r) \subset D$, where $\overline{B}(x_0, r) = \bigl\{x \in \mathbb{R}^d \,:\, \|x - x_0\| \leq r\bigr\}$. Then

$$0 = f(x_0, i_0) = \sum_{j \in \mathcal{S}} \int_{\partial B(x_0,r)} f(y, j) P^v_{x_0,i_0}\bigl(X(\tau_{B(x_0,r)}) \in dy, S(\tau_{B(x_0,r)}) = j\bigr).$$

Then, by Lemma 4.1, we can show using standard arguments [16, Chap. 6] that the support of the measure $P^v_{x_0,i_0}\bigl(X(\tau_{B(x_0,r)}) \in dy, S(\tau_{B(x_0,r)}) = j\bigr)$ is $\partial B(x_0, r) \times \mathcal{S}$. Hence,

$$f(y, j) = 0, \qquad \text{for all } y \in \partial B(x_0, r),\ j \in \mathcal{S}\,.$$

It follows that the set $\bigl\{y \,:\, f(y, j) = 0,\ j \in \mathcal{S}\bigr\}$ is open in D and since D is connected the result follows. □

We next state Harnack's inequality for L^v-harmonic functions, which extends a very important result in partial differential equations. This inequality plays a crucial role in proving the existence of a solution to the HJB equation via the vanishing discount method, as is done in Section 6. As far as we know, this result is not known in the literature on partial differential equations. The detailed proof of Harnack's inequality is quite elaborate and can be found in the Appendix. The proof follows the method introduced for diffusions by Krylov and Safonov [19], for deriving estimates for the oscillation of a harmonic function. For the system of coupled elliptic operators characterizing switching diffusions considerable complications arise in trying to follow the same methodology, due to the vector-valued nature of the L^v-harmonic functions. A crucial step in the proof is 'coupling' together the oscillations of the distinct components of the harmonic function. The irreducibility of the matrix $\widetilde{\Lambda}$ is essential in accomplishing this task.

Theorem 4.1 (Harnack's Inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $K \subset \Omega$ a closed set. There exists a constant $C > 0$, depending only on Ω, K, the dimension d, N, the bounds m, m_0 and the Lipschitz constant γ_0 introduced in Assumption 3.1, such that for any nonnegative function $f \in W^{2,p}_{\ell oc}(\Omega \times \mathcal{S})$, $p \in [1, \infty)$, satisfying $L^v f = 0$ in $\Omega \times \mathcal{S}$, for some Markov policy v,*

$$f(x, i) \leq C f(y, j)\,, \qquad \forall\ x, y \in K\,, \quad \forall\ i, j \in \mathcal{S}\,.$$

We now discuss the recurrence properties of switching diffusions. Our treatment closely follows [3], therefore we skip the details in several places. A point $(x, i) \in \mathbb{R}^d \times \mathcal{S}$ is said to be *recurrent* if given any $\varepsilon > 0$,

$$P^v_{x,i}\bigl(X(t_n) \in B(x, \varepsilon),\ S(t_n) = i,\ \text{for a sequence } t_n \uparrow \infty\bigr) = 1. \tag{4.16}$$

A point (x, i) is *transient* if

$$P^v_{x,i}\big(\|X(t)\| \to \infty, \text{ as } t \to \infty\big) = 1. \tag{4.17}$$

If all points of the switching diffusion are recurrent, then it is called recurrent. A transient switching diffusion is similarly defined. Note that the discrete component of the process has been ignored in the definition (4.17). The reason for doing so is that, in view of Assumption 3.1 (iii), we can show that provided that the continuous component visits a bounded set infinitely often with probability 1, then the discrete component is recurrent. More generally, a switching diffusion exhibits a dichotomy in that it is either recurrent or transient as we will later show.

Lemma 4.4. *The following statements are equivalent.*

(i) *The switching diffusion is recurrent;*
(ii) $P^v_{x,i}\big(X(t) \in D,\ S(t) = j,\ \textit{for some } t \geq 0\big) = 1$, *for any open set* $D \subset \mathbb{R}^d$ *and any* $j \in \mathcal{S}$.

Proof. We prove (i) → (ii) (the converse is easier). We distinguish two cases:
Case 1. Let $x \in D$, $i \neq j$. Let $B = B(x, \varepsilon)$ and B_1 be bounded open sets such that $\overline{B} \subset B_1$ and $\overline{B}_1 \subset D$. Let

$$\eta_1 = \inf\big\{t \geq 0 \,:\, X(t) \in \partial B_1\big\},$$

and inductively, for $n = 1, 2, \ldots,$

$$\begin{aligned}\eta_{2n} &= \inf\big\{t > \eta_{2n-1} \,:\, X(t) \in \partial B\big\}\\ \eta_{2n+1} &= \inf\big\{t > \eta_{2n} \,:\, X(t) \in \partial B_1\big\}.\end{aligned}$$

Then, by recurrence, $\eta_n < \infty$, $P^v_{x,i}$ − a.s. Note that

$$y, \ell \mapsto P^v_{y,\ell}\big(\tau_{(\overline{B}\times\{j\})^c} < \tau_{B_1}\big)$$

is L^v-harmonic in $B_1 \times \mathcal{S}$ and not identically zero. Therefore by Lemma 4.3,

$$\inf_{(y,\ell)\in\overline{B}\times\mathcal{S}} P^v_{y,\ell}\big(\tau_{(\overline{B}\times\{j\})^c} < \tau_{B_1}\big) > \delta_1 > 0, \tag{4.18}$$

for some $\delta_1 > 0$. Next we define

$$\begin{aligned}A_0 &= \big\{S(t) = j,\ \text{for some } t \in [0, \eta_1)\big\}\\ A_n &= \big\{S(t) = j,\ \text{for some } t \in [\eta_{2n}, \eta_{2n+1})\big\}.\end{aligned}$$

By (4.18) and the strong Markov property,

$$P^v_{x,i}\big(A_0^c\big) \leq (1 - \delta_1), \qquad P^v_{x,i}\Big(\bigcap_{k=0}^{n} A_k^c\Big) \leq (1 - \delta_1)^{n+1}.$$

Now,

$$\begin{aligned}
P^v_{x,i}\bigl(X(t) \in D,\, S(t) = j,\ \text{for no } t \geq 0\bigr) \\
&\leq P^v_{x,i}\bigl(X(t) \in \overline{B}_1,\ S(t) = j,\ \text{for no } t \geq 0\bigr) \\
&\leq \lim_{n\to\infty} P^v_{x,i}\Bigl(\bigcap_{k=0}^{n} A_k^c\Bigr) = 0.
\end{aligned}$$

Case 2. Suppose $x \notin D$ and let $B = B(x,\varepsilon)$, B_1 and D_1 be bounded open sets such that $B \bigcap D = \varnothing$, $\overline{B}_1 \subset D$ and $\overline{B} \bigcup \overline{B}_1 \subset D_1$. Let

$$\begin{aligned}
\eta'_1 &= \tau_{D_1} \\
\eta'_{2n} &= \bigl\{t > \eta'_{2n-1} \ :\ X(t) \in \partial B\bigr\} \\
\eta'_{2n+1} &= \bigl\{t > \eta'_{2n} \ :\ X(t) \in \partial D_1\bigr\}.
\end{aligned}$$

Let $\delta_2 > 0$ be such that

$$\inf_{(y,\ell)\in\partial D_1\times\mathcal{S}} P^v_{y,\ell}\bigl(\tau_{(\overline{B}_1\times\{j\})^c} < \tau_{(\overline{B}\times\{i\})^c}\bigr) > \delta_2 > 0.$$

Define

$$A'_n = \bigl\{X(t) \in \overline{B}_1, S(t) = j,\ \text{for some } t \in [\eta_{2n-1}, \eta_{2n})\bigr\}.$$

Then, as in the previous case,

$$P^v_{x,i}\bigl(X(t) \in D,\ S(t) = j,\ \text{for no } t \geq 0\bigr) = 0. \quad \square$$

In view of Lemma 4.4, the following results can be proved the same way as in [3], [4].

Lemma 4.5. *The following statements are equivalent.*

(i) *The switching diffusion is recurrent.*

(ii) $P^v_{x,i}\bigl(X(t) \in D,\ \textit{for some } t \geq 0\bigr) = 1$, *for all* $x \in \mathbb{R}^d$, $i \in \mathcal{S}$ *and any non-empty open set* D.

(iii) *There exists a compact set* $K \subset \mathbb{R}^d$ *such that* $P^v_{x,i}\bigl(X(t) \in K,\ \textit{for some } t \geq 0\bigr) = 1$ *for all* $(x,i) \in \mathbb{R}^d \times \mathcal{S}$.

(iv) $P^v_{x,i}\bigl(X(t_n) \in D,\ \textit{for a sequence } t_n \uparrow \infty\bigr) = 1$, *for all* $x \in \mathbb{R}^d$, $i \in \mathcal{S}$ *and any non-empty open set* D.

(v) *There exists a point* $z \in \mathbb{R}^d$, *a pair of numbers* r_0, r_1, $0 < r_0 < r_1$, *and a point* $y \in \partial B(z, r_1)$ *such that* $P^v_{y,i}\bigl(\tau_{\overline{B}(z,r_0)^c} < \infty\bigr) = 1$, *for any* $i \in \mathcal{S}$.

Theorem 4.2. *For any Markov policy, the switching diffusion is either recurrent or transient.*

A recurrent switching diffusion admits a unique (up to a constant multiple) σ-finite invariant measure. The switching diffusion is called *positive recurrent* if it is recurrent and admits a finite invariant measure.

A Markov policy v is called *stable* if the corresponding process is positive recurrent; the corresponding invariant probability measure is denoted by η_v.

As is well known from the general theory of dynamical systems, even if L_i^v generates a positive recurrent diffusion, for each $i \in \mathcal{S}$, and the parametric Markov chain is ergodic, there is no reason to expect that the policy v is stable, i.e., the switching diffusion is positive recurrent. Indeed, as the following example shows, the hybrid process can be anything from transient to positive recurrent.

Example 4.1. We first consider a piecewise deterministic system with state dependent Markovian switching. Let $E_+, E_- \subset \mathbb{R}^2$ be defined as follows:

$$\begin{aligned} E_+ &= \big\{(x_1, x_2) \,:\, x_1 > 0\big\} \bigcup \big\{x_2 \le 0, x_1 = 0\big\} \\ E_- &= \big\{(x_1, x_2) \,:\, x_1 < 0\big\} \bigcup \big\{x_2 \ge 0, x_1 = 0\big\}. \end{aligned}$$

Let

$$A_0 = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -3 & 1 \\ -1 & -3 \end{bmatrix}.$$

Consider two stable dynamical systems $\mathcal{D}_0$ and $\mathcal{D}_1$ defined by

$$\mathcal{D}_0 : \quad \dot{x} = \begin{cases} A_0 x, & x \in E_+ \\ A_1 x, & x \in E_- \end{cases}$$

and

$$\mathcal{D}_1 : \quad \dot{x} = \begin{cases} A_1 x, & x \in E_+ \\ A_0 x, & x \in E_- . \end{cases}$$

For $\delta > 0$, let Z be a (parameterized) Markov chain taking values in $\{0, 1\}$ with rate matrix

$$\begin{bmatrix} -\delta & \delta \\ \frac{1}{\delta} & -\frac{1}{\delta} \end{bmatrix} \text{ on } E_+, \quad \text{and} \quad \begin{bmatrix} -\frac{1}{\delta} & \frac{1}{\delta} \\ \delta & -\delta \end{bmatrix} \text{ on } E_-$$

and consider the dynamical system

$$\mathcal{D} := \mathcal{D}_Z .$$

If we define η by

$$\eta = \begin{cases} Z, & x \in E_+ \\ 1 - Z, & x \in E_- , \end{cases}$$

then η is Markovian with rate matrix

$$\begin{bmatrix} -\delta & \delta \\ \frac{1}{\delta} & -\frac{1}{\delta} \end{bmatrix},$$

and $\mathcal{D}$ can represented as

$$\dot{x} = A_\eta x.$$

Define

$$T_0(t) = \{\tau \le t \,:\, \eta(\tau) = 0\}$$
$$T_1(t) = \{\tau \le t \,:\, \eta(\tau) = 1\},$$

and $\lambda_0(t) = m(T_0(t))$, $\lambda_1(t) = m(T_1(t))$, where m is the Lebesgue measure on $\mathbb{R}_+$. Then, the solution to $\mathcal{D}$ can be expressed as

$$x(t) = \exp\bigl(2\lambda_0(t) - 3\lambda_1(t)\bigr) \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x(0).$$

By the ergodic theory of Markov processes [25, Chap. 1], as $t \to \infty$,

$$\lambda_1(t) \sim \frac{\delta^2 t}{1+\delta^2}\,, \qquad \lambda_0(t) \sim \frac{t}{1+\delta^2}\,.$$

Thus,

$$2\lambda_0(t) - 3\lambda_1(t) \sim \frac{2-3\delta^2}{1+\delta^2}\, t\,.$$

Therefore, $\mathcal{D}$ is stable for $\delta < \sqrt{\frac{2}{3}}$ and unstable for $\delta \ge \sqrt{\frac{2}{3}}$. The matrices A_0, A_1 can be suitably altered to exhibit various other possibilities.

Now let $X(t)$ be defined as $dX(t) = A_{\eta(t)}X(t)dt + \sigma dW(t)$, where $W(\cdot)$ is a standard 2-dimensional Wiener process and $\sigma\sigma'$ is a 2×2 positive definite matrix with constant entries. Then it is easily shown that the stability (instability) of $\mathcal{D}$ implies the positive recurrence (transience) of $X(t)$. Note that in this example the drift is unbounded. However, in the study of recurrence, boundedness of the drift can be replaced by local boundedness.

Remark 4.1. In view of the above example, it is clear that two positive recurrent processes with suitable switching may result in a transient process. Similarly, the random combination of two transient processes may give rise to a positive recurrent process. This phenomenon can be exploited in many practical situations such as fault-tolerant control systems, flexible manufacturing systems, etc. In a control system with multiple modes, we can trade off the stability of some (or all) nodes to gain a desired degree of flexibility. Addition of a few redundant nodes and/or the incorporation of a suitable switching mechanism among the nodes could result in global stability of the system, thereby gaining flexibility without sacrificing reliability.

A general criterion for positive recurrence of a switching diffusion is provided by the following theorem:

Theorem 4.3. *Let z, r_0, r_1 be as in Lemma 4.5(v). Then the switching diffusion is positive recurrent if*

$$\sup_{y\in\partial B(z,r_1),\, i\in\mathcal{S}} E^v_{y,i}\bigl[\tau_{\overline{B}(z,r_0)^c}\bigr] < \infty\,. \tag{4.19}$$

The proof is standard [3]. Note that it may be very difficult to verify (4.19) for general b, σ, λ. One usually verifies (4.19) by constructing a Lyapunov function [3]. For switching diffusions such a construction seems difficult, since it involves solving a system of ordinary differential equations in closed form. However, we present some criteria for positive recurrence and discuss some implications.

(C1) There exists a $w \in C^2(\mathbb{R}^d \times \mathcal{S})$, $w \geq 0$ such that

(i) $w(x,i) \to \infty$, as $\|x\| \to \infty$.

(ii) For each $v \in \Pi_M$, $E^v_{x,i}\big[w\big(X(t), S(t)\big)\big]$ and $E^v_{x,i}\big|L^v w\big(X(t), S(t)\big)\big|$ are locally bounded.

(iii) There exists $p > 0$, $q > 0$ such that $L^u w(x,i) \leq p - q w(x,i)$, for each $u \in U$.

(C2) There exists a C^2 function $w : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}_+$ such that

(i) $\lim_{\|x\| \to \infty} w(x,i) = +\infty$.

(ii) There exists $a > 0$ and $\varepsilon > 0$ such that for $\|x\| > a$, $L^u w(x,i) < -\varepsilon$, for all $u \in U$, $i \in \mathcal{S}$, and $\|\nabla w(x,i)\|^2 \geq m^{-1}$, where m is the constant in Assumption 3.1 (ii).

(iii) $w(x,i)$ and $\|\nabla w(x,i)\|$ have polynomial growth.

Theorem 4.4. *Under either (C1) or (C2), the process $\big(X(\cdot), S(\cdot)\big)$ under any Markov policy v is positive recurrent. Thus, all Markov policies are stable.*

Proof. Under (C1), the result follows from [25, Theorem 25, p. 70]. Under (C2), the technique of the proof of [6, Lemma 6.2.2, p. 150] can be closely paralleled to draw the desired conclusion. □

Remark 4.2. If $\sigma \equiv I$ and $\bar{b}$ is such that $\langle \bar{b}(x,i,u), x\rangle < -(d+1)/2$ for all $i \in \mathcal{S}$ and $\|x\|$ sufficiently large, then $w(x) = \|x\|^2$ is a Lyapunov function for the system. We can construct several examples using this idea. Note that in this case all the diffusion generators L^u_i give rise to positive recurrent diffusions and have a common Lyapunov function, (i.e., one which is independent of i). If all L^u_i have a common Lyapunov function, then switching does not destabilize the hybrid system. Of course, this is a very strong condition and is rarely met.

5. Existence of an Optimal Policy. In this section we establish the existence of a stable, nonrandomized Markov optimal policy under certain conditions. We follow the methodology developed in [6], [8], [9], [10] for controlled diffusions. For switching diffusions, similar techniques carry through with some extra effort. Therefore, we present the main ideas skipping some of the technical details.

Let Π_{SM} and Π_{SMD} denote the set of stable Markov and stable nonrandomized Markov policies respectively. Since we are searching for an optimal policy in Π_{SMD}, it is natural to assume that Π_{SM} is nonempty. Let $v \in \Pi_{SM}$. Then

$$\rho_v := \sum_{i \in \mathcal{S}} \int_{\mathbb{R}^d} c\big(x, i, v(x,i)\big)\eta_v(dx, i) = \lim_{T \to \infty} \frac{1}{T}\int_0^T c\big(X(s), S(s), v(X(s), S(s))\big) ds\,, \quad \text{a.s.} \tag{5.1}$$

Let

$$\rho^* := \inf_{v\in\Pi_{SM}} \bigl\{\rho_v\bigr\}. \tag{5.2}$$

We assume that $\rho^* < \infty$. We now state a condition on the cost function which penalizes unstable behavior.

(C3) Assume that for each $i \in \mathcal{S}$,

$$\liminf_{\|x\|\to\infty} \bigl\{ \inf_{u\in U} \bar{c}(x,i,u)\bigr\} > \rho^*. \tag{5.3}$$

Intuitively, (5.3) penalizes trajectories lying outside the set $\inf_{u\in U}\bigl\{\bar{c}(x,i,u)\bigr\} \le \rho^*$, forcing an optimal process to spend a non-vanishing fraction of time in a bounded neighborhood of this compact set. This behavior results in the stability of every optimal policy. If $\bar{c}(x,i,u) = K(\|x\|)$ for some increasing function $K : \mathbb{R}_+ \to \mathbb{R}_+$ then it can be easily seen that (5.3) holds. Such cost functions arise quite often in practice. Condition (C3) is referred to as the *near-monotonicity* condition [6, Chap. 6].

For $v \in \Pi_{SM}$ (or Π_{SMD}), we define the ergodic occupation measure $\mu[v] \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d\times\mathcal{S}\times U)$ as

$$\mu[v](dx,i,du) = \eta_v(dx,i)v(x,i)(du). \tag{5.4}$$

Let

$$\begin{aligned} I_1 &= \bigl\{\mu[v] \;:\; v\in\Pi_{SM}\bigr\} \\ I_2 &= \bigl\{\mu[v] \;:\; v\in\Pi_{SMD}\bigr\}. \end{aligned}$$

The following results can be proved as in [10], [14].

Lemma 5.1. *The sets I_1, I_2 are closed, I_1 is convex and the set of extreme points of I_1 lies in I_2.*

Let $v(\cdot)$ be an arbitrary admissible policy. Define the $\boldsymbol{\mathcal{P}}(\mathbb{R}^d\times\mathcal{S}\times U)$-valued empirical process $\mu_t(v)$, for $t>0$ by

$$\mu_t(v)\bigl(A\times\{i\}\times B\bigr) = \frac{1}{t}\int_0^t I\bigl\{X(s)\in A, S(s)=i\bigr\}\, v(s)(B)\,ds\,, \tag{5.5}$$

with $A\in\mathfrak{B}(\mathbb{R}^d)$, $B\in\mathfrak{B}(U)$ and $i\in\mathcal{S}$. Let $\overline{\mathbb{R}}^d = \mathbb{R}^d\bigcup\{\infty\}$ be the one-point compactification of $\mathbb{R}^d$. We identify $\mu_t(v)$ with an element of $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d\times\mathcal{S}\times U)$ by assigning zero mass at $\{\infty\}\times\mathcal{S}\times U$. Since $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d\times\mathcal{S}\times U)$ is compact, $\{\mu_t(v)\}$, viewed as a $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d\times\mathcal{S}\times U)$-valued process, converges to a sample path dependent compact limit set in $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d\times\mathcal{S}\times U)$. Note that any element $\mu\in\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d\times\mathcal{S}\times U)$ can be decomposed as

$$\mu(C) = \delta_\mu\,\mu'\bigl(C\bigcap(\mathbb{R}^d\times\mathcal{S}\times U)\bigr) + (1-\delta_\mu)\mu''\bigl(C\bigcap(\{\infty\}\times\mathcal{S}\times U)\bigr)\,, \tag{5.6}$$

for $C\in\mathfrak{B}\bigl(\overline{\mathbb{R}}^d\times\mathcal{S}\times U\bigr)$. In this decomposition $\delta_\mu\in[0,1]$ is always uniquely defined, and $\mu'\in\boldsymbol{\mathcal{P}}(\mathbb{R}^d\times\mathcal{S}\times U)$ (respectively $\mu''\in\boldsymbol{\mathcal{P}}(\{\infty\}\times\mathcal{S}\times U)$) is also unique if $\delta_\mu>0$ (resp. $\delta_\mu<1$). We may render μ', μ'' unique at all times by imposing an arbitrary fixed choice thereof when $\delta_\mu=0$, respectively, 1.

Combining the results in [20] with the technique in [6, Lemma 6.1.1, p. 144], we establish the following lemma.

Lemma 5.2. *If $\mu \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d \times \mathcal{S})$ satisfies*

$$\sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} L^v f(x,i)\mu(dx,i) = 0\,, \qquad \forall\ f \in H, \tag{5.7}$$

for some Markov policy v, where H is a dense subset of $C_0^2(\mathbb{R}^d \times \mathcal{S})$, then $\mu = \eta_v$.

Proof. Using the usual approximation procedure we can show that (5.7) is true for all $f \in C_b^2(\mathbb{R}^d \times \mathcal{S})$. Let $\bigl(X(\cdot), S(\cdot)\bigr)$ be the process corresponding to the policy v with initial law μ. The law μ_t of this process, for $t > 0$, satisfies the Kolmogorov forward equation

$$\sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} f(x,i)\mu_t(dx,i) = \sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} f(x,i)\mu(dx,i) + \sum_{i\in\mathcal{S}} \int_0^t \int_{\mathbb{R}^d} L^v f(x,i)\mu_s(dx,i)\,ds\,,$$

for all $f \in C_b^2(\mathbb{R}^d \times \mathcal{S})$. The uniqueness of the solution to the above equation is established in [26]. Since $\mu_t \equiv \mu$ is a solution to (5.7), it follows that $\mu = \eta_v$. $\square$

We disintegrate $\mu' \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d \times \mathcal{S} \times U)$ as

$$\mu'(dx,i,du) = \mu^*(dx,i)v_\mu(x,i)(du)\,, \tag{5.8}$$

where μ^* is the marginal of μ' on $\mathbb{R}^d \times \mathcal{S}$ and v_μ is a version of the regular conditional law defined μ^* – a.s. We select an arbitrary version and keep it fixed henceforth. Using the martingale stability theorem, the following characterization of the limit points of $\{\mu_t(\cdot)\}$ can be established as in [6, Lemma 6.1.2].

Lemma 5.3. *Outside a set of zero probability, each limit point μ of $\{\mu_t(\cdot)\}$ for which $\delta_\mu > 0$ satisfies $\mu^* = \eta_{v_\mu}$.*

We now establish the existence of an optimal policy under (C3). Since the proof closely follows the steps in [6, Theorem 6.1.1], we only present a brief sketch.

Theorem 5.1. *Under (C3), there exists a stable Markov policy which is a.s. optimal.*

Proof. Let $v_n \in \Pi_{SM}$ be such that

$$\int \bar{c}\,d\mu[v_n] \downarrow \rho^*.$$

We extend $\mu[v_n]$ to $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d \times \mathcal{S} \times U)$ in the usual manner and denote it also by $\mu[v_n]$. Let μ_∞ be a limit point of $\{\mu[v_n]\}$ and denote $v_\infty = v_{\mu_\infty}$, where v_{μ_∞} is obtained from μ_∞ by the decomposition in (5.6) and (5.8). Then, for $f \in C_0^2(\mathbb{R}^d \times \mathcal{S})$,

$$\sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} L^{v_n} f(x,i)\eta_{v_n}(dx,i) = \sum_{i\in\mathcal{S}} \int_{\overline{\mathbb{R}}^d\times U} L^u f(x,i)\mu[v_n](dx,i,du) = 0\,.$$

Hence,

$$\sum_{i\in\mathcal{S}} \int_{\overline{\mathbb{R}}^d\times U} L^u f(x,i)\mu_\infty(dx,i,du) = 0\,.$$

Thus, by Lemmas 5.2 and 5.3, $\mu^*_\infty = \eta_{v_\infty}$, if $\delta_{\mu_\infty} > 0$. Using (C3), we can demonstrate as in [6, Lemma 6.1.3] that this is indeed the case. Therefore,

$$\min_{v\in\Pi_{SM}} \int \overline{c}\, d\mu[v] = \int \overline{c}\, d\mu[v_\infty] = \rho^*.$$

Finally, following the technique in [6, Lemma 6.1.3], we can now show that for an arbitrary policy u,

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T c\big(X(s), S(s), u(s)\big) ds \geq \rho^*, \quad \text{a.s.}$$

which establishes the optimality of v_∞ in a much stronger sense. $\square$

Theorem 5.2. *Under (C3) there exists a $v^* \in \Pi_{SMD}$ which is a.s. optimal.*

Proof. We have already established the existence of $v_\infty \in \Pi_{SM}$ which is a.s. optimal. We argue as in [7, p. 58]. Embed I_1 in $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d \times \mathcal{S} \times U)$ by assigning zero mass at $\{\infty\} \times \mathcal{S} \times U$. Let $\overline{I}_1$ denote the closure of I_1 in $\boldsymbol{\mathcal{P}}(\overline{\mathbb{R}}^d \times \mathcal{S} \times U)$. Then $\overline{I}_1$ is a compact convex set. By Choquet's theorem [24], each element μ of $\overline{I}_1$ is the barycenter of a probability measure m supported on the set of extreme points of $\overline{I}_1$. Now, each extreme point of I_1 must be an extreme point of $\overline{I}_1$, since otherwise it would be assigning a strictly positive mass to $\{\infty\} \times \mathcal{S} \times U$. If m assigns a strictly positive mass to extreme points of $\overline{I}_1$, which are not extreme points of I_1, then μ must assign a strictly positive probability to $\{\infty\} \times \mathcal{S} \times U$, which is not true. Thus, m must be supported on the set I_1^e consisting of the extreme points of I_1. In particular,

$$\int \overline{c}\, d\mu[v_\infty] = \int_{I_1^e} \left(\int \overline{c}\, d\nu\right) m(d\nu)\,.$$

It follows that there exists a $v^* \in \Pi_{SMD}$ such that

$$\int \overline{c}\, d\mu[v_\infty] = \int \overline{c}\, d\mu[v^*]$$

and since $v_\infty \in \Pi_{SM}$ is optimal, the optimality of $v^* \in \Pi_{SMD}$ follows. $\square$

We now investigate the existence of an optimal Markov policy under the blanket stability conditions in (C1)–(C2).

Lemma 5.4. *Under either (C1) or (C2), for any admissible policy $v \in \Pi$, the empirical process $\{\mu_t(v)\}$ defined in (5.5) is tight.*

The proof of Lemma 5.4 closely follows the arguments in the proof of [6, Theorem 6.2.2]. Topologize the space Π_M as in [6], [14]. We now state another result, the proof of which closely follows [14, Theorem 3.3, Lemma 4.4].

Lemma 5.5. *Under either (C1) or (C2), the sets I_1, I_2 are compact in total variation and the map $v \mapsto \mu[v]$ (as defined in (5.4)) is continuous.*

Theorem 5.3. *Under either (C1) or (C2), there exists a $v^* \in \Pi_{SMD}$ which is a.s. optimal.*

Proof. First note that under (C1) or (C2), $\Pi_{SM} = \Pi_M$ and $\Pi_{SMD} = \Pi_{SD}$. By Lemma 5.5, there exists a $\overline{v} \in \Pi_{SM}$ such that

$$\min_{v \in \Pi_{SM}} \int \overline{c}\, d\mu[v] = \int \overline{c}\, d\mu[\overline{v}].$$

In view of Lemma 5.4 and the decomposition and disintegration of the measure as defined in (5.6), (5.8), it suffices to confine our attention to Π_{SM} for optimality. Thus, the existence of an a.s. optimal $v^* \in \Pi_{SMD}$ then follows via Choquet's theorem as in Theorem 5.2. $\square$

6. Hamilton-Jacobi-Bellman Equations. In this section, we study the HJB equations and characterize the optimal policy in terms of their solution. We introduce the following condition:

(C4) The cost function $\overline{c}$ is bounded, continuous and Lipschitz in its first argument uniformly with respect to the third.

We follow the vanishing discount approach, i.e., we derive the HJB equations for the ergodic criterion by taking the limit of the HJB equations for the discounted criterion as the discount factor approaches zero. The results and the broad outline of these proofs follow those of [9]. However, they differ in important technical details.

Let $V_\alpha(x,i)$ denote the discounted value function with discount factor $\alpha > 0$, i.e.,

$$V_\alpha(x,i) = \inf_{v \in \Pi} E^v_{x,i}\left[\int_0^\infty e^{-\alpha t} c\big(X(t), S(t), u(t)\big)\, dt\right], \qquad x \in \mathbb{R}^d, \quad i \in \mathcal{S}\,. \tag{6.1}$$

The following result is proved in [14].

Theorem 6.1. *Under (C4), V_α is the unique solution in $C^2\big(\mathbb{R}^d \times \mathcal{S}\big) \bigcap C_b\big(\mathbb{R}^d \times \mathcal{S}\big)$ of*

$$\inf_{u \in U} \big\{L^u V_\alpha(x,i) + \overline{c}(x,i,u)\big\} = \alpha V_\alpha(x,i). \tag{6.2}$$

For $i \in \mathcal{S}$, define

$$\begin{aligned} G_i &:= \big\{x \in \mathbb{R}^d \ :\ \inf_{u \in U} \overline{c}(x,i,u) \le \rho^*\big\}, \\ G &:= \bigcup_{i \in \mathcal{S}} G_i\,. \end{aligned} \tag{6.3}$$

Observe that by (C3), G is compact.

The following result plays a very crucial role.

Lemma 6.1. *Under (C3) and (C4), there exists $\alpha_0 \in (0,1)$ such that if $\alpha \in (0, \alpha_0]$, $\inf_{(x,i) \in \mathbb{R}^d \times \mathcal{S}} V_\alpha(x,i)$ is attained on the set G as defined in (6.3).*

Proof. Let $v_\alpha \in \Pi_{MD}$ be an optimal policy for the discount factor α. By the results of [14], for $i \in \mathcal{S}$,

$$\begin{aligned} &\sum_{k=1}^d \overline{b}_k(x,i,v_\alpha(x,i)) \frac{\partial V_\alpha(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x, v_\alpha(x,i)) V_\alpha(x,j) + \overline{c}(x,i,v_\alpha(x,i)) \\ &\qquad = \inf_{u \in U} \left\{ \sum_{k=1}^d \overline{b}_k(x,i,u) \frac{\partial V_\alpha(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x,u) V_\alpha(x,j) + \overline{c}(x,i,u) \right\} \quad \text{a.e.} \end{aligned} \tag{6.4}$$

We let $\|x_n\| \to \infty$ in $\mathbb{R}^d$ and fix $i \in \mathcal{S}$. For given α, let $\big(X^n(\cdot), S^n(\cdot)\big)$ be the process under the policy v_α with $X^n(0) = x_n$ and $S^n(0) = i$. We can show as in [21] that $\{X^n(\cdot) - x_n\}$ are tight as $C\big([0,\infty); \mathbb{R}^d\big)$-valued random variables. Dropping to a subsequence and using Skorohod's theorem [16, p. 9] we may assume that they are defined on a common probability space and converge a.s. in $C\big([0,\infty); \mathbb{R}^d\big)$ to some process $Y(\cdot)$. Hence, $\|X^n(t)\| \to \infty$, uniformly in $t \in [0,T]$ for each $T < \infty$, a.s. By (C3), there exist $\varepsilon > 0$ and $M > 0$, such that

$$\inf_{u\in U} \big\{\overline{c}(x,i,u)\big\} > \rho^* + 2\varepsilon\,, \qquad \text{if} \quad \|x\| > M\,, \quad \forall\; i \in \mathcal{S}\,.$$

We select a constant T_α such that

$$(\rho^* + 2\varepsilon)\big(1 - e^{-\alpha T_\alpha}\big) > \rho^* + \varepsilon\,,$$

i.e., $e^{-\alpha T_\alpha} < \frac{\varepsilon}{(\rho^*+2\varepsilon)}$. Since

$$V_\alpha(x_n, i) \ge E^{v_\alpha}_{x_n,i}\left[\int_0^{T_\alpha} e^{-\alpha t} c\big(X^n(t), S^n(t), v_\alpha(X^n(t), S^n(t))\big)\,dt\right],$$

it follows that

$$V_\alpha(x_n, i) > \frac{\rho^* + \varepsilon}{\alpha}\,, \tag{6.5}$$

for n sufficiently large. On the other hand, by a standard Tauberian theorem,

$$\limsup_{\alpha\to 0} \big\{\alpha V_\alpha(x,i)\big\} \le \rho^*, \qquad \forall\; (x,i) \in \mathbb{R}^d \times \mathcal{S}\,. \tag{6.6}$$

Fix $x_0 \in \mathbb{R}^d$. By (6.6), there exists $\alpha_0 = \alpha_0(x_0)$ such that $V_\alpha(x_0,\cdot) \le \frac{\rho^*+\frac{\varepsilon}{2}}{\alpha}$, for all $\alpha \le \alpha_0$. Hence, it follows from (6.5) that if $\alpha \le \alpha_0$, then $\inf_{x\in\mathbb{R}^d} V_\alpha(x,i)$ is attained in a set $\big\{x \in \mathbb{R}^d \;:\; \|x\| \le R(\alpha)\big\}$, for all $i \in \mathcal{S}$. Let

$$x_{\alpha,i} := \arg\min_{x\in\mathbb{R}^d} \big\{V_\alpha(x,i)\big\}\,, \qquad (x_\alpha, i_\alpha) := \arg\min_{i\in\mathcal{S}} \big\{V_\alpha(x_{\alpha,i}, i)\big\}\,.$$

Using (6.2) and the fact that, at a minimum, the gradient of $V_\alpha(\cdot,i)$ vanishes and its Hessian is positive semi-definite, we have, for $\alpha \le \alpha_0$,

$$\inf_{u\in U} \left\{\overline{c}\big(x_{\alpha,i}, i, u\big) + \sum_{j\in\mathcal{S}} \overline{\lambda}_{ij}\big(x_{\alpha,i}, u\big) V_\alpha\big(x_{\alpha,i}, j\big)\right\} \le \alpha V_\alpha(x_{\alpha,i}, i). \tag{6.7}$$

In turn, from (6.7),

$$\inf_{u\in U} \big\{\overline{c}(x_\alpha, i_\alpha, u)\big\} \le \alpha V_\alpha(x_\alpha, i_\alpha)\,, \qquad \forall\; \alpha \le \alpha_0\,. \tag{6.8}$$

We claim that $\alpha V_\alpha(x_\alpha, i_\alpha) \le \rho^*$ for all $\alpha > 0$. Indeed, for any $v \in \Pi_{SM}$,

$$V_\alpha(x,i) \le E^{v}_{x,i}\left[\int_0^{\infty} e^{-\alpha t} c\big(X(t), S(t), v(X(t), S(t))\big)\,dt\right], \qquad \forall\; (x,i) \in \mathbb{R}^d \times \mathcal{S}\,. \tag{6.9}$$

Integrating both sides of (6.9) with respect to $\eta_v(dx,i)$ and using Fubini's theorem, we obtain

$$\sum_{i\in\mathcal{S}}\int_{\mathbb{R}^d} V_\alpha(x,i)\eta_v(dx,i) \le \frac{\rho_v}{\alpha}\,.$$

Hence,

$$V_\alpha(x_\alpha,i_\alpha) \le \frac{\rho^*}{\alpha}\,. \tag{6.10}$$

From (6.8),

$$\inf_{u\in U}\bigl\{c(x_\alpha,i_\alpha,u)\bigr\} \le \rho^*,$$

concluding that $(x_\alpha,i_\alpha)\in G\times\mathcal{S}$. $\square$

Lemma 6.2. *Under (C3) and (C4), the map* $(x,y,i,j)\mapsto |V_\alpha(x,i)-V_\alpha(y,j)|$ *is bounded on compact subsets, uniformly in* $\alpha\in(0,\alpha_0]$.

Proof. In view of Lemma 6.1, it suffices to prove that $\overline{V}_\alpha(\cdot,\cdot) = V_\alpha(\cdot,\cdot) - V_\alpha(x_\alpha,i_\alpha)$ is uniformly bounded on compacta. By (6.2) and (6.4),

$$L^{v_\alpha}V_\alpha(x,i) = \alpha V_\alpha(x,i) - \bar{c}\bigl(x,i,v_\alpha(x,i)\bigr) \quad \text{a.e.}$$

Let $R>0$ be large enough so that $G\subset B(0,R)$. Let $\bigl(X(\cdot),S(\cdot)\bigr)$ be the process under the policy v_α and define $\tau = \inf\bigl\{t\ge 0\ :\ X(t)\notin B(0,2R)\bigr\}$. Then for $x\in B(0,R)$, using the strong Markov property

$$\begin{aligned}
V_\alpha(x,i) &= E^{v_\alpha}_{x,i}\biggl[\int_0^\infty e^{-\alpha t}\bar{c}\bigl(X(t),S(t),v_\alpha(X(t),S(t))\bigr)dt\biggr]\\
&= E^{v_\alpha}_{x,i}\biggl[\int_0^\tau e^{-\alpha t}\Bigl\{\bar{c}\bigl(X(t),S(t),v_\alpha(X(t),S(t))\bigr) - \alpha V_\alpha\bigl(X(\tau),S(\tau)\bigr)\Bigr\}dt\biggr]\\
&\qquad + E^{v_\alpha}_{x,i}\Bigl[V_\alpha\bigl(X(\tau),S(\tau)\bigr)\Bigr].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\bigl|V_\alpha(x,i) - E^{v_\alpha}_{x,i}V_\alpha\bigl(X(\tau),S(\tau)\bigr)\bigr|\\
&\qquad = \biggl|E^{v_\alpha}_{x,i}\int_0^\tau e^{-\alpha t}\Bigl\{\bar{c}\bigl(X(t),S(t),v_\alpha(X(t),S(t))\bigr) - \alpha V_\alpha\bigl(X(\tau),S(\tau)\bigr)\Bigr\}dt\biggr|.
\end{aligned}$$

Using (C4) and Lemma 4.2, we deduce that there exists a constant C_1 (independent of α) such that

$$\bigl|V_\alpha(x,i) - E^{v_\alpha}_{x,i}V_\alpha\bigl(X(\tau),S(\tau)\bigr)\bigr| \le C_1\,, \qquad \forall\ (x,i)\in B(0,R)\times\mathcal{S}\,. \tag{6.11}$$

We write

$$\begin{aligned}
V_\alpha(x,i) - V_\alpha(x_\alpha,i_\alpha) &= \Bigl(V_\alpha(x,i) - E^{v_\alpha}_{x,i}V_\alpha\bigl(X(\tau),S(\tau)\bigr)\Bigr)\\
&\qquad + \Bigl(E^{v_\alpha}_{x,i}V_\alpha\bigl(X(\tau),S(\tau)\bigr) - V_\alpha(x_\alpha,i_\alpha)\Bigr).
\end{aligned} \tag{6.12}$$

Let

$$f(x,i) = E^{v_\alpha}_{x,i} V_\alpha\big(X(\tau), S(\tau)\big) - V_\alpha(x_\alpha, i_\alpha).$$

We observe that $f \geq 0$ and $L^{v_\alpha} f = 0$ in $W^{2,p}\big(B(0,2R) \times \mathcal{S}\big)$, $2 \leq p < \infty$. Then, by Theorem 4.1, there exists a constant C_2 (independent of α) such that, in view of (6.11),

$$f(x,i) \leq C_2 f(x_\alpha, i_\alpha) \leq C_1 C_2\,, \qquad \forall\ (x,i) \in B(0,R) \times \mathcal{S}\,.$$

Hence,

$$V_\alpha(x,i) - V_\alpha(x_\alpha, i_\alpha) \leq C_1(1 + C_2)\,, \qquad \forall\ (x,i) \in B(0,R) \times \mathcal{S}\,. \quad \square$$

Corollary 6.1. *For any $\varepsilon > 0$ and any compact $K \subset \mathbb{R}^d$, there exists $\alpha_\varepsilon \in (0, \alpha_0]$ such that for all $x \in K$, $i \in \mathcal{S}$ and $\alpha \in (0, \alpha_\varepsilon)$,*

$$\alpha V_\alpha(x,i) < \rho^* + \varepsilon. \tag{6.13}$$

Proof. Follows directly from Lemma 6.2 and (6.10). $\square$

Theorem 6.2. *Under (C3) and (C4), there exists a function $V \in C^2(\mathbb{R}^d \times \mathcal{S})$ and a scalar $\rho \in \mathbb{R}$ such that for some fixed $i_0 \in \mathcal{S}$,*

$$\rho \leq \rho^*, \qquad V(0, i_0) = 0, \qquad \inf_{(x,i) \in \mathbb{R}^d \times \mathcal{S}} V(x,i) > -\infty \tag{6.14}$$

and the pair (V, ρ) satisfies the HJB equations given by

$$\inf_{u \in U} \big\{ L^u V(x,i) + \overline{c}(x,i,u) \big\} = \rho. \tag{6.15}$$

Moreover, among all pairs $(\varphi, \rho) \in W^{2,p}_{\ell oc}(\mathbb{R}^d \times \mathcal{S}) \times \mathbb{R}$, $2 \leq p < \infty$, satisfying (6.15), (V, ρ^*), is the unique one satisfying (6.14).

Proof. Set $\overline{V}_\alpha(x,i) = V_\alpha(x,i) - V_\alpha(0, i_0)$. Then $\overline{V}(0, i_0) = 0$ and by (6.2), (6.4),

$$L^{v_\alpha} \overline{V}_\alpha(x,i) = \alpha V_\alpha(x,i) - \overline{c}(x,i,v_\alpha(x,i)).$$

By Corollary 6.1, Lemma 6.2 and the interior estimates for solutions of uniformly elliptic systems [22, pp. 398–402], we can show using a standard bootstrap argument that for any $R > 0$, $2 \leq p < \infty$,

$$\sup_{\alpha \in (0, \alpha_\varepsilon)} \Big\| \overline{V}_\alpha(\cdot,\cdot) \Big\|_{W^{2,p}(B(0,R) \times \mathcal{S})} \leq C,$$

for some constant C. Since $W^{2,p}_{\ell oc} \hookrightarrow W^{1,p}_{\ell oc}$ is compact for $p \geq 1$, $\{\overline{V}_\alpha(\cdot), \alpha \in (0, \alpha_\varepsilon)\}$ is sequentially compact in $W^{1,p}_{\ell oc}$. Let $\alpha_n \to 0$ in $(0, \alpha_\varepsilon)$. By dropping to a subsequence, if necessary, let $\overline{V}_{\alpha_n} \to V$ in $W^{1,p}_{\ell oc}$ for some V. By the Sobolev imbedding theorem, this convergence is also uniform on compact subsets of $\mathbb{R}^d$. Let ρ be a limit point of $\alpha_n V_{\alpha_n}(0, i_0)$

and hence of $\alpha_n V_{\alpha_n}(x,i)$ for any $(x,i) \in \mathbb{R}^d \times \mathcal{S}$, in view of Lemma 6.2. By (6.13), $\rho \leq \rho^*$. It can be shown as in [2], [22, p. 420] that

$$\inf_{u \in U} \left\{ \sum_{k=1}^{d} \bar{b}_k(x,i,u) \frac{\partial \overline{V}_{\alpha_n}(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x,u) \overline{V}_{\alpha_n}(x,j) + \bar{c}(x,i,u) \right\}$$
$$\xrightarrow[n\to\infty]{} \inf_{u \in U} \left\{ \sum_{k=1}^{d} \bar{b}_k(x,i,u) \frac{\partial V(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x,u) V(x,j) + \bar{c}(x,i,u) \right\}$$

in $L^p_{\ell oc}$ strongly. From the above discussion, it follows that $V \in W^{1,p}_{\ell oc}$, for any $2 \leq p < \infty$, and V satisfies (6.15) in D' (i.e., in the sense of distributions). By elliptic regularity, $V \in W^{2,p}_{\ell oc}$, $2 \leq p < \infty$. In turn, by the Sobolev imbedding theorem, $V \in C^{1,\gamma}(\mathbb{R}^d \times \mathcal{S})$, for $0 < \gamma < 1$, γ arbitrarily close to 1, and hence by (C4), it is easy to see that

$$\inf_{u \in U} \left\{ \sum_{k=1}^{d} \bar{b}_k(x,i,u) \frac{\partial V(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x,u) V(x,j) + \bar{c}(x,i,u) \right\}$$

is in $C^{0,\gamma}(\mathbb{R}^d \times \mathcal{S})$. By elliptic regularity [15, p. 287] applied to (6.15), we conclude that $V \in C^{2,\gamma}(\mathbb{R}^d \times \mathcal{S})$. Clearly, $V(0, i_0) = 0$. It suffices to show that V is bounded below. For any $x \in \mathbb{R}^d$, $i \in \mathcal{S}$,

$$\begin{aligned} V(x,i) &= \lim_{n\to\infty} \big[V_{\alpha_n}(x,i) - V_{\alpha_n}(0,i_0)\big] \\ &\geq \lim_{n\to\infty} \big[V_{\alpha_n}(x_{\alpha_n},i) - V_{\alpha_n}(0,i_0)\big] + \lim_{n\to\infty} \big[V_{\alpha_n}(x_{\alpha_n},i_{\alpha_n}) - V_{\alpha_n}(x_{\alpha_n},i)\big]. \end{aligned} \tag{6.16}$$

Using Lemmas 6.1 and 6.2, it follows from (6.16) that for each $i \in \mathcal{S}$,

$$\inf_{(x,i)\in\mathbb{R}^d\times\mathcal{S}} V(x,i) > -\infty$$

and the proof of the first part of the theorem is complete. The second assertion can be shown by following the methodology in [9]. □

Further, based on Lemmas 6.1, 6.2 and Theorem 4.1, the following theorem can be proved using the techniques presented in [9]. We therefore skip the proof.

Theorem 6.3. *Assume (C3) and (C4). Let $v^* \in \Pi_{MD}$ be such that for each i*

$$\begin{aligned} &\inf_{u \in U} \left\{ \sum_{k=1}^{d} \bar{b}_k(x,i,u) \frac{\partial V(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}(x,u) V(x,j) + \bar{c}(x,i,u) \right\} \\ &= \sum_{k=1}^{d} \bar{b}_k\big(x,i,v^*(x,i)\big) \frac{\partial V(x,i)}{\partial x_k} + \sum_{j \in \mathcal{S}} \overline{\lambda}_{ij}\big(x,v^*(x,i)\big) V(x,j) + \bar{c}\big(x,i,v^*(x,i)\big) \quad a.e. \end{aligned} \tag{6.17}$$

Then $v^* \in \Pi_{SMD}$. The scalar ρ in (6.15) equals ρ^* and v^* is a.s. optimal. Moreover, $v \in \Pi_{SMD}$ is a.s. optimal if and only if it satisfies (6.17).

Remark 6.1. The boundedness condition on the cost function $\bar{c}$ may be relaxed. For unbounded $\bar{c}$ we can use a suitable truncation procedure to approximate $\bar{c}$ by a sequence of bounded functions. Then the arguments in [9, p. 202] can be paralleled to establish the results in Theorems 6.2–6.3.

We now study the HJB equation under (C1) and (C4). Recall that under (C1), $\Pi_M = \Pi_{SM}$.

Lemma 6.3. *Let w satisfy (C1). Then for any $v \in \Pi_{SM}$,*

(i) $\displaystyle\sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} w(x,i)\eta_v(dx,i) < \infty.$

(ii) $\displaystyle\lim_{t\to\infty} \frac{1}{t} E^v_{x,i}\big[w\big(X(t),S(t)\big)\big] = 0.$

Proof. Let $R > 0$ and τ_R the exit time of $X(t)$ from $B(0,R)$. Then by Ito's formula

$$E^v_{x,i}\Big[w\big(X(t\wedge\tau_R),S(t\wedge\tau_R)\big)\Big] - w(x,i) = E^v_{x,i}\Big[\int_0^{t\wedge\tau_R} L^v w\big(X(s),S(s)\big)ds\Big].$$

Letting $R\to\infty$, we have

$$E^v_{x,i}\big[w\big(X(t),S(t)\big)\big] - w(x,i) = E^v_{x,i}\Big[\int_0^{t} L^v w\big(X(s),S(s)\big)ds\Big].$$

Therefore, by using (C1), we have

$$\frac{d}{dt}E^v_{x,i}\big[w\big(X(t),S(t)\big)\big] \le p - qE^v_{x,i}\big[w\big(X(t),S(t)\big)\big].$$

Then by Gronwall's inequality,

$$E^v_{x,i}\big[w\big(X(t),S(t)\big)\big] \le \frac{p}{q} + w(x,i)e^{-qt}. \tag{6.18}$$

Both (i) and (ii) follow directly from (6.18). $\square$

Lemma 6.4. *Assume (C1) holds. Let $a > 0$ be such that,*

$$L^u w(x,i) \le -1\,, \qquad \textit{for all} \quad \|x\| > a\,,\ u\in U\,,\ i\in\mathcal{S}\,.$$

If

$$\tau_a := \inf\big\{t\ge 0 \,:\, \|X(t)\|\le a\big\}\,, \tag{6.19}$$

then, for all $v\in\Pi_M$, $\|x\|>a$ and $i\in\mathcal{S}$,

$$E^v_{x,i}\big[\tau_a\big] \le w(x,i)\,. \tag{6.20}$$

Proof. Let $v\in\Pi_M$. Choose $R>0$ such that $a<\|x\|<R$. Let

$$\tau'_R = \inf\big\{t\ge 0 \,:\, X(t)\notin B(0,R)\setminus B(0,a)\big\}.$$

Then by Ito's formula

$$E^v_{x,i}\Big[w\big(X(t\wedge\tau'_R),S(t\wedge\tau'_R)\big)\Big]=w(x,i)+E^v_{x,i}\Bigg[\int_0^{t\wedge\tau'_R}L^v w\big(X(s),S(s)\big)ds\Bigg].$$

Therefore,

$$E^v_{x,i}\Big[w\big(X(t\wedge\tau'_R),S(t\wedge\tau'_R)\big)\Big]\le w(x,i)-E^v_{x,i}\big[t\wedge\tau'_R\big].$$

Thus,

$$E^v_{x,i}\big[t\wedge\tau'_R\big]\le w(x,i). \tag{6.21}$$

Letting first $t\to\infty$ and then $R\to\infty$, invoking Fatou's lemma at each step, we obtain (6.20). $\square$

Theorem 6.4. *Under (C1) and (C4), the HJB equation (6.15) admits a unique solution (V,ρ) in the class $C^2(\mathbb{R}^d\times\mathcal{S})\bigcap\mathcal{O}(w)$, satisfying $V(0,i_0)=0$ for some fixed $i_0\in\mathcal{S}$.*

Proof. Let $v^*\in\Pi_{SMD}$ be a.s. optimal. The existence of such a v^* is guaranteed by Theorem 5.3. Let

$$K_1=\sup_{x,i,u}\big\{\overline{c}(x,i,u)\big\}$$
$$K_2=\sup_{v\in\Pi_{SMD}}\int\overline{c}\,d\mu[v].$$

We select an arbitrary sequence of smooth functions $\psi_n:\mathbb{R}^d\to[0,K_1+4K_2]$, $n\ge1$, that are zero on $B(0,n)$ and equal to K_1+4K_2 on the complement of $B(0,n+1)$, and define

$$c_{1n}(x,i,u)=\tfrac{1}{2}\big[\overline{c}(x,i,u)+\psi_n(x)\big]$$
$$c_{2n}(x,i,u)=\tfrac{1}{2}\big[\psi_n(x)-\overline{c}(x,i,u)\big].$$

Then for a sufficiently large n, c_{1n} and c_{2n} both satisfy the penalizing condition (C3). We select one such term of the sequence from now on and drop the subscript n for notational convenience. Let $\big(X(\cdot),S(\cdot)\big)$ be the process under the policy v^*. For $\alpha>0$, we define

$$V_{\alpha,1}(x,i)=E^{v^*}_{x,i}\Bigg[\int_0^\infty e^{-\alpha t}c_1\big(X(t),S(t),v^*(X(t),S(t))\big)dt\Bigg],$$

$$V_{\alpha,2}(x,i)=E^{v^*}_{x,i}\Bigg[\int_0^\infty e^{-\alpha t}c_2\big(X(t),S(t),v^*(X(t),S(t))\big)dt\Bigg],$$

$$V_{\alpha}(x,i)=E^{v^*}_{x,i}\Bigg[\int_0^\infty e^{-\alpha t}\overline{c}\big(X(t),S(t),v^*(X(t),S(t))\big)dt\Bigg].$$

Then we can modify the arguments in the proof of Lemma 6.2 to conclude that for a fixed $i_0\in\mathcal{S}$, $\big(V_{\alpha,1}(x,i)-V_{\alpha,1}(0,i_0)\big)$ and $\big(V_{\alpha,2}(x,i)-V_{\alpha,2}(0,i_0)\big)$ are bounded on compacta uniformly in $\alpha\in(0,\alpha_0]$, for some $\alpha_0>0$. Hence,

$$\begin{aligned}\overline{V}_\alpha(x,i):=&\,V_\alpha(x,i)-V_\alpha(0,i_0)\\ =&\big[V_{\alpha,1}(x,i)-V_{\alpha,1}(0,i_0)\big]-\big[V_{\alpha,2}(x,i)-V_{\alpha,2}(0,i_0)\big]\end{aligned}$$

is bounded on compact sets, uniformly in $\alpha \in (0, \alpha_0]$. Arguing as in the proof of Theorem 6.2 we conclude that $\overline{V}_\alpha(x,i) \to V(x,i)$, as $\alpha \to 0$, uniformly on compacta and in $W^{2,p}_{\ell oc}(\mathbb{R}^d \times \mathcal{S})$, for any $p \in [2,\infty)$, and that the limit V satisfies

$$L^{v^*} V(x,i) + \overline{c}(x,i,v^*(x,i)) = \rho^*,$$

with $V(0,i_0) = 0$. Using the strong Markov property, relative to the stopping time τ_a in (6.19), we obtain

$$\begin{aligned}\overline{V}_\alpha(x,i) = E^{v^*}_{x,i}\biggl[\int_0^{\tau_a} e^{-\alpha t}\bigl[\overline{c}\bigl(X(t),S(t),v^*(X(t),S(t))\bigr] - \alpha V_\alpha(0,i_0)\bigr)dt\biggr]\\ + E^{v^*}_{x,i}\Bigl[e^{-\alpha\tau_a}\overline{V}_\alpha\bigl(X(\tau_a),S(\tau_a)\bigr)\Bigr].\end{aligned}$$

Hence, by Lemma 6.4, for $\alpha \in (0,\alpha_0]$ and $\|x\| > a$,

$$\begin{aligned}\bigl|\overline{V}_\alpha(x,i)\bigr| &\le C_1 + C_2 E^{v^*}_{x,i}\bigl[\tau_a\bigr]\\ &\le C_1 + C_2 w(x,i),\end{aligned}$$

where C_1, C_2 are positive constants independent of α. Passing to the limit as $\alpha \to 0$, it follows that V is in the class $\mathcal{O}(w)$. Next we let $v \in \Pi_{SMD}$ be such that for each $i \in \mathcal{S}$,

$$\begin{aligned}&\sum_{k=1}^d \overline{b}_k(x,i,v(x,i))\frac{\partial V(x,i)}{\partial x_k} + \sum_{j\in\mathcal{S}} \overline{\lambda}_{ij}(x,v(x,i))V(x,j) + \overline{c}(x,i,v(x,i))\\ &\qquad = \inf_{u\in U}\left\{\sum_{k=1}^d \overline{b}_k(x,i,u)\frac{\partial V(x,i)}{\partial x_k} + \sum_{j\in\mathcal{S}} \overline{\lambda}_{ij}(x,u)V(x,j) + \overline{c}(x,i,u)\right\} \quad \text{a.e.}\end{aligned}$$

Suppose that for some $i' \in \mathcal{S}$, there exist $\delta > 0$, such that the set

$$D = \Bigl\{x \in \mathbb{R}^d \;:\; L^v V(x,i') \le \rho^* - \overline{c}\bigl(x,i',v(x,i')\bigr) - \delta\Bigr\}$$

has positive Lebesgue measure. By Ito's formula

$$E^v_{x,i}\bigl[V\bigl(X(t),S(t)\bigr)\bigr] - V(x,i) = E^v_{x,i}\biggl[\int_0^t L^v V\bigl(X(s),S(s)\bigr)ds\biggr].$$

This is justified because V is $\mathcal{O}(w)$. Therefore,

$$\begin{aligned}E^v_{x,i}\bigl[V\bigl(X(t),S(t)\bigr)\bigr] - V(x,i) \le E^v_{x,i}\biggl[\int_0^t \Bigl[\rho^* - \overline{c}\bigl(X(s),S(s),v(X(s),S(s))\bigr)\Bigr]ds\biggr]\\ - \delta E^v_{x,i}\biggl[\int_0^t I\bigl\{X(s)\in D, S(s)=i'\bigr\}ds\biggr].\end{aligned}$$

Dividing by t, letting $t \to \infty$ and using Lemma 6.3, we have

$$\rho_v \le \rho^* - \delta\eta_v\bigl(D \times \{i'\}\bigr).$$

Lemma 4.1 implies that η_v is mutually absolutely continuous with respect to the Lebesgue measure. Therefore, $\eta_v\big(D \times \{i'\}\big) > 0$. Hence, $\rho_v < \rho^*$ which contradicts the optimality of v^*. Thus, for each $i \in \mathcal{S}$,

$$\inf_{u\in U}\big\{L^u V(x,i) + \overline{c}(x,i,u)\big\} = \rho^* \quad \text{a.e.} \tag{6.24}$$

Similar arguments as in the proof of Theorem 6.2 establish that $V \in C^{2,\gamma}(\mathbb{R}^d \times \mathcal{S})$, where $0 < \gamma < 1$, γ arbitrarily close to 1. We now proceed to show uniqueness. Let (V', ρ') be another solution of (6.15) in the desired class satisfying $V'(0, i_0) = 0$. Using Ito's formula and Lemma 6.3, it again follows that $\rho' = \rho^*$. Therefore,

$$L^{v^*}\big(V'(x,i) - V(x,i)\big) \geq 0.$$

Let $\big(X(t), S(t)\big)$ be the process governed by v^* and with initial law η_{v^*}. Then,

$$M(t) := V'\big(X(t), S(t)\big) - V\big(X(t), S(t)\big)$$

is a submartingale satisfying

$$\sup_{t\geq 0} E^{v^*}|M(t)| \leq C_1' + C_2' \sum_{i\in\mathcal{S}} \int_{\mathbb{R}^d} w(x,i)\eta_{v^*}(dx, i) < \infty,$$

by Lemma 6.3, where C_1', C_2' are suitable constants. Here we are using the fact that both V and V' are of $\mathcal{O}(w)$. By the submartingale convergence theorem, $M(t)$ converges a.s. Since $\big(X(t), S(t)\big)$ is ergodic and irreducible under v^*, it follows that $V'(x,i) - V(x,i)$ must be constant a.s. This constant must be zero, since $V'(0, i_0) - V(0, i_0) = 0$. $\square$

Remark 6.2. For the stable case we have carried out our analysis under the Lyapunov condition (C1). Analogous results can be derived under the condition (C2).

7. Conclusions. We have analyzed the optimal control of switching diffusions with a pathwise average cost criterion. Under certain conditions we have established the existence of a stable, nonrandomized Markov policy which is a.s. optimal in the class of all admissible policies. Also, we demonstrate the existence of a unique solution to the associated HJB equations in C^2, under varying conditions, and the optimal policy is characterized as a minimizing selector of the Hamiltonian. We have applied our results to a manufacturing model of Bielecki and Kumar and have exhibited that our methodology affords both greater generality and ease of solution. By studying the recurrence and ergodic properties of switching diffusions we have also obtained two new results in p.d.e. viz. a strong maximum principle and Harnack's inequality for a weakly coupled elliptic system.

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Appendix. This Appendix is devoted to the proof of Theorem 4.1.

Given a domain $\Omega \subset \mathbb{R}^d$, a real function $\boldsymbol{u}$ defined on $\Omega \times \mathcal{S}$ is viewed as a vector valued function $\boldsymbol{u} = (u_1, \dots, u_N)$, with each component u_i being a real function on Ω.

Consider a second order operator L defined by (note that L_k is different from the operator in (3.7)):

$$\begin{aligned} (L\boldsymbol{u})_k(x) &:= L_k u_k(x) + \sum_{\substack{j\in\mathcal{S}\\ j\neq k}} c_{kj}(x) u_j(x)\,, \qquad k \in \mathcal{S} \\ L_k &:= \sum_{i,j=1}^{d} a^k_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^k_i(x) \frac{\partial}{\partial x_i} + c_{kk}(x)\,. \end{aligned} \tag{A.1}$$

Let $m, \overline{m}, \overline{\gamma}$ and ε_Ω be given positive constants, the last depending on the choice of a bounded domain Ω. We denote by $\mathfrak{L} = \mathfrak{L}(m, \overline{m}, \overline{\gamma}, \varepsilon_\Omega)$ the class of all such operators L, with coefficients $a^k_{ij}(\cdot) \in C^{0,1}(\mathbb{R}^d)$ and $b^k_i(\cdot), c_{kj}(\cdot) \in L^\infty(\mathbb{R}^d)$, satisfying:

(A.2) $m\|\zeta\|^2 \le \sum_{i,j=1}^{d} a^k_{ij}(x)\zeta_i\zeta_j \le \overline{m}\|\zeta\|^2$, for all $x, \zeta \in \mathbb{R}^d$, $k \in \mathcal{S}$.

(A.3) $\|b^k_i\|_\infty \le \overline{m}$, $\|c_{k\ell}\|_\infty \le \overline{m}$ and $\|a^k_{ij}(x) - a^k_{ij}(y)\|_\infty \le \overline{\gamma}\|x-y\|$, for all $x, y \in \mathbb{R}^d$, $i, j \in \{1, \dots, d\}$, $k, \ell \in \mathcal{S}$.

(A.4) $\sum_{i\in\mathcal{S}} c_{ki}(\cdot) = 0$ and $c_{kj} \ge 0$, for $j \neq k$.

(A.5) The matrix $\boldsymbol{C}(x;\varepsilon_\Omega) := \big[c_{ij}(x) \,:\, c_{ij}(x) \ge \varepsilon_\Omega, i \neq j\big]$ is irreducible at each $x \in \Omega$.

We denote by $\mathfrak{U}_\Omega$ the class of all nonnegative functions $\boldsymbol{u} \in W^{2,d}_{\ell oc}(\Omega\times\mathcal{S}) \bigcap C^0(\overline{\Omega}\times\mathcal{S})$, satisfying $L\boldsymbol{u} = 0$ in Ω, for some $L \in \mathfrak{L}$. If $\xi \in \mathbb{R}$, then $\boldsymbol{u} \ge \xi$ is to be interpreted as $u_i \ge \xi$, for all $i \in \mathcal{S}$, and if $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, then $\boldsymbol{u} \ge \boldsymbol{\xi} \iff u_i \ge \xi_i$, for all $i \in \mathcal{S}$. For better clarity, we denote all $\mathbb{R}^N$-valued quantities by a bold letter. Also operations such as 'inf' on $\mathbb{R}^N$-valued functions are meant to be componentwise. If Γ is a closed subset of Ω we define, for $x \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^N_+$,

$$\boldsymbol{\Psi}_x\big(\mathfrak{U}_\Omega, \Gamma; \boldsymbol{\xi}\big) := \inf_{\boldsymbol{u}\in\mathfrak{U}_\Omega} \big\{\boldsymbol{u}(x) \,:\, \boldsymbol{u} \ge \boldsymbol{\xi} \text{ on } \Gamma\big\}\,.$$

Deviating from the usual vector space notation, if D is a cube in $\mathbb{R}^d$ and $\delta > 0$, δD denotes the cube which is *concentric* to D and whose edges are δ times as long. For a measurable set $A \subset \mathbb{R}^d$, $|A|$ denotes the Lebesgue measure of A, while $\mathcal{B}(A)$ and $L^d(A)$ denote the sets of real-valued, measurable functions on A such that

$$\|f\|_{\mathcal{B}(A)} := \operatorname*{ess\,sup}_{x\in A} |f(x)| < \infty\,, \qquad \forall\ f \in \mathcal{B}(A)$$

and

$$\|f\|_{d;A} := \left(\int_A |f(x)|^d dx\right)^{1/d} < \infty\,, \qquad \forall\ f \in L^d(A)\,.$$

We use quite frequently the following comparison principle, which can be viewed as a weaker version of the maximum principle in that it holds even without condition (A.5): If $\boldsymbol{\varphi}, \boldsymbol{\psi} \in W^{2,d}_{\ell oc}(\Omega \times \mathcal{S}) \bigcap C^0(\overline{\Omega} \times \mathcal{S})$ satisfy $L\boldsymbol{\varphi} \le L\boldsymbol{\psi}$ in Ω and $\boldsymbol{\varphi} \ge \boldsymbol{\psi}$ on $\partial\Omega$, then $\boldsymbol{\varphi} \ge \boldsymbol{\psi}$ in $\overline{\Omega}$. The same comparison principle holds for $\varphi, \psi \in W^{2,d}_{\ell oc}(\Omega) \bigcap C^0(\overline{\Omega})$ relative to the set of operators $\{L_k\}_{k \in \mathcal{S}}$ as defined in (A.1).

We start with a measure theoretic result, announced in [19].

Lemma A.1. *Let $K \subset \mathbb{R}^d$ be a cube, $\Gamma \subset K$ a closed subset and $0 < \alpha < 1$. Define*

$$\boldsymbol{\mathcal{Q}} := \bigl\{ Q \,:\, Q \textit{ is a subcube of } K \textit{ and } |Q \textstyle\bigcap \Gamma| \ge \alpha |Q| \bigr\}$$

$$\widetilde{\Gamma} := \bigcup_{Q \in \boldsymbol{\mathcal{Q}}} \bigl(3Q \textstyle\bigcap K \bigr) .$$

Then either $\widetilde{\Gamma} = K$, or $|\widetilde{\Gamma}| \ge \frac{1}{\alpha} |\Gamma|$.

Proof. If $|\Gamma| \ge \alpha |K|$, then $K \in \boldsymbol{\mathcal{Q}}$ and $\widetilde{\Gamma} = K$. So we assume $|\Gamma| < \alpha |K|$ or equivalently, $K \notin \boldsymbol{\mathcal{Q}}$. We subdivide K into 2^d congruent subcubes with disjoint interiors. We select the ones in $\boldsymbol{\mathcal{Q}}$, while the remaining ones are similarly subdivided and the process is repeated indefinitely. Let $\boldsymbol{\mathcal{Q}}_0$ be the collection thus obtained and with $\widehat{Q}$ denoting the ancestor of Q, we define

$$\widehat{\Gamma} := \bigcup_{Q \in \boldsymbol{\mathcal{Q}}_0} \widehat{Q}.$$

Clearly, $\widehat{Q} \subset 3Q \bigcap K$; hence, $\widetilde{\Gamma} \supset \widehat{\Gamma}$. Note that, discarding repetitions, $\widehat{\Gamma}$ can be represented as a disjoint union of cubes $\widehat{Q}$ which are not in $\boldsymbol{\mathcal{Q}}$. Therefore, each member $\widehat{Q}$ of this union satisfies $|\widehat{Q} \bigcap \Gamma| < \alpha |\widehat{Q}|$, and by σ-additivity we obtain

$$|\widehat{\Gamma} \textstyle\bigcap \Gamma| < \alpha |\widehat{\Gamma}| \le \alpha |\widetilde{\Gamma}| .$$

By the regularity properties of the Lebesgue measure $|\widehat{\Gamma} \bigcap \Gamma| = |\Gamma|$ and the proof is complete. □

Next we state without proof a variant of the weak maximum principle of A. D. Aleksandroff.

Lemma A.2. *There exist constants $C_1 > 0$ and $\kappa_0 \in (0, 1]$ such that if $D \subset \mathbb{R}^d$ is any cube of volume $|D| \le \kappa_0$ and $\varphi \in W^{2,d}_{\ell oc}(D) \bigcap C^0(\overline{D})$, $f \in L^d(D)$ satisfy $L_k \varphi \ge f$ in D and $\varphi = 0$ on ∂D, for some $L \in \mathfrak{L}$, then*

$$\sup_{x \in D} \bigl\{ \varphi(x) \bigr\} \le C_1 |D|^{1/d} \|f\|_{d;D} .$$

For the remainder of this Appendix, D will denote an open cube in $\mathbb{R}^d$ of volume not exceeding the constant κ_0 in Lemma A.2.

Lemma A.3. *There exist constants $\beta_0 > 0$ and $\alpha_0 < 1$ such that, if Γ is a closed subset of some cube $D \subset \mathbb{R}^d$, satisfying $|\Gamma| \geq \alpha_0 |D|$, then*

$$\inf_{x \in \frac{1}{3}D} \boldsymbol{\Psi}_x\bigl(\mathfrak{U}_D, \Gamma; \boldsymbol{\xi}\bigr) \geq \beta_0 \boldsymbol{\xi}\,, \qquad \forall\ \boldsymbol{\xi} \in \mathbb{R}^N_+\,.$$

Proof. Observe that if $\boldsymbol{u} \in \mathfrak{U}_D$, then each component u_k satisfies $L_k u_k \leq 0$ in D. Define $\varphi', \varphi'' \in W^{2,d}_{\ell oc}(D) \bigcap C^0(\overline{D})$ by:

$$L_k \varphi'(x) = -I_\Gamma(x), \quad L_k \varphi''(x) = -I_{\Gamma^c}(x)\,, \qquad \text{in } D$$

$$\text{and} \quad \varphi'(x) = \varphi''(x) = 0\,, \qquad \text{on } \partial D\,.$$

Then $\varphi := \varphi' + \varphi''$ satisfies $L_k \varphi = -1$ in D and $\varphi = 0$ on ∂D. Without loss of generality, suppose D is centered at the origin and consider the function

$$\psi(x) := \prod_{i=1}^d \bigl(|D|^{2/d} - 4x_i^2\bigr).$$

Note that $\psi = 0$ on ∂D and $\psi > 0$ in D. In addition, there exists a positive constant C_2 such that

$$\inf_{x \in \frac{1}{3}D} \bigl\{\psi(x)\bigr\} \geq C_2 |D|^{2/d} \|L_k \psi\|_{\mathcal{B}(D)}\,, \qquad \forall\ L \in \mathfrak{L}\,.$$

Therefore, by the comparison principle,

$$\varphi(x) \geq \frac{\psi(x)}{\|L_k \psi\|_{\mathcal{B}(D)}} \geq C_2 |D|^{2/d}\,, \qquad \forall\ x \in \tfrac{1}{3}D\,. \tag{A.6}$$

Using Lemma A.2, we obtain

$$\begin{aligned} \varphi' &\leq C_1 |D|^{1/d} |\Gamma|^{1/d} = C_1 |D|^{2/d} \Bigl(\tfrac{|\Gamma|}{|D|}\Bigr)^{1/d} \\ \varphi'' &\leq C_1 |D|^{1/d} |\Gamma^c|^{1/d} = C_1 |D|^{2/d} \Bigl(1 - \tfrac{|\Gamma|}{|D|}\Bigr)^{1/d}. \end{aligned} \tag{A.7}$$

By (A.6) and (A.7),

$$\varphi'(x) \geq C_2 |D|^{2/d} - C_1 |D|^{2/d} \Bigl(1 - \tfrac{|\Gamma|}{|D|}\Bigr)^{1/d}, \qquad \forall\ x \in \tfrac{1}{3}D\,.$$

On the other hand, since $L_k \varphi' = 0$ in $D \setminus \Gamma$ and $\varphi' = 0$ on ∂D, the comparison principle yields

$$\inf_{x \in \frac{1}{3}D} \bigl\{u_k(x)\bigr\} \geq \xi_k \frac{C_2 - C_1 \Bigl(1 - \tfrac{|\Gamma|}{|D|}\Bigr)^{1/d}}{C_1 \Bigl(\tfrac{|\Gamma|}{|D|}\Bigr)^{1/d}}\,. \tag{A.8}$$

Selecting α_0 to satisfy

$$\alpha_0 \geq 1 - \Bigl(\frac{C_2}{2C_1}\Bigr)^d,$$

(A.8) yields

$$\inf_{x \in \frac{1}{3}D} \bigl\{u_k(x)\bigr\} \geq \frac{C_2 \xi_k}{2C_1}.$$

Hence, the claim follows with $\beta_0 = \frac{C_2}{2C_1}$. $\square$

Lemma A.4. *For each $\delta > 0$, there exists a constant $k'_\delta > 0$ such that if $Q \subset (1-\delta)D$ is a subcube of an open cube $D \subset \mathbb{R}^d$, then*

$$\boldsymbol{\Psi}_x\big(\mathfrak{U}_D, \tfrac{1}{3}Q; \boldsymbol{\xi}\big) \geq k'_\delta \boldsymbol{\xi}\,, \qquad \forall\; x \in 3Q \textstyle\bigcap (1-\delta)D\,, \quad \forall\; \boldsymbol{\xi} \in \mathbb{R}^N_+\,.$$

Proof. Let $B(r) \subset \mathbb{R}^d$, denote the ball of radius r centered at the origin. We claim that there exists a constant $m_0 > 0$, such that if $r \leq 1$, then

$$\inf_{x \in B(\frac{3r}{4})} \boldsymbol{\Psi}_x\big(\mathfrak{U}_{B(r)}, B\big(\tfrac{r}{4}\big); \boldsymbol{\xi}\big) \geq m_0\, \boldsymbol{\xi}\,, \qquad \forall\; \boldsymbol{\xi} \in \mathbb{R}^N_+\,. \tag{A.9}$$

In order to establish (A.9) we use the function

$$\varphi(x) := \exp\left\{a\left(1 - \tfrac{\|x\|^2}{r^2}\right)\right\} - 1\,, \qquad a := \tfrac{\overline{m}}{m}(16d+2)\,, \qquad x \in B(r)\,,$$

which satisfies $L_k\varphi(x) \geq 0$ for all $L \in \mathfrak{L}$, provided $\|x\| \geq \frac{r}{4}$ and $r \leq 1$. By the comparison principle, (A.9) holds with

$$m_0 = \frac{e^{\frac{7a}{16}} - 1}{e^{\frac{15a}{16}} - 1}\,.$$

It follows that if $B(r,y)$ is a ball of radius r centered at y, and x is an arbitrary point in D such that the distance between ∂D and the line segment joining x and y is at least r, then

$$\boldsymbol{\Psi}_x\big(\mathfrak{U}_D, B\big(\tfrac{r}{4}, y\big); \boldsymbol{\xi}\big) \geq (m_0)^\ell\, \boldsymbol{\xi}\,, \qquad \text{with} \quad \ell = \left\lceil \tfrac{4\|x-y\| - r}{2r} \right\rceil\,, \qquad \forall\; \boldsymbol{\xi} \in \mathbb{R}^N_+\,. \tag{A.10}$$

Choosing $r = \min\{\frac{2}{3}, \frac{\delta}{2}\}|Q|^{1/d}$ and applying (A.10), an easy calculation shows that the result holds with

$$k'_\delta := m_0^{\ell(\delta)}\,, \qquad \ell(\delta) := \left\lceil \frac{6\sqrt{d}}{\min\{1,\delta\}} \right\rceil. \qquad \square$$

Lemma A.5. *Suppose that there exist constants ε and θ, such that if $\Gamma \subset (1-\delta)D$ is a closed subset of some cube D and $\boldsymbol{\xi} \in \mathbb{R}^N_+$, then*

$$\inf_{x \in \frac{1}{3}D} \boldsymbol{\Psi}_x\big(\mathfrak{U}_D, \Gamma; \boldsymbol{\xi}\big) \geq \varepsilon \boldsymbol{\xi}, \quad \textit{whenever} \quad |\Gamma| \geq \theta |D|\,.$$

Then there exists a constant $k_\delta > 0$ such that

$$\inf_{x \in \frac{1}{3}D} \boldsymbol{\Psi}_x\big(\mathfrak{U}_D, \Gamma; \boldsymbol{\xi}\big) \geq \varepsilon k_\delta \boldsymbol{\xi}, \quad \textit{whenever} \quad |\Gamma| \geq \alpha_0\theta |D|\,,$$

where α_0 is the constant in Lemma A.3.

Proof. Suppose $|\Gamma| \geq \alpha_0\theta|D|$ and let $y \in \widetilde{\Gamma}$, with $\widetilde{\Gamma}$ as defined in Lemma A.1 corresponding to $\alpha = \alpha_0$ and $K = (1-\delta)D$. Then there exists a subcube $Q \subset K$ such that $|\Gamma \bigcap Q| \geq \alpha_0 |Q|$ and $y \in 3Q \bigcap K$. We use the identities,

$$\boldsymbol{\Psi}_x\big(\mathfrak{U}_D, \Gamma; \boldsymbol{\xi}\big) \geq \boldsymbol{\Psi}_x\Big(\mathfrak{U}_D, \widetilde{\Gamma};\, \inf_{y \in \Gamma} \boldsymbol{\Psi}_y\big(\mathfrak{U}_D, \Gamma; \boldsymbol{\xi}\big)\Big) \tag{A.11}$$

and

$$\boldsymbol{\Psi}_y\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big) \ge \boldsymbol{\Psi}_y\Big(\mathfrak{U}_D,\tfrac{1}{3}Q;\ \inf_{z\in\frac{1}{3}Q}\ \boldsymbol{\Psi}_z\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big)\Big) \tag{A.12}$$
$$\ge \boldsymbol{\Psi}_y\Big(\mathfrak{U}_D,\tfrac{1}{3}Q;\ \inf_{z\in\frac{1}{3}Q}\ \boldsymbol{\Psi}_z\big(\mathfrak{U}_Q,\Gamma\bigcap Q;\boldsymbol{\xi}\big)\Big)\,.$$

From Lemma A.3, we have

$$\inf_{z\in\frac{1}{3}Q}\ \boldsymbol{\Psi}_z\big(\mathfrak{U}_Q,\Gamma\bigcap Q;\boldsymbol{\xi}\big) \ge \beta_0\boldsymbol{\xi}\,. \tag{A.13}$$

From Lemma A.4, we obtain $\boldsymbol{\Psi}_y\big(\mathfrak{U}_D,\frac{1}{3}Q;\beta_0\boldsymbol{\xi}\big) \ge \beta_0 k'_\delta\boldsymbol{\xi}$, for all $y\in 3Q\bigcap K$. Hence, combining (A.12) and (A.13) yields

$$\inf_{y\in\Gamma}\ \boldsymbol{\Psi}_y\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big) \ge k_\delta\boldsymbol{\xi}\,,\qquad \text{with}\quad k_\delta := \beta_0 k'_\delta\,. \tag{A.14}$$

From Lemma A.1, $|\widetilde{\Gamma}| \ge \frac{1}{\alpha_0}|\Gamma| \ge \theta|D|$. Therefore, by hypothesis,

$$\inf_{x\in\frac{1}{3}D}\ \boldsymbol{\Psi}_x\big(\mathfrak{U}_D,\widetilde{\Gamma};k_\delta\boldsymbol{\xi}\big) \ge \varepsilon k_\delta\boldsymbol{\xi}\,,$$

which along with (A.11) and (A.14) yield the desired result. $\square$

Theorem A.1. *The following estimates hold:*

(i) *Let D be a cube and $\Gamma\subset(1-\delta)D$ a closed subset. Then for all $\boldsymbol{\xi}\in\mathbb{R}^N_+$,*

$$\inf_{x\in\frac{1}{3}D}\ \boldsymbol{\Psi}_x\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big) \ge \beta_0\left(\tfrac{|\Gamma|}{|D|}\right)^{\rho(\delta)}\boldsymbol{\xi}\,,\qquad \rho(\delta) := \frac{\log k_\delta}{\log\alpha_0}, \tag{A.15}$$

where the constants α_0, β_0 and k_δ are as in Lemma A.3 and Lemma A.5.

(ii) *There exists a real function F defined in $[0,1]$, with $F(\theta)>0$, if $\theta>0$ such that if $\Gamma\subset D$ is a closed subset of a cube D, then*

$$\inf_{x\in\frac{1}{3}D}\ \boldsymbol{\Psi}_x\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big) \ge F\left(\tfrac{|\Gamma|}{|D|}\right)\boldsymbol{\xi}\,,\qquad \forall\ \boldsymbol{\xi}\in\mathbb{R}^N_+\,. \tag{A.16}$$

Proof. Part (i) is direct consequence of Lemmas A.3 and A.5. For part (ii), choose $\delta = \frac{|\Gamma|}{4d|D|}$. Then,

$$\frac{|\Gamma\bigcap(1-\delta)D|}{|D|} \ge \frac{|\Gamma|}{|D|} - \big(1-(1-\delta)^d\big) \ge \frac{|\Gamma|}{|D|} - d\delta \ge \frac{3|\Gamma|}{4|D|}\,. \tag{A.17}$$

Since

$$\boldsymbol{\Psi}_x\big(\mathfrak{U}_D,\Gamma;\boldsymbol{\xi}\big) \ge \boldsymbol{\Psi}_x\big(\mathfrak{U}_D,\Gamma\bigcap(1-\delta)D;\boldsymbol{\xi}\big)\,,$$

the bound in (A.16) follows from (A.15) and (A.17), with

$$F(\theta) := \beta_0\left(\tfrac{3\theta}{4}\right)^{\rho(\frac{\theta}{4d})}\,.\quad\square$$

Definition A.1. If $A\subset\Omega$ we define the oscillation of a function $\boldsymbol{u}\in C^0(\overline{\Omega}\times\mathcal{S})$ over A by

$$\operatorname{osc}(\boldsymbol{u};A) = \max_{k\in\mathcal{S}}\ \sup_{x\in A}\big\{u_k(x)\big\} - \min_{k\in\mathcal{S}}\ \inf_{x\in A}\big\{u_k(x)\big\}\,.$$

The oscillation of a function in $C^0(\overline{\Omega})$ is defined in the usual manner.

Theorem A.2. *If D is a cube, $\boldsymbol{u} \in \mathfrak{U}_D$ and $q = F\big(\frac{1}{2}\big)$, with $F(\cdot)$ as defined in Theorem A.1 (ii), then*

$$\operatorname{osc}(u_k; \tfrac{1}{3}D) \le \big(1 - \tfrac{q}{2}\big) \operatorname{osc}(\boldsymbol{u}; D)\,, \qquad \forall\ k \in \mathcal{S}\,.$$

Proof. Let

$$\begin{aligned} M_k^a &:= \sup_{x \in \frac{1}{3}D} \big\{u_k(x)\big\}\,, \qquad & M^a &:= \max_{k \in \mathcal{S}}\ M_k^a \\ m_k^a &:= \inf_{x \in \frac{1}{3}D} \big\{u_k(x)\big\}\,, \qquad & m^a &:= \min_{k \in \mathcal{S}}\ m_k^a \end{aligned}$$

and M^b, m^b be the corresponding quantities relative to D. Consider the sets

$$\begin{aligned} \varGamma_1^{(k)} &:= \Big\{x \in D\ :\ u_k(x) \le \tfrac{M^b + m^b}{2}\Big\} \\ \varGamma_2^{(k)} &:= \Big\{x \in D\ :\ u_k(x) \ge \tfrac{M^b + m^b}{2}\Big\}. \end{aligned}$$

Suppose $|\varGamma_2^{(k)}| \ge \frac{1}{2}|D|$. Since $\boldsymbol{u} - m^b$ is nonnegative and $u_k - m^b \ge \frac{M^b - m^b}{2}$ in $\varGamma_2^{(k)}$, applying Theorem A.1 (ii), yields

$$u_k(x) - m^b \ge q\frac{M^b - m^b}{2}, \qquad \forall\ x \in \tfrac{1}{3}D\,.$$

Consequently, $m_k^a \ge m^b + q\frac{M^b - m^b}{2}$ and since $M^a \le M^b$, we obtain

$$M^a - m_k^a \le M^b - m^b - q\tfrac{M^b - m^b}{2} \le \big(1 - \tfrac{q}{2}\big)(M^b - m^b)\,. \tag{A.18}$$

On the other hand, if $|\varGamma_1^{(k)}| \ge \frac{1}{2}|D|$, then using the nonnegative function $M^b - \boldsymbol{u}$, we similarly obtain

$$M_k^a - m^a \le \big(1 - \tfrac{q}{2}\big)(M^b - m^b)\,, \tag{A.19}$$

and the result follows by (A.18)–(A.19). $\square$

Theorem A.3. *There exists a constant $M_1 > 0$ such that, for any $\boldsymbol{u} \in \mathfrak{U}_D$,*

$$\sup_{x \in \frac{1}{9}D} \big\{u_i(x)\big\} \le M_1 \max_{k \in \mathcal{S}}\ \inf_{x \in \frac{1}{9}D} \big\{u_k(x)\big\}\,, \qquad \forall\ i \in \mathcal{S}\,.$$

Proof. Let β_0 be as given in Lemma A.3 and with $\rho(\cdot)$ and q as in (A.15) and Theorem A.2, respectively, define

$$\rho := \frac{1}{d\rho(\frac{2}{3})} \qquad \text{and} \qquad q_0 := \frac{(1 - \frac{q}{4})}{(1 - \frac{q}{2})}\,. \tag{A.20}$$

We claim that the value of the constant M_1 may be chosen as

$$M_1 := \frac{4q_0}{q\beta_0} \left[\frac{27N^{1/d}}{2\big(q_0^\rho - 1\big)}\right]^{1/\rho}. \tag{A.21}$$

We argue by contradiction. Suppose $\boldsymbol{u} \in \mathfrak{U}_D$ violates this bound and let $\{x^{(1)}, \dots, x^{(N)}\}$ denote the points in $\frac{1}{9}\overline{D}$ where the minima of $\boldsymbol{u}$ are attained, i.e.,

$$\inf_{x \in \frac{1}{9}D} \bigl\{u_k(x)\bigr\} = u_k(x^{(k)})\,, \qquad k \in \mathcal{S}\,.$$

Without loss of generality, suppose that $\max_{k\in\mathcal{S}}\bigl\{u_k(x^{(k)})\bigr\} = 1$ ($\boldsymbol{u}$ can always be scaled to satisfy this) and that for some $y_0 \in \frac{1}{9}D$ and $k_0 \in \mathcal{S}$, $u_{k_0}(y_0) = M > aM_1$, with $a > 1$. Using the estimate for the growth of the oscillation of $\boldsymbol{u}$ in Theorem A.2, we will show that $\boldsymbol{u}$ has to be unbounded in $\frac{1}{3}D$. By hypothesis, $\frac{M}{a}$ exceeds M_1 in (A.21) and in order to facilitate the construction which follows we choose to express this as

$$\frac{1}{9} + 3N^{1/d}\Bigl(\frac{4a}{q\beta_0 M}\Bigr)^{\rho}\sum_{n=0}^{\infty}\Bigl(\frac{1}{q_0}\Bigr)^{n\rho} < \frac{1}{3}\,. \tag{A.22}$$

For $\xi > 0$, define

$$\mathcal{D}_k^{(\xi)} := \bigl\{z \in \tfrac{1}{3}\overline{D} \,:\, u_k(z) \geq \xi\bigr\}\,, \qquad \mathcal{D}^{(\xi)} := \bigcup_{k\in\mathcal{S}} \mathcal{D}_k^{(\xi)}\,.$$

If $\mathbf{1}_k \in \mathbb{R}^N_+$ stands for the vector whose k-th component is equal to 1 and the others 0, then

$$\boldsymbol{u}(x^{(k)}) \geq \boldsymbol{\Psi}_{x^{(k)}}\bigl(\mathfrak{U}_D, \mathcal{D}_k^{(\xi)}; \xi\mathbf{1}_k\bigr)\,, \qquad \forall\ k \in \mathcal{S}\,, \tag{A.23}$$

while, on the other hand, Theorem A.1 yields,

$$\boldsymbol{\Psi}_{x^{(k)}}\bigl(\mathfrak{U}_D, \mathcal{D}_k^{(\xi)}; \xi\mathbf{1}_k\bigr) \geq \beta_0 \left(\frac{|\mathcal{D}_k^{(\xi)}|}{|D|}\right)^{\rho(\frac{2}{3})} \xi\mathbf{1}_k\,, \qquad \forall\ k \in \mathcal{S}\,. \tag{A.24}$$

By (A.23)–(A.24) and using (A.20), we obtain the estimate

$$|\mathcal{D}^{(\xi)}| \leq \sum_{k\in\mathcal{S}} |\mathcal{D}_k^{(\xi)}| \leq \sum_{k\in\mathcal{S}} \left(\frac{u_k(x^{(k)})}{\xi\beta_0}\right)^{\rho d} |D| \leq N\left(\frac{1}{\xi\beta_0}\right)^{\rho d} |D|\,, \qquad \forall\ \xi > 0\,. \tag{A.25}$$

Choosing $\xi = \frac{qM}{4}$, we have by (A.25)

$$\Bigl|\Bigl\{x \in \tfrac{1}{3}D \,:\, \max_{k\in\mathcal{S}}\{u_k(x)\} \geq \tfrac{qM}{4}\Bigr\}\Bigr| \leq N\left(\frac{4}{q\beta_0 M}\right)^{\rho d} |D|\,.$$

Hence, if Q_0 is a cube of volume $|Q_0| = N\bigl(\frac{4a}{q\beta_0 M}\bigr)^{\rho d}|D|$ centered at y_0, then

$$\operatorname{osc}(u_{k_0}; Q_0) \geq \bigl(1 - \tfrac{q}{4}\bigr)M\,. \tag{A.26}$$

By Theorem A.2, we obtain from (A.26)

$$\operatorname{osc}(\boldsymbol{u}; 3Q_0) \geq \frac{(1-\frac{q}{4})}{(1-\frac{q}{2})}M = q_0 M\,. \tag{A.27}$$

Since $\boldsymbol{u}$ is nonnegative, (A.27) implies that there exists $y^{(1)} \in 3Q_0$ and $k_1 \in \mathcal{S}$ such that

$$u_{k_1}(y^{(1)}) \geq q_0 M \,.$$

Note that (A.22) implies that $3Q_0 \subset \frac{1}{3}D$. Therefore, we can repeat the argument, now choosing $\xi = q_0 \frac{qM}{4}$ in (A.25) and a cube Q_1 of volume $N\big(\frac{4a}{q_0 q \beta_0 M}\big)^{\rho d}|D|$ centered at $y^{(1)}$, to conclude that there exists $y^{(2)} \in 3Q_1$ and $k_2 \in \mathcal{S}$ such that $u_{k_2}(y^{(2)}) \geq q_0^2 M$. Inductively, we can construct a sequence $\big\{y^{(n)}, k_n, Q_n\big\}_{n=0}^{\infty}$ satisfying, for all $n = 0, 1, \dots,$

$$\begin{aligned} y^{(0)} = y_0 \in \tfrac{1}{9}\overline{D} \textstyle\bigcap Q_0 \,, \qquad & y^{(n)} \in Q_n \textstyle\bigcap 3Q_{n-1} \,, \\ |Q_n|^{1/d} = N^{1/d} \big(\tfrac{1}{q_0}\big)^{n\rho} & \big(\tfrac{4a}{q\beta_0 M}\big)^{\rho} |D|^{1/d} \,, \\ u_{k_n}(y^{(n)}) & \geq q_0^n M \,. \end{aligned} \tag{A.28}$$

The inequality in (A.22) guarantees that $y^{(n)} \in \frac{1}{3}D$, for all n. But (A.28) implies that $\boldsymbol{u}$ is unbounded in $\frac{1}{3}D$ which is a contradiction. $\square$

Remark A.1. By the comparison principle, Lemmas A.3–A.5 and Theorem A.1 clearly hold unmodified for the class of L_k-superharmonic, nonnegative functions, i.e., functions $u \in W^{2,d}_{\ell oc}(D) \bigcap C^0(\overline{D})$, satisfying $L_k u \leq 0$ in D, for some $k \in \mathcal{S}$ and $L \in \mathfrak{L}$. This fact will be used in the next result.

Lemma A.6. *Let $L \in \mathfrak{L}$, $k \in \mathcal{S}$ and suppose φ is a solution to the Dirichlet problem $L_k \varphi = -f$ in a cube $D \subset \mathbb{R}^d$, with $\varphi = 0$ on ∂D, with f satisfying*

$$0 \leq f(x) \leq M \,, \quad \forall\, x \in D \qquad \textit{and} \qquad \|f\|_{d;D} \geq \varepsilon > 0 \,,$$

for some constants M and ε. Then there exists a constant $C' = C'(M, \varepsilon, m, \overline{m}, \overline{\gamma})$ such that

$$\inf_{x \in \frac{1}{3}D} \big\{\varphi(x)\big\} \geq C'.$$

Proof. First note that the Dirichlet problem as defined has a unique strong solution $\varphi \in W^{2,p}_{\ell oc}(D) \bigcap C^0(\overline{D})$, for all $p \in [d, \infty)$. We argue by contradiction. Suppose there exists a sequence of operators $\big\{L^{(n)}\big\}_{n=1}^{\infty} \subset \mathfrak{L}$ and a sequence of functions $\big\{f^{(n)}\big\}_{n=1}^{\infty}$, in accord with the hypotheses of the Lemma, such that the corresponding solutions $\big\{\varphi^{(n)}\big\}_{n=1}^{\infty}$ of $L_k^{(n)} \varphi^{(n)} = -f^{(n)}$ satisfy

$$\inf_{x \in \frac{1}{3}D} \big\{\varphi^{(n)}(x)\big\} < \frac{1}{n^2} \,, \qquad n = 1, 2, \dots$$

Thus, by Theorem A.1,

$$\Big|\big\{x \in D \,:\, \varphi^{(n)}(x) \geq \tfrac{1}{n}\big\}\Big| \leq \left(\frac{1}{\beta_0 n}\right)^{\rho d} |D| \,,$$

with ρ as defined in (A.20). Since the sequence $\varphi^{(n)}$ is bounded in $L^\infty(D)$ (by Lemma A.2), it follows that $\varphi^{(n)} \to 0$ in $L^p(D)$, as $n \to \infty$, for all $p \in [1, \infty)$. Let $D' = \delta D$, with $\delta < 1$, be a subcube of D, and let $\| \cdot \|_{2,p;D'}$ denote the standard norm of $W^{2,p}(D')$. We use the well known estimate

$$\|\varphi^{(n)}\|_{2,p;D'} \le C''\bigl(\|\varphi^{(n)}\|_{p;D} + \|f^{(n)}\|_{p;D}\bigr)\,,$$

for some constant $C'' = C''(|D|, p, \delta, d, m, \overline{m}, \overline{\gamma})$, to conclude that the first and second derivatives of $\varphi^{(n)}$ converge weakly to 0 in $L^p(D')$, for all $p \in [1, \infty)$. In turn, since $W_0^{2,p}(D') \hookrightarrow W_0^{1,p}(D')$ is compact for $p > d$, using the standard approximation argument we deduce that $\frac{\partial \varphi^{(n)}}{\partial x_i}$ converges in $L^p(D')$ strongly, for all $i = 1, \dots, d$. Also, since the second-order coefficients of $L_k^{(n)}$ are uniformly Lipschitz, we can extract a subsequence, along which they converge uniformly. Combining all the previous arguments, we deduce that the sequence $\bigl\{L_k^{(n)} \varphi^{(n)}\bigr\}$ converges weakly to 0 in $L^p(D')$, $p \in [1, \infty)$. On the other hand, if we choose $\delta \ge \Bigl(1 - \frac{\varepsilon}{2M|D|}\Bigr)^{1/d}$, an easy calculation yields,

$$\int_{D'} f^{(n)}(x)\, dx \ge \frac{\varepsilon}{2}\,, \qquad n = 1, 2, \dots\,,$$

resulting in a contradiction. $\square$

We pause to note that (A.5) has not been utilized in any of the results obtained thus far. It will be used in the next result to provide the necessary 'coupling' between distinct components of the harmonic function.

Lemma A.7. *For each cube $D \subset \mathbb{R}^d$ there exists a constant $M_2 > 0$ such that, for any $\boldsymbol{u} \in \mathfrak{U}_D$*

$$\inf_{x \in \frac{1}{9}D} \bigl\{u_i(x)\bigr\} \le M_2 \inf_{x \in \frac{1}{9}D} \bigl\{u_j(x)\bigr\}\,, \qquad \forall\, i, j \in \mathcal{S}\,.$$

Proof. Let ε_D be the constant in hypothesis (A.5). Define a collection of functions $\{\varphi_{ij}(x),\ i, j \in \mathcal{S}\} \subset W^{2,d}_{\ell oc}(\frac{1}{3}D) \bigcap C^0(\frac{1}{3}\overline{D})$, relative to some $L \in \mathfrak{L}$, by

$$\begin{aligned} &L_i \varphi_{ij}(x) = -c_{ij}(x) \quad \text{in } \tfrac{1}{3}D\,, \qquad \text{and} \quad \varphi_{ij}(x) = 0 \quad \text{on } \partial\bigl(\tfrac{1}{3}D\bigr)\,, \qquad \text{if} \quad i \ne j \\ &\quad \varphi_{ij}(x) = 0\,, \qquad \text{if} \quad i = j\,, \end{aligned} \tag{A.29}$$

and let $\boldsymbol{\Phi}(x)$, $\boldsymbol{C}(x)$ denote the matrices with elements $\bigl\{\varphi_{ij}(x)\bigr\}$ and $\bigl\{c_{ij}(x)\bigr\}_{i \ne j}$, respectively. By (A.4), there exists a constant irreducible matrix $\boldsymbol{C}_D \subset \mathbb{R}^{N \times N}$, with elements equal to 0 or 1 such that

$$\Bigl|\bigl\{x \in \tfrac{1}{3}D \;:\; \boldsymbol{C}(x) \ge \varepsilon_D \boldsymbol{C}_D\bigr\}\Bigr| \ge \tfrac{1}{N^2 3^d}|D|\,. \tag{A.30}$$

It follows, by (A.29), (A.30) and Lemma A.6 that there exists a constant $\varepsilon'_D > 0$, such that

$$\boldsymbol{\Phi}(x) \ge \varepsilon'_D \boldsymbol{C}_D\,, \qquad \forall\, x \in \tfrac{1}{9}D\,, \tag{A.31}$$

and (A.31) holds relative to any $L \in \mathfrak{L}$ used to generate φ_{ij}. Therefore, if $\boldsymbol{u} \in \mathfrak{U}_D$ and we define $\underline{\boldsymbol{u}} := \inf_{x \in \frac{1}{9}D} \boldsymbol{u}(x)$ and $\underline{\boldsymbol{u}}' := \inf_{x \in \frac{1}{3}D} \boldsymbol{u}(x)$, it is a direct consequence of the comparison principle that

$$\boldsymbol{u}(x) \geq \boldsymbol{\Phi}(x)\underline{\boldsymbol{u}}', \qquad \forall\, x \in \tfrac{1}{3}D. \tag{A.32}$$

On the other hand, by Theorem A.1,

$$\underline{\boldsymbol{u}}' \geq F\big(\tfrac{1}{9^d}\big)\underline{\boldsymbol{u}}. \tag{A.33}$$

By (A.31)–(A.33),

$$\boldsymbol{u}(x) \geq \varepsilon'_D F\big(\tfrac{1}{9^d}\big)\boldsymbol{C}_D\underline{\boldsymbol{u}}, \qquad \forall\, x \in \tfrac{1}{9}D,$$

which yields $\underline{\boldsymbol{u}} \geq \varepsilon'_D F\big(\tfrac{1}{9^d}\big)\boldsymbol{C}_D\underline{\boldsymbol{u}}$. In turn, the irreducibility of $\boldsymbol{C}_D$ implies that

$$\underline{\boldsymbol{u}}_i \geq \Big(\varepsilon'_D F\big(\tfrac{1}{9^d}\big)\Big)^{N-1}\underline{\boldsymbol{u}}_j, \qquad \forall\, i,j \in \mathcal{S}. \quad \square$$

Combining Theorem A.3 and Lemma A.7 and letting $M := M_1M_2$, we have:

Theorem A.4. *For each cube $D \subset \mathbb{R}^d$, $|D| \leq \kappa_0$ there exists a constant $M > 0$ such that, for any $\boldsymbol{u} \in \mathfrak{U}_D$*

$$u_i(y) \leq M u_j(x), \qquad \forall\, x, y \in \tfrac{1}{9}D, \quad \forall\, i,j \in \mathcal{S}.$$

Theorem 4.1 easily follows from Theorem A.4 by covering the domain $\varOmega$ with a collection of congruent cubes D of suitable size. For an elegant exposition of this technique see [11, pg. 153]. The existence of a constant $\varepsilon_\varOmega > 0$ satisfying (A.5) is guaranteed by the continuity and irreducibility conditions in Assumption 3.1 (i) and (iii), along with the compactness of U. Concerning (A.3), (A.4) and the upper bound in (A.2), observe that for each bounded domain $\varOmega$, Assumption 3.1 (i) implies the existence of constants $\overline{m}$ and $\overline{\gamma}$ satisfying all these conditions in $\varOmega$. This suffices for our purposes.