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Pricing American-Style Derivatives with European Call Options

Scott B. Laprise
BAE Systems, Advanced Information Technologies, 3811 N. Fairfax Drive, Arlington, Virginia 22203, scott.laprise@baesystems.com

Michael C. Fu
The Robert H. Smith School of Business, University of Maryland, College Park, Maryland 20742, mfu@rhsmith.umd.edu

Steven I. Marcus
Department of Electrical and Computer Engineering, University of Maryland, College Park, Maryland 20742, marcus@isr.umd.edu

Andrew E. B. Lim
Department of Industrial Engineering and Operations Research, University of California, Berkeley, California 94720, lim@ieor.berkeley.edu

Huiju Zhang
The Robert H. Smith School of Business, University of Maryland, College Park, Maryland 20742, joyzhang@glue.umd.edu

We present a new approach to pricing American-style derivatives that is applicable to *any* Markovian setting (i.e., not limited to geometric Brownian motion) for which European call-option prices are readily available. By approximating the value function with an appropriately chosen interpolation function, the pricing of an American-style derivative with *arbitrary* payoff function is converted to the pricing of a portfolio of *European call options*, leading to analytical expressions for those cases where analytical European call prices are available (e.g., the Merton jump-diffusion process). Furthermore, in many settings, the approach yields upper and lower analytical bounds that provably converge to the true option price. We provide computational results to illustrate the convergence and accuracy of the resulting estimators.

Key words: American-style derivatives; American options; European options; call options; early exercise; stochastic dynamic programming
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1. Introduction

The pricing of American-style derivatives remains one of the more challenging problems in derivatives finance. We define American-style derivatives, sometimes termed Bermudan derivatives, to be derivative contracts with early-exercise opportunities at a finite number of exercise dates prior to expiration (as opposed to continuously exercisable for a "pure" American derivative). The major difficulty in pricing such derivatives with "early-exercise" features lies in the determination of optimal early-exercise policies. Conversely, the pricing of European derivatives—specifically, European options—is a comparatively less difficult task. In particular, European call and put options can be priced analytically under numerous price processes beyond geometric Brownian motion; otherwise, they generally can be determined without too much difficulty via a numerical method such as numerical inversion or simulation. In this paper, we reduce the complexity of the pricing of an American-style derivative to that of pricing European call options. Specifically, by approximating the value functions with appropriately chosen linear segments in a dynamic programming recursion, we are able to price an American-style derivative by pricing a portfolio of European call options at varying strike prices, yielding *analytical* pricing formulas for underlying asset price processes with analytical European call-option prices.

We consider American-style derivatives written on a single underlying asset that follows a Markovian price process. One general approach to the pricing of such derivatives is to cast the problem in the framework of a stochastic dynamic programming problem and employ a backwards induction algorithm. However, due to the "curse of dimensionality," solving the dynamic programming equations directly can become prohibitively complex. Specifically, at any early-exercise date, the payoff from immediate exercise must be compared to the holding value, defined as the discounted conditional expectation of the payoff from keeping the derivative alive. Computing this conditional expectation generally requires the determination of the next-stage value function *over its entire asset space domain.*

Our approach to approximating this conditional expectation essentially follows from the ideas of numerical quadrature. We first compute the next-stage value function at a finite number of chosen "interpolating" points in the asset space domain, and then approximate this value function with a piecewise linear function over these points. Unlike the true value function, this approximate function can be easily determined over the entire asset space domain, and the computing of its conditional expectation is straightforward. An approximate holding value, defined as the discounted conditional expectation of the approximate next-stage value function, is then used in the dynamic programming equations. Although approximating the value function with a piecewise polynomial of higher order, e.g., a cubic spline, may result in a better fit, the resulting approximate holding value function will lack the structure that our method exploits.

We consider two methods for constructing the piecewise linear approximation of the value function. The first and simplest approach, which we refer to as the *secant interpolation*, consists of simply connecting the interpolating points with secant lines. The second approach, the *tangent interpolation*, involves the piecing together of adjacent tangent lines at the interpolating points. The key insight to either interpolation technique is that the resultant interpolation function can be expressed as a finite sum of European call-option payoffs; hence, the holding value at the previous early-exercise date can be approximated arbitrarily closely by a finite sum of European call-option prices. Thus, the far larger arsenal of numerical procedures (and approximations) that have been developed for pricing European call options can be directly applied to American-style derivatives via our methods. For a broad class of problem settings, the algorithm utilizing the secant interpolation results in an upper bound on the true derivative price, the algorithm with the tangent interpolation results in a lower bound, and both methods converge (as the number of interpolating points is increased) to the true price. Examples of such problem settings include American-style put and call options written on an underlying asset following a wide range of stochastic processes, including geometric Brownian motion and the Merton (1976) jump-diffusion model. Numerical experiments in §3 illustrate the convergence of the two algorithms for these examples.

The proposed approach can handle pure-jump and jump-diffusion processes, which can sometimes be problematic for the most popular pricing methods, such as partial differentiation equation (PDE) methods, binomial trees, and other lattice methods. To put our work in context, we begin by reviewing some of the (non-simulation-based) literature on pricing American options. Carr et al. (1992) decompose the value of an American put option in two ways: into the corresponding European put price plus the early-exercise premium, and also into its intrinsic value and time value. Their focus, however, is not so much on actual implementation of numerical procedures. In a closely related work, Kim (1990) provides an analytical valuation formula for American put-and-call options written on a continuous dividend-paying asset. However, only under certain special cases can the formulas be solved explicitly; otherwise, numerical techniques must be used. Ju (1998) approximates the early-exercise boundary with a multipiece exponential function, similar to the approach of Omberg (1987). Broadie and Detemple (1996) develop lower and upper bounds on the prices of standard American call and put options via the use of capped options. Amin (1993) develops an extension of the binomial method to handle the inclusion of jumps, making it applicable to jump-diffusion models. Similarly, Zhang (1997) also develops extensions for the PDE finite-difference method to jump-diffusion models, using variational inequalities in the pricing of equity options. Das (1997) develops analogous finite-difference methods for pricing bond options when the interest rate process follows a jump-diffusion model. Huang et al. (1996) approximate the early-exercise boundary using Richardson extrapolation, and use this to express the price of an American option as the price of the corresponding European option plus the early-exercise premium. Their development, however, is more problem dependent, in that each price process requires a separate analysis of the corresponding integrals used in approximating the boundary, and the method was developed for pure American options.

Another general way of extending beyond geometric Brownian motion is through Monte Carlo simulation. A number of Monte Carlo–based algorithms for pricing American-style derivatives have been proposed in the last decade. Many of these algorithms also cast the problem in a stochastic dynamic programming framework and attempt to approximate the holding values, i.e., the conditional expectations (see Tilley 1993, Fu et al. 2001, Tsitsiklis and Van Roy 2001, Longstaff and Schwartz 2001, Carriere 1996). Thus, these algorithms are perhaps the closest in spirit to our approach. Another simulation-based approach approximates optimal early-exercise policies directly rather than the dynamic programming equations (see Grant et al. 1996, Fu and Hu 1995). Finally, Broadie and Glasserman (1997a, b) develop algorithms that use simulated paths and backwards recursion to obtain upper and lower bounds.

To summarize our work and put it in the context of the just-cited literature, our approach has two major contributions: (i) It provides a new and interesting way of relating an American-style derivative to corresponding European call options; and (ii) it leads to computational algorithms that may offer computational savings over existing numerical approaches in the case of a relatively small number of finite exercise opportunities, providing bounds in many cases. In particular, it is significantly more computationally efficient than simulation-based approaches in the setting where analytical European call prices are available. In comparing with PDE and lattice-based methods, we note that both of these require specification of a discretization in both the time axis and state spaces, which is not required in our method. As a result, settings that require fine discretization of the asset price process (especially in the time dimension) are likely to be computationally burdensome for either the PDE or lattice-based methods, whereas this is not the case for our algorithms, which depend instead on the number and placement of the interpolating points (this can be thought of as analogous to a discretization in the state space, as well). Thus, we conjecture that for up to five exercise dates, our method would be superior to lattice/PDE methods. This is consistent with numerical comparisons to the algorithm of Amin (1993) for a Merton jump-diffusion put-option example, where the computation time for the lattice increases dramatically as a function of the discretization fineness. For the "pure" (continuously exercisable) American case, we use Richardson extrapolation on the prices obtained from our interpolation algorithms, and numerical results indicate that this is also competitive in terms of computation time and pricing accuracy. Similar conclusions hold in numerical comparisons with the method of Huang et al. (1996) on American put options, i.e., our method is more effective in the Bermudan setting, but is still a viable alternative for pricing the American.

The rest of the paper is organized as follows. In §2, we present the backwards recursion algorithm with the secant or tangent interpolation of the value function. Also, we establish criteria for the analytical upper and lower bounds, discuss convergence with respect to the number of interpolating points, and consider the optimal selection of the interpolating points. In §3, we apply our algorithms to American-style call and put options and provide numerical results. In §4, we offer some conclusions and discuss future applications of our methods. The appendix includes proofs of all propositions.

2. Backwards Recursion Algorithms

Let S_t denote the price at time t of an asset whose price dynamics follow a time-homogeneous, Markov process (time homogeneity can be relaxed) governed by

$$S_{t+\Delta t} = h(Z; S_t, \Theta), \quad \Delta t > 0,$$

where Θ is a vector of parameters including the risk-free interest rate r and the continuous dividend rate δ, and Z is some random vector independent of S_t and θ (independence can be relaxed in many cases, but is required for the accompanying upper- and lower-bound results). At time t, with $S_t = S$, we define $V_t^E(S, x, \eta)$ as the price of a European call option written on this asset with strike price x and time to maturity η:

$$V_t^E(S, x, \eta) = e^{-r\eta} E^Q[(S_{t+\eta} - x)^+ \mid S_t = S], \tag{1}$$

where Q denotes the appropriate risk-neutral (martingale) measure and $(a-b)^+ \equiv \max(a-b, 0)$. As we assume time homogeneity in the price process, the European call prices are also time homogeneous; thus, we henceforth drop the time indexing in (1) and write $V^E(S, x, \eta)$.

Next, given the asset price at time $t_0 = 0$, S_0, the price of an American-style derivative with expiration date T written on the asset can be expressed as the solution to the following optimal stopping-time problem:

$$\sup_\eta E^Q[e^{-r\eta} L_\eta(S_\eta) \mid S_0], \tag{2}$$

where $L_t(\cdot)$ represents the payoff at time t (we assume the payoff is only a function of the current asset price), and the supremum is over all stopping times $\eta \in (t_0, T]$. Henceforth, for ease of notation, we drop the superscript Q on the expectation, but maintain that all subsequent expectations are taken with respect to this measure. We restrict early-exercise opportunities to discrete points $\{t_i, i = 1, \ldots, N-1\}$, $0 = t_0 < t_1 < \cdots < t_N = T$, where t_N is the final exercise date, and, to simplify notation, assume a fixed timespan τ between exercise dates ($\tau = t_1 - t_0$). Further, without loss of generality, we assume that the form of the payoff function is independent of the exercise date, in which case we can drop the subscript on $L_t(\cdot)$. Finally, we abuse notation slightly by writing S_i for S_{t_i}, $i = 0, \ldots, N$.

We let $V_i(S)$ represent the value of the "live" derivative at date t_i as a function of the underlying asset price S. At the expiration date t_N, $V_N(S) = L(S)$. At previous dates, we can express $V_i(S)$ as the maximum of the derivative's holding value and exercise value:

$$V_i(S) = \max(L(S), H_i(S)), \tag{3}$$

where the holding value, $H_i(S)$, is the present value of the expected one-period-ahead derivative value:

$$H_i(S) = e^{-r\tau}E[V_{i+1}(S_{i+1}) \mid S_i = S]. \quad (4)$$

As we have assumed that the derivative cannot be exercised at t_0, the derivative price (2) can be expressed as $V_0(S_0) = H_0(S_0) = e^{-r\tau}E[V_1(S_1) \mid S_0]$.

Ideally, backwards recursion could be done on (3) and (4) to obtain the derivative price $V_0(S_0)$. However, prior to the expiration date, it is generally impossible to obtain the value function $V_i(S)$ over the entire asset space domain, yet this is necessary to calculate the holding value at the previous exercise date. An alternative approach that we adopt is to replace the current value function at each early-exercise date with a suitably chosen approximating value function that can be determined over the entire asset space, and then calculate the holding value at the previous exercise date as the discounted expectation of this approximating value function. The details of this general approach are as follows. At exercise date t_{N-1},

$$\begin{aligned} H_{N-1}(S) &= e^{-r\tau}E[V_N(S_N) \mid S_{N-1} = S] \\ &= e^{-r\tau}E[L(S_N) \mid S_{N-1} = S] \end{aligned} \quad (5)$$

is the value of a corresponding European option of length τ, with starting asset price at t_{N-1} equal to S and payoff $L(\cdot)$ at t_N. With V_{N-1} given by (3), we construct a function $\widehat{V}_{N-1}$ that *approximates* the value function V_{N-1}, and define $\widetilde{H}_{N-2}$ as the approximate holding-value function obtained by replacing V_{N-1} with $\widehat{V}_{N-1}$ in (4): $\widetilde{H}_{N-2}(S) = e^{-r\tau}E[\widehat{V}_{N-1}(S_{N-1}) \mid S_{N-2} = S]$.

Proceeding recursively, at exercise date t_i, $i = N-2, \ldots, 1$, we define $\widetilde{V}_i$ as the current value function obtained by replacing H_i with $\widetilde{H}_i$ in (3):

$$\widetilde{V}_i(S) = \max(L(S), \widetilde{H}_i(S)), \quad (6)$$

construct a function $\widehat{V}_i$ that approximates $\widetilde{V}_i$, and define

$$\widetilde{H}_{i-1}(S) = e^{-r\tau}E[\widehat{V}_i(S_i) \mid S_{i-1} = S]. \quad (7)$$

The derivative price estimate is then $\widetilde{V}_0(S_0) = \widetilde{H}_0(S_0)$ (if exercising is allowed at t_0, $\widetilde{V}_0(S_0)$ is given by (6)). We have assumed that the t_{N-1} holding value (5) can be computed without difficulty; otherwise, if determining the expectation of an approximate value function $\widehat{V}_N$ to $L(\cdot)$ is easier than finding the price of a European derivative with payoff $L(\cdot)$, the value function approximation could begin at expiration date t_N. Henceforth, we assume the approximation begins at t_{N-1} and, for notational convenience, let $\widetilde{H}_{N-1} \equiv H_{N-1}$ and $\widetilde{V}_{N-1} \equiv V_{N-1}$.

Careful construction of the approximating functions can result in upper or lower bounds:

Proposition 1. *Let $i = 0, \ldots, N-2$. If $\widehat{V}_j \le (\ge) \widetilde{V}_j$ for $j = i+1, \ldots, N-1$, then $\widetilde{V}_i \le (\ge) V_i$.*

Therefore, constructing $\widehat{V}_i$ as a lower (upper) bound to $\widetilde{V}_i$ at all early-exercise dates results in lower (upper) bounds on the true value functions; in particular, $\widetilde{V}_0(S_0)$ will be a lower (upper) bound on the true derivative price $V_0(S_0)$. Further, the following proposition, which states that errors in the approximation of $\widetilde{V}_i$ with $\widehat{V}_i$ only contribute linearly to the overall approximation error, will be used in later sections to establish convergence.

Proposition 2. *Let $i = 0, \ldots, N-2$. If $|\widehat{V}_j(S) - \widetilde{V}_j(S)| < \epsilon_j$ for $j = i+1, \ldots, N-1$, then $|\widetilde{V}_i(S) - V_i(S)| < \sum_{j=i+1}^{N-1} \epsilon_j$.*

Left unspecified up to this point is the selection of the approximating value function $\widehat{V}_i$. In our approach, $\widehat{V}_i$ is a *linear interpolation* of the current value function $\widetilde{V}_i$ at a selected finite number of interpolating points in the asset space. This piecewise linear interpolation function can be conveniently expressed as a summation of European call-option payoffs; thus the approximate holding value (7), as an expectation of this interpolation function, can be represented as a portfolio of European call options. Hence, the valuation of an American-style derivative is essentially reduced to the pricing of European call options. We consider two types of linear interpolation functions: The *secant interpolation* function interpolates with secant lines and the *tangent interpolation* function interpolates with tangent lines. In §2.1, we describe the *Secant Algorithm*, which utilizes the secant interpolation function in the above general approach, and, with Proposition 1, show that under certain conditions, the Secant Algorithm results in upper bounds on the true derivative price. In §2.2, we provide similar details for the *Tangent Algorithm*, which incorporates tangent interpolation, and show that under certain conditions the Tangent Algorithm results in lower bounds.

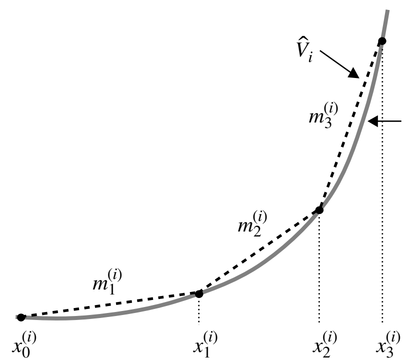
2.1. Secant Algorithm

At date t_i, $i = N-1, \ldots, 1$, we assume that $\widetilde{V}_i(\cdot)$ is continuous on $[0, \infty]$; for cases where this is not true, for example, if $\widetilde{V}_i(\cdot) = 0$ for some fixed interval of the domain space, the construction, while similar, would need to be updated on a case-by-case basis. We first choose $n_i + 1$ interpolating points $0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{n_i}^{(i)}$ in the asset space of $\widetilde{V}_i(\cdot)$, where $x_{n_i}^{(i)}$ is a chosen large value of the asset space. Then the secant interpolation function, $\widehat{V}_i$, linearly interpolates $\widetilde{V}_i$ with secant lines connecting the points $\{(x_j^{(i)}, \widetilde{V}_i(x_j^{(i)}))\}_{j=0}^{n_i}$; see Figure 1.

Specifically, for $S \in [x_0^{(i)}, x_{n_i}^{(i)}]$,

$$\widehat{V}_i(S) = m_j^{(i)}(S - x_{j-1}^{(i)}) + \widetilde{V}_i(x_{j-1}^{(i)}) \quad \text{if } x_{j-1}^{(i)} \le S < x_j^{(i)},\ j = 1, \ldots, n_i,$$

Figure 1 Secant Interpolation of Value Function



Notes. $\{x_j^{(i)}\}$ denote interpolating points, and $\{m_j^{(i)}\}$ denote slopes of secant lines.

where, for $j = 1, \ldots, n_i$,

$$m_j^{(i)} = \frac{\widetilde{V}_i(x_j^{(i)}) - \widetilde{V}_i(x_{j-1}^{(i)})}{x_j^{(i)} - x_{j-1}^{(i)}} \tag{8}$$

is the slope of the secant line from $(x_{j-1}^{(i)}, \widetilde{V}_i(x_{j-1}^{(i)}))$ to $(x_j^{(i)}, \widetilde{V}_i(x_j^{(i)}))$. For $S > x_{n_i}^{(i)}$, we let $m_{n_i+1}^{(i)}(S - x_{n_i}^{(i)}) + \widetilde{V}_i(x_{n_i}^{(i)})$ define the limiting line with left endpoint $(x_{n_i}^{(i)}, \widetilde{V}_i(x_{n_i}^{(i)}))$, where the limiting slope, $m_{n_i+1}^{(i)}$, while unconstrained, should be chosen with regard to the right-hand limit of $\widetilde{V}_i$. For many derivatives, the choice of $m_{n_i+1}^{(i)}$ is intuitive (e.g., the American call and put options considered in §3); otherwise, $m_{n_i+1}^{(i)}$ could simply be set equal to $m_{n_i}^{(i)}$ from (8). If $m_{n_i+1}^{(i)} < 0$, the limiting line intersects the S-axis at $x_{n_i}^{(i)} - \widetilde{V}_i(x_{n_i}^{(i)})/m_{n_i+1}^{(i)} > x_{n_i}^{(i)}$, and we let $x_{n_i+1}^{(i)} = x_{n_i}^{(i)} - \widetilde{V}_i(x_{n_i}^{(i)})/m_{n_i+1}^{(i)}$ and consider the S-axis as the "new" limiting line for $S > x_{n_i+1}^{(i)}$; i.e., $m_{n_i+2}^{(i)} = 0$. Otherwise, if $m_{n_i+1}^{(i)} \geq 0$, we let $x_{n_i+1}^{(i)} = \infty$. Thus, for $S > x_0^{(i)}$,

$$\widehat{V}_i(S) = \begin{cases} m_j^{(i)}(S - x_{j-1}^{(i)}) + \widetilde{V}_i(x_{j-1}^{(i)}) \\ \quad \text{if } x_{j-1}^{(i)} \leq S < x_j^{(i)},\ j = 1, \ldots, n_i + 1, \\ 0 \quad \text{if } S \geq x_{n_i+1}^{(i)}. \end{cases} \tag{9}$$

We can then prove the following simple result.

Proposition 3.

$$\widehat{V}_i(S) = \widetilde{V}_i(x_0^{(i)}) + \sum_{j=0}^{n_i+1} (m_{j+1}^{(i)} - m_j^{(i)})(S - x_j^{(i)})^+, \tag{10}$$

where $m_0^{(i)} = 0$, and, if $m_{n_i+1}^{(i)} \geq 0$, the last term of the summation is zero.

By (7), the approximate holding-value function can then be expressed as

$$\begin{aligned} &\widetilde{H}_{i-1}(S) \\ &\quad = e^{-r\tau}\widetilde{V}_i(x_0^{(i)}) + \sum_{j=0}^{n_i+1} (m_{j+1}^{(i)} - m_j^{(i)}) e^{-r\tau} \\ &\qquad \cdot E\big[(S_i - x_j^{(i)})^+ \mid S_{i-1} = S\big] \\ &\quad = e^{-r\tau}\widetilde{V}_i(x_0^{(i)}) + \sum_{j=0}^{n_i+1} (m_{j+1}^{(i)} - m_j^{(i)}) V^E(S, x_j^{(i)}, \tau), \end{aligned} \tag{11}$$

where $V^E(S, x_j^{(i)}, \tau)$, defined in (1), is the value of a European call option with initial price at t_{i-1} equal to S and maturing at t_i with strike price $x_j^{(i)}$. Thus, (11) expresses the approximate holding value, $\widetilde{H}_{i-1}$, as a portfolio of European call-option values of length τ with varying strike prices $\{x_j^{(i)}\}_{j=0}^{n_i+1}$. $V^E(S, x, \eta)$ can be evaluated either in a closed-form expression, as when the process follows geometric Brownian motion, or via simulation or another numerical method. The following summarizes the backwards recursion procedure.

Secant Algorithm

Step 0. Let $i = N - 1$.

Step 1. Choose interpolating points $0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{n_i}^{(i)}$; and, for $j = 0, \ldots, n_i$, compute $\widetilde{V}_i(x_j^{(i)})$ via (6), where $\widetilde{H}_{N-1} = H_{N-1}$ via (5), and, for $i < N - 1$, $\widetilde{H}_i$ is given by (11).

Step 2. Let $m_0^{(i)} = 0$, and, for $j = 1, \ldots, n_i$, compute $m_j^{(i)}$ via (8).

Step 3. Choose $m_{n_i+1}^{(i)}$. If $m_{n_i+1}^{(i)} < 0$, let $x_{n_i+1}^{(i)} = x_{n_i}^{(i)} - \widetilde{V}_i(x_n^{(i)})/m_{n_i+1}^{(i)}$, and $m_{n_i+2}^{(i)} = 0$.

Step 4. $i := i - 1$. If $i > 0$, return to Step 1. Otherwise, return $\widetilde{V}_0(S_0) = \widetilde{H}_0(S_0)$ via (11).

If $\widetilde{V}_i$, $i = 1, \ldots, N - 1$, is convex over its entire domain, a careful selection of the limiting slope $m_{n_i+1}^{(i)}$ will result in a price estimate $\widetilde{V}_0(S_0)$ that is an analytical upper bound on the true derivative price. To be more precise, as indicated graphically in Figure 1, $\widetilde{V}_i$ convex implies $\widehat{V}_i(S) \geq \widetilde{V}_i(S)$ for $S \leq x_{n_i}^{(i)}$, and, if the limiting secant line with slope $m_{n_i+1}^{(i)}$ is chosen such that $\widehat{V}_i(S) \geq \widetilde{V}_i(S)$ for $S > x_{n_i}^{(i)}$, we will have $\widehat{V}_i \geq \widetilde{V}_i$, which, with Proposition 1, implies upper bounds on the true value functions. Proposition 4 provides conditions under which the approximating value function, $\widetilde{V}_i$, is convex under the secant algorithm.

Proposition 4. *Suppose $L(\cdot)$ is convex. If either $L(\cdot)$ is nondecreasing and $h(Z; \cdot, \Theta)$ is convex, or $L(\cdot)$ is nonincreasing and $h(Z; \cdot, \Theta)$ is concave, then H_{N-1} and V_{N-1} are convex. For $i = N - 1, \ldots, 1$, if $\widetilde{V}_i$ and $h(Z; \cdot, \Theta)$ are convex, $m_{n_i+1}^{(i)} \geq m_{n_i}^{(i)}$, and $h(Z; \cdot, \Theta)$ is linear or $m_1^{(i)} \geq 0$, then $\widetilde{H}_{i-1}$ and $\widetilde{V}_{i-1}$ are convex.*

By Proposition 2, convergence of the secant algorithm with the number of interpolating points requires being able to arbitrarily bound $|\widehat{V}_i(S) - \widetilde{V}_i(S)|$ at each early-exercise date. Consider date t_i and assume $x = b$ is the chosen rightmost interpolating point; i.e., for $n_i + 1$ interpolating points, $b = x_{n_i}^{(i)}$. The quantity $|\widehat{V}_i(S) - \widetilde{V}_i(S)|$ can be arbitrarily bounded for $S \leq b$ by simply adding interpolating points in $[0, b]$; however, bounding $|\widehat{V}_i(S) - \widetilde{V}_i(S)|$ for $S > b$ directly depends on the choice of the limiting slope. For the American put-and-call options considered in §3, this tail error can be bounded and convergence follows.

2.2. Tangent Algorithm

The secant interpolation function only requires the evaluation of the current value function at a finite number of interpolating points. The tangent interpolation function is somewhat more difficult in that constructing tangent lines at these interpolating points also requires the first derivative of the current value function. However, for many processes, the structure of the approximate value function that develops throughout the algorithm allows for a relatively easy calculation of its first derivative.

The details of the tangent interpolation closely parallel those for the secant interpolation provided in §2.1. At date t_i, $i = N-1, \dots, 1$, we again assume the current value function, $\widetilde{V}_i(\cdot)$, to be continuous on $[0, \infty]$, and we first choose $n_i + 1$ interpolating points $0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{n_i}^{(i)}$. We then construct the tangent interpolation function $\widehat{V}_i$ on the interpolating points by piecing together adjacent tangent lines at each of the points $\{(x_j^{(i)}, \widetilde{V}_i(x_j^{(i)}))\}_{j=0}^{n_i}$; see Figure 2. Specifically, for $j = 0, \dots, n_i$, $m_j^{(i)}(S - x_j^{(i)}) + \widetilde{V}_i(x_j^{(i)})$ is the tangent line to $\widetilde{V}_i(\cdot)$ at $x_j^{(i)}$, where $m_j^{(i)} = (\partial \widetilde{V}_i(S)/\partial S)|_{S=x_j^{(i)}}$ (we defer discussion of the determination of $\partial \widetilde{V}_i(S)/\partial S$ for now). Letting $(y_{-1}^{(i)}, z_{-1}^{(i)}) \equiv (x_0^{(i)}, \widetilde{V}_i(x_0^{(i)}))$ and $(y_{n_i}^{(i)}, z_{n_i}^{(i)}) \equiv (x_{n_i}^{(i)}, \widetilde{V}_i(x_{n_i}^{(i)}))$, for $j = 0, \dots, n_i - 1$, we define $(y_j^{(i)}, z_j^{(i)})$ as the point of intersection of the tangent lines at $x_j^{(i)}$ and $x_{j+1}^{(i)}$, respectively; i.e.,

$$y_j^{(i)} = \frac{m_{j+1}^{(i)} x_{j+1}^{(i)} - m_j^{(i)} x_j^{(i)} - \widetilde{V}_i(x_{j+1}^{(i)}) + \widetilde{V}_i(x_j^{(i)})}{m_{j+1}^{(i)} - m_j^{(i)}}. \tag{12}$$

Without loss of generality, we assume that adjacent tangent lines do indeed intersect; in the rare instance where we have parallel, nonintersecting adjacent tangent lines, we could adjust the chosen interpolating points. Similarly, we can assume that for $j = 0, \dots, n_i - 1$, $x_j^{(i)} \le y_j^{(i)} \le x_{j+1}^{(i)}$. Next, we note that for $j = 0, \dots, n_i$, the tangent line to $\widetilde{V}_i(\cdot)$ at $x_j^{(i)}$ can also be written as either $m_j^{(i)}(S - y_{j-1}^{(i)}) + z_{j-1}^{(i)}$ or $m_j^{(i)}(S - y_j^{(i)}) + z_j^{(i)}$. Thus, for $S \in [x_0^{(i)}, x_{n_i}^{(i)}]$, the tangent interpolation function, $\widehat{V}_i$, can be expressed as

$$\widehat{V}_i(S) = m_j^{(i)}(S - y_{j-1}^{(i)}) + z_{j-1}^{(i)} \qquad \text{if } y_{j-1}^{(i)} \le S < y_j^{(i)},\ j = 0, \dots, n_i.$$

For $S > x_{n_i}^{(i)} \equiv y_{n_i}^{(i)}$, the construction of $\widehat{V}_i$ is identical up to notation to that for the secant interpolation function. Again choosing the limiting slope $m_{n_i+1}^{(i)}$ with respect to the right-hand limit of $\widetilde{V}_i$ ($m_{n_i+1}^{(i)} = m_{n_i}^{(i)}$ is the simplest alternative), we let $m_{n_i+1}^{(i)}(S - y_{n_i}^{(i)}) + z_{n_i}^{(i)}$ define the limiting line with left endpoint $(y_{n_i}^{(i)}, z_{n_i}^{(i)}) = (x_{n_i}^{(i)}, \widetilde{V}_i(x_{n_i}^{(i)}))$. If $m_{n_i+1}^{(i)} < 0$, we let $y_{n_i+1}^{(i)} = y_{n_i}^{(i)} - z_{n_i}^{(i)}/m_{n_i+1}^{(i)}$ and consider the S-axis as the "new" limiting line for $S > y_{n_i+1}^{(i)}$; i.e., $m_{n_i+2}^{(i)} = 0$. Otherwise, if $m_{n_i+1}^{(i)} \ge 0$, we let $y_{n_i+1}^{(i)} = \infty$. Thus, for $S \ge x_0^{(i)}$,

$$\widehat{V}_i(S) = \begin{cases} m_j^{(i)}(S - y_{j-1}^{(i)}) + z_{j-1}^{(i)} \\ \qquad \text{if } y_{j-1}^{(i)} \le S < y_j^{(i)},\ j = 0, \dots, n_i + 1, \\ 0 \quad \text{if } S \ge y_{n_i+1}^{(i)}. \end{cases}$$

The following result parallels Proposition 3.

Proposition 5.

$$\widehat{V}_i(S) = \widetilde{V}_i(x_0^{(i)}) + \sum_{j=-1}^{n_i+1} (m_{j+1}^{(i)} - m_j^{(i)})(S - y_j^{(i)})^+,$$

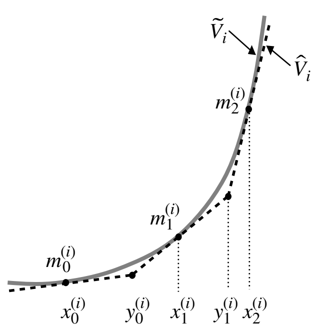
where $m_{-1}^{(i)} = 0$, and, if $m_{n_i+1}^{(i)} \ge 0$, the last term of the summation is zero.

Thus, analogous to the secant algorithm, the approximate holding value can be expressed as

$$\widetilde{H}_{i-1}(S) = e^{-r\tau} \widetilde{V}_i(x_0^{(i)}) + \sum_{j=-1}^{n_i+1} (m_{j+1}^{(i)} - m_j^{(i)}) V^E(S, y_j^{(i)}, \tau) \tag{13}$$

by (7), where $V^E(S, x, \eta)$ is given by (1).

Figure 2 Tangent Interpolation of Value Function



Notes. $\{x_j^{(i)}\}$ denote interpolating points, $\{m_j^{(i)}\}$ denote slopes of tangent lines, and $\{y_j^{(i)}\}$ denote intersection points of tangent lines.

We now consider the calculation of the tangent line slopes. For $i = N-1, \ldots, 1$, by (6),

$$\frac{\partial}{\partial S}\widetilde{V}_i(S) = \begin{cases} \dfrac{\partial}{\partial S}\widetilde{H}_i(S) & \text{if } \widetilde{H}_i(S) > L(S), \\[2ex] \dfrac{\partial}{\partial S}L(S) & \text{if } \widetilde{H}_i(S) < L(S), \end{cases} \tag{14}$$

at all points of differentiability of $\widetilde{V}_i$. Generally, the only points of nondifferentiability of $\widetilde{V}_i$ will be when $\widetilde{H}_i(S) = L(S)$; thus, for most cases, we would only select interpolating points for $\widehat{V}_i(\cdot)$ where the holding and exercising values do not equal. Assuming $\partial L(S)/\partial S$ is easily calculated, we consider $\partial \widetilde{H}_i(S)/\partial S$. First, at t_{N-1}, for a smooth price process where the dominated convergence theorem could be applied, we have by (5),

$$\frac{\partial}{\partial S}H_{N-1}(S) = e^{-r\tau}E\left[\frac{\partial}{\partial S}L(S_N)\,\middle|\, S_{N-1} = S\right]. \tag{15}$$

Next, at t_i, $i = N-2, \ldots, 1$, by (13),

$$\frac{\partial}{\partial S}\widetilde{H}_i(S) = \sum_{j=-1}^{n_{i+1}+1}\left(m_{j+1}^{(i+1)} - m_j^{(i+1)}\right)\frac{\partial}{\partial S}V^E\left(S, y_j^{(i+1)}, \tau\right). \tag{16}$$

Thus, the determination of the tangent line slopes over the entire recursion only requires $\partial H_{N-1}(S)/\partial S$ and $\partial V^E(S, x, \eta)/\partial S$. In general, for a smooth price process where European call-option values can be determined analytically, $\partial H_{N-1}(S)/\partial S$ and $\partial V^E(S, x, \eta)/\partial S$ can also be determined analytically, or at least easily approximated by a finite difference estimate.

Finally, if computing $\partial H_{N-1}(S)/\partial S$ is more difficult than computing $\partial V^E(S, x, \eta)/\partial S$, it may be beneficial to begin the tangent interpolation, i.e., the value function approximation, at the expiration date t_N, rather than at t_{N-1}. In this case, $\partial H_{N-1}(S)/\partial S$ can be computed via (16). We again provide a step-by-step summary of the backwards recursion algorithm.

Tangent Algorithm

Step 0. Let $i = N-1$.

Step 1. Choose interpolating points $0 = x_0^{(i)} < x_1^{(i)} < \cdots < x_{n_i}^{(i)}$; and, for $j = 0, \ldots, n_i$, compute $\widetilde{V}_i(x_j^{(i)})$ via (6), where $\widetilde{H}_{N-1} = H_{N-1}$ via (5), and, for $i < N-1$, $\widetilde{H}_i$ is given by (13).

Step 2. Let $m_{-1}^{(i)} = 0$, and, for $j = 0, \ldots, n_i$, calculate $m_j^{(i)} = (\partial \widetilde{V}_i(S)/\partial S)|_{S=x_j}$ via (14), where $\partial \widetilde{H}_{N-1}(S)/\partial S = \partial H_{N-1}(S)/\partial S$ via (5) and/or (15), and, for $i < N-1$, $\partial \widetilde{H}_i(S)/\partial S$ is given by (16).

Step 3. Let $y_{n_i}^{(i)} = x_{n_i}^{(i)}$, $y_{-1}^{(i)} = x_0^{(i)}$, and, for $j = 0, \ldots, n_i - 1$, compute $y_j^{(i)}$ via (12).

Step 4. Choose $m_{n_i+1}^{(i)}$. If $m_{n_i+1}^{(i)} < 0$, let $y_{n_i+1}^{(i)} = x_{n_i}^{(i)} - \widetilde{V}_i(x_n^{(i)})/m_{n_i+1}^{(i)}$ and $m_{n_i+2}^{(i)} = 0$.

Step 5. $i := i-1$. If $i > 0$, return to Step 1. Otherwise, return $\widetilde{V}_0(S_0) = \widetilde{H}_0(S_0)$ via (13).

Analogous to the secant algorithm where convexity of the approximate value function may lead to upper bounds, convexity may lead to lower bounds for the tangent algorithm. Specifically, if for $i = 1, \ldots, N-1$, $\widetilde{V}_i$ is convex over its entire domain, then $\widehat{V}_i(S) \le \widetilde{V}_i(S)$ for $S \le x_{n_i}^{(i)}$ (see Figure 2) and, if slope $m_{n_i+1}^{(i)}$ is chosen such that $\widehat{V}_i(S) \le \widetilde{V}_i(S)$ for $S > x_{n_i}^{(i)}$, we will have that $\widehat{V}_i \le \widetilde{V}_i$, which, with Proposition 1, implies lower bounds on the true value functions. In particular, $\widetilde{V}_0(S_0)$ will be a lower bound on the true derivative price. Proposition 6 parallels Proposition 4 and provides conditions for which $\widetilde{V}_i$ is convex under the tangent algorithm.

Proposition 6. *Suppose $L(\cdot)$ is convex. If either $L(\cdot)$ is nondecreasing and $h(Z;\cdot,\Theta)$ is convex, or $L(\cdot)$ is nonincreasing and $h(Z;\cdot,\Theta)$ is concave, then H_{N-1} and V_{N-1} are convex. For $i = N-1, \ldots, 1$, if $\widetilde{V}_i$ and $h(Z;\cdot,\Theta)$ are convex, $m_{n_i+1}^{(i)} \ge m_{n_i}^{(i)}$, and $h(Z;\cdot,\Theta)$ is linear or $m_0^{(i)} \ge 0$, then $\widetilde{H}_{i-1}$ and $\widetilde{V}_{i-1}$ are convex.*

In particular, by Propositions 4 and 6, if $h(Z;\cdot,\Theta)$ is linear and $L(\cdot)$ is convex and monotone, a careful choice of the limiting slopes at each early-exercise date will result in upper and lower bounds on the true derivative price via the secant and tangent algorithms, respectively. Convergence of the tangent algorithm with the number of interpolating points can be established via Proposition 6 using a similar argument as for the secant algorithm, again depending on the choice of the limiting slopes.

The accuracy of the secant or tangent algorithm is clearly dependent on the selection of the interpolating points. If possible, the interpolating points should be chosen so as to minimize the error in replacing the current value function $\widetilde{V}_i$ with the interpolation function $\widehat{V}_i$. In this regard, more interpolating points should be concentrated on the areas of the state space where $\widetilde{V}_i$ is most convex or concave. One simple heuristic where the interpolating points are chosen iteratively is as follows: Given a current set of points and the corresponding secant or tangent lines, additional interpolating points are inserted into those areas where the absolute difference between the slopes of adjacent secant or tangent lines is large, as these areas should correspond to areas of higher convexity or concavity. Further discussion of the selection of interpolating points can be found in Laprise (2002), which includes alternative heuristics for iteratively selecting the interpolating points, discusses the possible accumulation of errors as a result of poorly chosen interpolating points, and, for the secant interpolation algorithm, shows that using the same set of interpolating points at each early-exercise date may result in a reduction of overall computational time.

3. American-Style Call and Put Options

We illustrate the numerical properties of the algorithms by pricing two common American-style derivatives: (dividend-paying) call and put options. By following a specific approach to the selection of interpolating points and limiting slopes, we prove that for a wide range of underlying stochastic processes, the secant and tangent interpolation algorithms result in analytical upper and lower bounds, respectively, and argue that both algorithms converge to the true option price with the number of interpolating points. We also prove that for each option, optimal early-exercise policies are threshold policies and the algorithms provide analytical bounds on the true thresholds. Included in this class of stochastic processes are geometric Brownian motion and the Merton (1976) jump diffusion, which we consider as examples in §§3.3 and 3.4, respectively.

The put-and-call options are assumed to be written on an underlying asset that possibly pays continous dividends at a rate of $\delta \geq 0$. For the call option, we assume $\delta > 0$; otherwise, as is well known, the optimal exercise policy is to never early exercise and the American-style option is essentially a European option. For the propositions in this section, we consider stochastic processes that satisfy the following assumptions:

Assumptions.

Assumption 1. $0 \leq \partial V^E(S, x, \eta)/\partial S \leq e^{-\delta\eta}$.

Assumption 2. $V^E(0, x, \eta) = 0$.

Assumption 3. *For the call option, $h(Z; \cdot, \Theta)$ is convex; for the put option, $h(Z; \cdot, \Theta)$ is linear.*

We note that $h(Z; \cdot, \Theta)$ linear will also satisfy Assumption 3 for the call option. Assumption 1 is a mild condition equivalent to nonnegative and nonpositive deltas for the European call and put options, respectively. Further, for a multiplicative process, i.e., $S_{t+\Delta t} = S_t X_{t,\, t+\Delta t}$ where $X_{t,\, t+\Delta t}$ (>0) is a random variable independent of all $\{S_u,\, u \leq t\}$, Assumptions 2 and 3 are trivially satisfied. The stock price models we consider are multiplicative.

The results in this section implicitly assume that exact European call values and European call delta values are available, whereas only the call values are needed for the secant algorithm. However, we note that only the claims in this section, e.g., the upper and lower bounds, are dependent upon these exact values and the satisfaction of the above assumptions; the specific procedures detailed are applicable for a wide range of underlying stock price models.

3.1. Call Option

In this case, $L(S) = (S - K)^+$, and we assume $\delta > 0$. The holding value at the latest early-exercise date is simply the value of a European call option:

$$\begin{aligned} H_{N-1}(S) &= e^{-r\tau} E[(S-K)^+ \mid S_{N-1} = S] \\ &= V^E(S, K, \tau). \end{aligned} \tag{17}$$

Thus, in applying the secant or tangent algorithm to the American call option, the European call option is the only option that needs pricing. For the tangent algorithm, because $H_{N-1}(S) = V^E(S, K, \tau)$, the discussion in §2.2 implies that the determination of the tangent line slopes only requires the delta of the European call option, $\partial V^E(S, x, \eta)/\partial S$.

We now show that the optimal exercise policy at t_{N-1} is a threshold policy. First, $V_{N-1}(S) = H_{N-1}(S)$ for $S \leq K$, and by (17) and Assumption 1, $\partial H_{N-1}(S)/\partial S \leq e^{-\delta\tau} < 1$. Thus, as $\partial(S - K)/\partial S = 1$, there exists a finite $s^*_{N-1} > K$ such that $H_{N-1}(s^*_{N-1}) = L(s^*_{N-1})$ and $H_{N-1}(S) > (<) L(S)$ for $S < (>) s^*_{N-1}$; i.e., the option should only be exercised if $S \geq s^*_{N-1}$ and

$$V_{N-1}(S) = \begin{cases} H_{N-1}(S) & \text{if } S < s^*_{N-1}, \\ L(S) = S - K & \text{if } S \geq s^*_{N-1}. \end{cases} \tag{18}$$

For both algorithms, we take advantage of linearity in the value functions and choose interpolating points accordingly. For ease of notation, we assume a fixed number of interpolating points at each early-exercise date and omit the subscript on n_i.

Secant Algorithm. We apply the secant algorithm via the general summary provided in §2.1 with some added specifications. As $V_{N-1}(S)$ is linear with slope 1.0 for $S \geq s^*_{N-1}$, we let $x_n^{(N-1)} = s^*_{N-1}$ and $m_{n+1}^{(N-1)} = 1.0$ (as H_{N-1} is given by (17), we assume s^*_{N-1} can be determined numerically via some rootfinding method). After choosing interpolating points between $x_0^{(N-1)} = 0$ and $x_n^{(N-1)}$, $\widetilde{H}_{N-2}$ and $\widetilde{V}_{N-2}$ can be determined via (11) and (6), respectively. Moreover, $\widetilde{H}_{N-2}$ also invokes a threshold policy for the approximate value function at t_{N-2}; i.e.,

$$\widetilde{V}_{N-2}(S) = \begin{cases} \widetilde{H}_{N-2}(S) & \text{if } S < \tilde{s}^*_{N-2}, \\ L(S) = S - K & \text{if } S \geq \tilde{s}^*_{N-2}, \end{cases}$$

for some finite $\tilde{s}^*_{N-2} > K$. Proceeding recursively, at t_i, given that $\widetilde{V}_i$ admits a threshold policy with threshold $\tilde{s}^*_i$, if we let $x_n^{(i)} = \tilde{s}^*_i$ and $m_{n+1}^{(i)} = 1.0$, the resultant $\widetilde{H}_{i-1}$ invokes a threshold policy at t_{i-1}. The following proposition formalizes this result, and via Proposition 4, states that the approximate value functions provide upper bounds on the true value functions; in particular, $\widetilde{V}_0(S_0) \geq V_0(S_0)$. For Propositions 7, 8, and 9, $\widetilde{H}_{N-1} \equiv H_{N-1}$ is given by (17), $\tilde{s}^*_{N-1} \equiv s^*_{N-1}$ is given by (18), and s^*_0 and $\tilde{s}^*_0$ are taken to be infinity.

Proposition 7 (Secant Algorithm for American Call). *Let* $i = 0, \ldots, N-1$. *If* $x_n^{(j)} = \tilde{s}_j^*$ *and* $m_{n+1}^{(j)} = 1.0$ *for* $j = i+1, \ldots, N-1$, *there exists a finite* $\tilde{s}_i^* > K$ *such that*

$$\widetilde{V}_i(S) = \begin{cases} \widetilde{H}_i(S) & \text{if } S < \tilde{s}_i^*, \\ L(S) = S - K & \text{if } S \geq \tilde{s}_i^*. \end{cases}$$

Further, $\widetilde{V}_i \geq V_i$ *and* $\widetilde{H}_i \geq H_i$.

By Proposition 7, for $i = 1, \ldots, N-1$, $\widehat{V}_i(S) = \widetilde{V}_i(S)$ for $S > \tilde{s}_i^* = x_n^{(i)}$. Hence, by Proposition 2 and the argument in §2.1, the secant algorithm converges with the number of interpolating points. Next, Proposition 8 states that the true optimal early-exercise policies are threshold policies and the approximate thresholds provide upper bounds on the true thresholds.

Proposition 8. *For* $i = 1, \ldots, N-1$, *there exists an* s_i^*, *where* $K \leq s_i^* \leq \tilde{s}_i^*$, *such that*

$$V_i(S) = \begin{cases} H_i(S) & \text{if } S < s_i^*, \\ L(S) = S - K & \text{if } S \geq s_i^*. \end{cases}$$

Tangent Algorithm. We also include some added specifications in the application of the tanget algorithm. At t_{N-1}, we again take advantage of the linearity of V_{N-1} and let $x_n^{(N-1)} = s_{N-1}^*$ and $m_{n+1}^{(N-1)} = 1.0$. However, because $V_{N-1}(\cdot)$ is nondifferentiable at s_{N-1}^*, we let $m_n^{(N-1)} = \partial H_{N-1}(S)/\partial S|_{S=x_n^{(N-1)}}$, where H_{N-1} is given by (17). Further, as $H_{N-1}(0) = 0$ by (17) and Assumption 2, we set $m_0^{(N-1)} = 0$ to ensure lower bounds. Then, as for the secant algorithm, $\widetilde{H}_{N-2}$, given by (13), invokes a threshold policy for $\widetilde{V}_{N-2}$, and proceeding recursively, constructing the tangent interpolation function as above at each early-exercise date results in threshold policies throughout. Further, the approximate value functions and thresholds provide lower bounds on the true value functions and thresholds, respectively.

Proposition 9 (Tangent Algorithm for American Call). *Let* $i = 0, \ldots, N-1$. *If* $x_n^{(j)} = \tilde{s}_j^*$, $m_0^{(j)} = 0$, $m_n^{(j)} = \partial \widetilde{H}_j(S)/\partial S|_{S=x_n^{(j)}}$, *and* $m_{n+1}^{(j)} = 1.0$ *for* $j = i+1, \ldots, N-1$, *there exists a finite* $\tilde{s}_i^* > K$ *such that*

$$\widetilde{V}_i(S) = \begin{cases} \widetilde{H}_i(S) & \text{if } S < \tilde{s}_i^*, \\ L(S) = S - K & \text{if } S \geq \tilde{s}_i^*. \end{cases}$$

Further, $\widetilde{V}_i \leq V_i$, $\widetilde{H}_i \leq H_i$, *and* $\tilde{s}_i^* \leq s_i^*$.

In particular, $\widetilde{V}_0(S_0) \leq V_0(S_0)$. As previously argued for the secant algorithm, Proposition 9 implies the convergence of the tangent algorithm with the number of interpolating points.

3.2. Put Option

In this case, $L(S) = (K - S)^+$. The holding value at t_{N-1} is the value of a European put option, and by put-call parity, we have:

$$H_{N-1}(S) = Ke^{-r\tau} - Se^{-\delta\tau} + V^E(S, K, \tau). \tag{19}$$

Therefore, as is the case for the American call option, the European call option is the only option that needs pricing when applying the secant or tangent algorithm to the American put pricing problem. Further, (19) implies that the only requirement for determining the tangent line slopes is $\partial V^E(S, x, \eta)/\partial S$.

As for the call option, the optimal early-exercise policy at t_{N-1} is a threshold policy. In particular, for $S \geq K$, $H_{N-1}(S) > L(S) = 0$, and, by (19) and Assumption 2, $H_{N-1}(0) = Ke^{-r\tau} \leq K = L(0)$. Further, by (19) and Assumption 1, $\partial H_{N-1}(S)/\partial S \geq -1$. Therefore, there exists an $s_{N-1}^* < K$ such that $H_{N-1}(S) < L(S)$ for $S < s_{N-1}^*$ and $H_{N-1}(S) \geq L(S)$ for $S \geq s_{N-1}^*$; i.e.,

$$V_{N-1}(S) = \begin{cases} L(S) = K - S & \text{if } S < s_{N-1}^*, \\ H_{N-1}(S) & \text{if } S \geq s_{N-1}^*. \end{cases} \tag{20}$$

Thus, the option should only be exercised if $S \leq s_{N-1}^*$; in particular, s_{N-1}^* is positive unless $r = 0$. In applying our algorithms, we again take advantage of linearity in the value functions.

Secant Algorithm. As $V_{N-1}(S)$ is linear with slope -1.0 for $S \leq s_{N-1}^*$, we let $x_1^{(N-1)} = s_{N-1}^*$ and $m_1^{(N-1)} = -1.0$. Next, by (19) and (20), $V_{N-1}(S)$ approaches zero with increasing S; thus, given some small tolerance ϵ, we choose $x_n^{(N-1)} > K$ large enough such that $H_{N-1}(x_n^{(N-1)}) < \epsilon$, and, to ensure upper bounds, let $m_{n+1}^{(N-1)} = 0$. Then, after choosing interpolating points between $x_1^{(N-1)}$ and $x_n^{(N-1)}$, $\widetilde{H}_{N-2}$ can be determined via (11). In particular, as $m_{n+1}^{(N-1)} \geq 0$, the last term in the summation of (11) is zero, and, as $x_0^{(N-1)} = 0$, $V_{N-1}(x_0^{(N-1)}) = L(0) = K$, and $V^E(S, 0, \tau) = Se^{-\delta\tau}$, $\widetilde{H}_{N-2}$ can be written as:

$$\widetilde{H}_{N-2}(S) = Ke^{-r\tau} - Se^{-\delta\tau} + \sum_{j=1}^{n} \left(m_{j+1}^{(N-1)} - m_j^{(N-1)}\right) V^E\!\left(S, x_j^{(N-1)}, \tau\right). \tag{21}$$

Comparing (19) and (21), we note that the European call option component of the holding value at t_{N-1} consists of one call option, while the respective component for the approximate holding value at t_{N-2} consists of a portfolio of call options. Next, it can be shown that $\widetilde{H}_{N-2}$ invokes a threshold policy in the approximate value function at t_{N-2}, i.e.,

$$\widetilde{V}_{N-2}(S) = \begin{cases} L(S) = K - S & \text{if } S < \tilde{s}_{N-2}^*, \\ \widetilde{H}_{N-2}(S) & \text{if } S \geq \tilde{s}_{N-2}^*, \end{cases}$$

for some $\tilde{s}^*_{N-2} < K$. Also, as $V^E(S, x, \eta)$ approaches $Se^{-\delta\tau} - xe^{-r\eta}$ for $S \gg x$ and, in (21), $\sum_{j=1}^n (m_{j+1}^{(N-1)} - m_j^{(N-1)}) = 1$, $\lim_{S\to\infty} \widetilde{H}_{N-2}(S) = 0$; hence $\lim_{S\to\infty} \widetilde{V}_{N-2}(S) = 0$. Proceeding recursively, at t_i, given that $\widetilde{V}_i$ admits a threshold policy with threshold $\tilde{s}^*_i$ and $\lim_{S\to\infty} \widetilde{V}_i(S) = 0$, we let $x_1^{(i)} = \tilde{s}^*_i$, $m_1^{(i)} = -1.0$, and $m_{n+1}^{(i)} = 0$, and choose interpolating points $x_2^{(i)}, \ldots, x_n^{(i)}$, where $x_n^{(i)} > K$ is arbitrarily large. Then, the resultant $\widetilde{H}_{i-1}$ again triggers a threshold policy at t_{i-1}, has a form identical to that seen in (21), and thus approaches zero with increasing S. The following proposition formalizes this result, and via Proposition 4, states that the approximate value functions provide upper bounds on the true value functions; in particular, $\widetilde{V}_0(S_0) \geq V_0(S_0)$. For Propositions 10, 11, and 12, $\widetilde{H}_{N-1} \equiv H_{N-1}$ is given by (19), $\tilde{s}^*_{N-1} \equiv s^*_{N-1}$ is given by (20), and s^*_0 and $\tilde{s}^*_0$ are taken to be zero.

Proposition 10 (Secant Algorithm for American Put). *Let $i = 0, \ldots, N-1$. If $x_1^{(j)} = \tilde{s}^*_j$, $m_1^{(j)} = -1.0$, $x_n^{(j)} > K$, and $m_{n+1}^{(j)} = 0$ for $j = i+1, \ldots, N-1$, there exists an $\tilde{s}^*_i < K$ such that $\widetilde{H}_i(S) < (\geq) L(S)$ for $S < (\geq) \tilde{s}^*_i$; i.e.,*

$$\widetilde{V}_i(S) = \begin{cases} L(S) = K - S & \text{if } S < \tilde{s}^*_i, \\ \widetilde{H}_i(S) & \text{if } S \geq \tilde{s}^*_i, \end{cases}$$

where, for $i < N-1$, $\widetilde{H}_i$ can be written in the following form:

$$\widetilde{H}_i(S) = Ke^{-r\tau} - Se^{-\delta\tau} + \sum_{p=1}^{n} \big(m_{p+1}^{(i+1)} - m_p^{(i+1)}\big) V^E\big(S, x_p^{(i+1)}, \tau\big). \quad (22)$$

Further, $\widetilde{V}_i \geq V_i$ and $\widetilde{H}_i \geq H_i$.

We next consider convergence with the number of interpolating points. For $i = 1, \ldots, N-2$, by Proposition 10, $\lim_{S\to\infty} \widetilde{V}_i(S) = 0$. Thus, as previously argued for $i = N-1$, for arbitrary tolerance ϵ, we can choose $x_n^{(i)}$ large enough such that $\widetilde{V}_i(x_n^{(i)}) < \epsilon$; in which case, given that $m_{n+1}^{(i)} = 0$, $|\widehat{V}_i(S) - \widetilde{V}_i(S)| < \epsilon$ for $S > x_n^{(i)}$. So, for sufficiently large $x_n^{(i)}$ and enough interpolating points, $|\widehat{V}_i(S) - \widetilde{V}_i(S)|$ can be arbitrarily bounded which implies convergence.

Next, Proposition 11 states that the true optimal early-exercise policies are threshold policies and the approximate thresholds provide lower bounds on the true thresholds.

Proposition 11. *For $i = 1, \ldots, N-1$, there exists an s^*_i, where $\tilde{s}^*_i \leq s^*_i < K$, such that*

$$V_i(S) = \begin{cases} L(S) = K - S & \text{if } S < s^*_i, \\ H_i(S) & \text{if } S \geq s^*_i. \end{cases}$$

Tangent Algorithm. As for the secant algorithm, we set $x_1^{(N-1)} = s^*_{N-1}$; however, with the nondifferentiability of $V_{N-1}(S)$ at $S = s^*_{N-1}$, we set $m_1^{(N-1)} = \partial H_{N-1}(S)/\partial S|_{S=x_1^{(N-1)}}$. Also, we let $m_0^{(N-1)} = -1.0$ and $(y_0^{(N-1)}, z_0^{(N-1)}) = (x_1^{(N-1)}, V_{N-1}(x_1^{(N-1)}))$. Further, we choose $x_n^{(N-1)}$ arbitrarily large and let the limiting slope $m_{n+1}^{(N-1)} = m_n^{(N-1)}$. As H_{N-1}, hence V_{N-1}, is a decreasing function, $m_{n+1}^{(N-1)} < 0$, which implies $y_{n+1}^{(N-1)} < \infty$ and $m_{n+2}^{(N-1)} = 0$. Then, $\widetilde{H}_{N-2}$ can be determined via (13), and, in particular, as $y_{-1}^{(N-1)} = x_0^{(N-1)} = 0$, $\widetilde{V}_{N-1}(x_0^{(N-1)}) = L(0) = K$, and $V^E(S, 0, \tau) = Se^{-\delta\tau}$, $\widetilde{H}_{N-2}$ can be written in the following form:

$$\widetilde{H}_{N-2}(S) = Ke^{-r\tau} - Se^{-\delta\tau} + \sum_{j=0}^{n+1} \big(m_{j+1}^{(N-1)} - m_j^{(N-1)}\big) V^E\big(S, y_j^{(N-1)}, \tau\big).$$

Thus, we have a similar form for the approximate holding value at t_{N-2} using the tangent interpolation as when using the secant interpolation (21). Recursively constructing $\widehat{V}_i$ in this same manner, we find that the approximate holding values admit threshold policies, the approximate value functions provide lower bounds on the true value functions, and the approximate thresholds provide upper bounds on the true thresholds.

Proposition 12 (Tangent Algorithm for American Put). *Let $i = 0, \ldots, N-1$. If $x_1^{(j)} = \tilde{s}^*_j$, $y_0^{(j)} = x_1^{(j)}$, $m_0^{(j)} = -1.0$, $m_1^{(j)} = \partial \widetilde{H}_j(S)/\partial S|_{S=x_1^{(j)}}$, and $m_{n+1}^{(j)} = m_n^{(j)}$ for $j = i+1, \ldots, N-1$, there exists an $\tilde{s}^*_i < K$ such that $\widetilde{H}_i(S) < (\geq) L(S)$ for $S < (\geq) \tilde{s}^*_i$; i.e.,*

$$\widetilde{V}_i(S) = \begin{cases} L(S) = K - S & \text{if } S < \tilde{s}^*_i, \\ \widetilde{H}_i(S) & \text{if } S \geq \tilde{s}^*_i, \end{cases}$$

where $\widetilde{H}_i$ can be written in the following form:

$$\widetilde{H}_i(S) = Ke^{-r\tau} - Se^{-\delta\tau} + \sum_{p=0}^{n+1} \big(m_{p+1}^{(i+1)} - m_p^{(i+1)}\big) V^E\big(S, y_p^{(i+1)}, \tau\big).$$

Further, $\widetilde{V}_i \leq V_i$, $\widetilde{H}_i \leq H_i$, and $\tilde{s}^*_i \geq s^*_i$.

In particular, $\widetilde{V}_0(S_0) \geq V_0(S_0)$. As argued for the secant algorithm, $\widetilde{V}_i$ is a decreasing function approaching zero; hence, with $m_{n+1}^{(i)} = m_n^{(i)} < 0$, for any ϵ where $\widetilde{V}_i(x_n^{(i)}) < \epsilon$, $|\widehat{V}_i(S) - \widetilde{V}_i(S)| < \epsilon$ for $S > x_n^{(i)}$, implying convergence with number of interpolating points.

In the next two sections, we price call and put options via the procedures detailed in §§3.1 and 3.2 on assets following geometric Brownian motion (3.3) and Merton jump diffusion (3.4).

3.3. Geometric Brownian Motion

We assume the underlying stock price process follows geometric Brownian motion: $dS_t = S_t[(r-\delta)dt + \sigma dW_t]$, where W_t is a standard Brownian motion process and σ is the volatility. This leads to the multiplicative lognormal price process:

$$S_{t+\Delta t} = h(Z; S_t, \Theta) = S_t e^{(r-\delta-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z}, \tag{23}$$

where $Z \sim N(0, 1)$ and $\Theta = \{r, \delta, \sigma\}$. The European call-option price is given by the Black-Scholes formula:

$$V^E(S, x, \eta) = Se^{-\delta\eta}\Phi(a + \sigma\sqrt{\eta}) - xe^{-r\tau}\Phi(a),$$

where

$$a = \frac{\log(S/x) + (r - \delta - \sigma^2/2)\eta}{\sigma\sqrt{\eta}}, \tag{24}$$

and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Thus, the European call-option delta is given by

$$\frac{\partial}{\partial S} V^E(S, x, \eta) = e^{-\delta\eta}\Phi(a + \sigma\sqrt{\eta}), \tag{25}$$

where a is given in (24). By (25), Assumption 1 is satisfied, and as the price process (23) is multiplicative, Assumptions 2 and 3 are satisfied. Hence, Propositions 7–12 apply; in particular, the secant and tangent algorithms provide analytical upper and lower bounds, respectively, on the American-style call and put option prices.

Table 1 shows the results of applying the algorithms to a three-year call option, exercisable every 0.5 years, with strike price $K = 100$, $\sigma = 0.2$, $r = 0.05$, and $\delta = 0.04$. For each choice of n (# interpolating points), Table 1 displays option price estimates for a range of starting asset prices, the corresponding threshold estimates (threshold values are independent of the starting prices, and the $t_5 = 2.5$ threshold is omitted, as it is obtained independently), and CPU times (in seconds). The final row, labeled "Eur," displays the corresponding European call prices, i.e., no early-exercise opportunities, given by (24) with $\eta = 3.0$.

Our results show that the upper and lower bounds via the secant and tangent algorithms, respectively, tighten quickly with the number of interpolating points. For example, with just 100 interpolating points, the upper and lower bounds are within a penny of the true price. The convergence is even faster for the threshold values. To get an idea of the relative computational burden with respect to other methods, we also estimated the prices using a binomial lattice on the same computational platform. The results indicate comparable price accuracy-computation trade-offs. For example, in the $S_0 = 100$ case, a discretization of daily increments led to a lattice price of \$13.552 taking 0.36 CPU seconds, falling somewhere between the $n = 50$ and $n = 100$ results of the tangent algorithm in terms of both accuracy and computation. A thrice finer discretization yields a more accurate price of \$13.554 taking 3.95 CPU seconds, corresponding roughly to the $n = 200$ tangent algorithm. Also, other numerical experiments reported in Laprise (2002) indicate only linear growth in the computation time of our algorithm with the number of exercise dates.

Analogous to Table 1, Table 2 shows the results of applying the secant and tangent algorithms to a put option with the same parameters as for the call, except for the dividend rate set to zero. The row labeled "Eur" displays the corresponding European put prices given by (19) and (24) with $\eta = 3.0$. Again, the bounds on the option price tighten quickly with the number of interpolating points; see also Figure 3 illustrating the convergence for the $S_0 = 100$ case. The in-the-money prices are the most accurate; e.g., for $S_0 = 60$, only 10 interpolating points are needed for the upper and lower bounds to bracket the true price to within a penny, whereas over 200 points are required for comparable accuracy for $S_0 = 140$. The threshold bounds for the put option are even tighter than for the call (Table 1).

Table 1 Bermudan Call Option on Asset Under Geometric Brownian Motion

		Option price					Thresholds				
Alg type	n	$S_0 = 60$	$S_0 = 90$	$S_0 = 100$	$S_0 = 110$	$S_0 = 140$	$t_1 = 0.5$	$t_2 = 1.0$	$t_3 = 1.5$	$t_4 = 2.0$	CPU
Tan	10	0.718	8.431	13.355	19.331	42.129	157.53	153.40	148.15	141.35	0.03
	20	0.825	8.582	13.507	19.487	42.251	158.19	153.87	148.54	141.61	0.07
	50	0.864	8.624	13.547	19.526	42.280	158.38	154.02	148.65	141.69	0.24
	100	0.870	8.630	13.553	19.532	42.285	158.41	154.04	148.66	141.70	0.85
	200	0.874	8.631	13.554	19.533	42.286	158.42	154.05	148.67	141.70	3.30
Sec	200	0.880	8.633	13.556	19.535	42.288	158.43	154.06	148.68	141.70	2.20
	100	0.887	8.637	13.559	19.538	42.290	158.44	154.07	148.68	141.71	0.62
	50	0.909	8.648	13.571	19.551	42.303	158.56	154.13	148.74	141.77	0.16
	20	1.056	8.751	13.674	19.656	42.384	159.17	154.98	149.37	141.88	0.05
	10	1.298	9.132	14.104	20.145	42.881	163.42	155.71	149.77	142.47	0.02
Eur		0.873	8.548	13.371	19.179	40.741					

Notes. $K = 100$, $t_i = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ yrs, n: number of interpolating points, "Eur": European price.

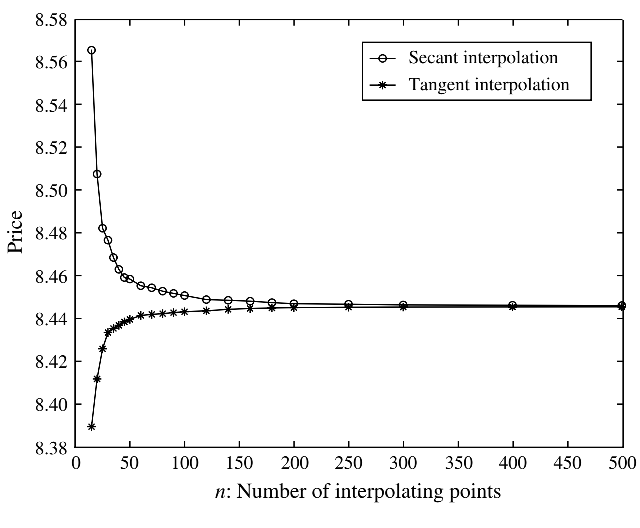
Table 2 **Bermudan Put Option on Asset Under Geometric Brownian Motion**

		Option price					Thresholds				
Alg type	n	$S_0=60$	$S_0=90$	$S_0=100$	$S_0=110$	$S_0=140$	$t_1=0.5$	$t_2=1.0$	$t_3=1.5$	$t_4=2.0$	CPU
Tan	10	37.553	12.831	8.329	5.332	1.250	83.18	84.11	85.37	87.22	0.03
	20	37.554	12.884	8.412	5.453	1.434	83.09	84.04	85.33	87.20	0.06
	50	37.554	12.902	8.440	5.489	1.476	83.06	84.03	85.32	87.20	0.25
	100	37.554	12.904	8.444	5.494	1.491	83.06	84.03	85.32	87.20	0.86
	200	37.554	12.905	8.445	5.497	1.497	83.06	84.03	85.32	87.20	3.50
Sec	200	37.554	12.906	8.447	5.500	1.506	83.06	84.02	85.32	87.20	2.30
	100	37.554	12.909	8.451	5.506	1.522	83.06	84.02	85.32	87.20	0.60
	50	37.554	12.913	8.459	5.518	1.550	83.05	84.02	85.32	87.20	0.18
	20	37.554	12.947	8.508	5.577	1.611	82.99	83.98	85.29	87.19	0.05
	10	37.559	13.172	8.759	5.810	1.740	82.59	83.82	85.18	87.15	0.03
Pure Am	4	36.26	12.30	8.01	5.20	1.41					0.11
	6	37.55	12.65	8.22	5.35	1.45					0.21
	12	38.23	12.96	8.44	5.50	1.49					1.0
Eur		27.967	10.240	6.995	4.710	1.367					

Notes. $K=100$, $t_i=0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ yrs, "pure Am": estimated pure American price, n: number of interpolating points for Tan/Sec and number of intervals for pure Am, "Eur": European price.

The "pure Am" entries indicate a price estimate for the continuously exercisable American put using the method of Huang et al. (1996), where # intervals indicates the number of intervals used in approximating the early-exercise boundary. The results indicate that the computational burden is comparable, so in the Bermudan case, the precision (including bounds) of our method makes it very highly recommended, but if the pure American price is the goal, then the choice is not so clear, as applying our method would require Richardson extrapolation. Computational comparisons with pricing via a binomial lattice were similar to those of the previous example; however, in terms of practical implementation, our algorithms have two important advantages. First, lattice methods generally do not provide any information on the level of precision of their estimates, while the secant and tangent algorithms provide bounds. Second, our algorithms can provide price estimates for *any* starting asset price in one pass of the algorithm; i.e., each row in Tables 1 and 2 is computed from a single recursion, whereas a lattice method would require computation over a different lattice for each starting price.

Figure 3 **Convergence of Bounds for American Put Option** ($S_0=100$)



3.4. Merton Jump Diffusion

The jump-diffusion model from Merton (1976) can be written as follows:

$$\begin{aligned} S_{t+\Delta t} &= h(Z; S_t, \Theta) \\ &= S_t e^{(r-\delta-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}Z_0+\sum_{j=1}^{N(\Delta t)}(\gamma Z_j-\gamma^2/2)}, \end{aligned} \tag{26}$$

where $Z_j \sim N(0,1)$ i.i.d., $N(\Delta t) \sim \text{Poisson}(\lambda\Delta t)$, the jump sizes are i.i.d. lognormally distributed: $\text{LN}(-\gamma^2/2, \gamma^2)$, and r, δ, and σ retain their definitions from the geometric Brownian motion process (23); here $\theta = \{r, \delta, \sigma, \gamma, \lambda\}$. Merton (1976) derives the following closed-form solution for the price of a European call option written on an asset following (26):

$$V^E(S, x, \eta) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\eta}(\lambda\eta)^n}{n!} V_n^E(S, x, \eta), \tag{27}$$

where $V_n^E(S, x, \eta)$ is the modified Black-Scholes formula (24) with σ^2 replaced by $\nu_n^2 = \sigma^2 + n\gamma^2/\eta$. Thus, $V^E(S, x, \eta)$ is a weighted sum of Black-Scholes prices. Further, by (27), the European call delta is given by

$$\begin{aligned} \frac{\partial}{\partial S} V^E(S, x, \eta) &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\eta}(\lambda\eta)^n}{n!} \frac{\partial}{\partial S} V_n^E(S, x, \eta) \\ &= \sum_{n=0}^{\infty} \frac{e^{-\lambda\eta}(\lambda\eta)^n}{n!} \Phi(a_n + \sigma\sqrt{\eta}), \end{aligned} \tag{28}$$

Table 3 Bermudan Put Option on Asset Under Merton Jump Diffusion

	$N=2$, $\tau=1/4$		$N=3$, $\tau=1/6$		$N=4$, $\tau=1/8$		$N=6$, $\tau=1/12$	
Alg	Price	CPU	Price	CPU	Price	CPU	Price	CPU
Tan	8.612	0.04	8.684	1.62	8.720	2.80	8.756	4.85
Sec	8.614	0.03	8.686	1.19	8.725	1.85	8.765	3.24
Sim	8.597 (0.010)	27.90	8.678 (0.009)	41.73	8.715 (0.009)	50.30	8.753 (0.009)	69.40

Notes. $S_0=100$, $K=100$, $T=0.5$, $r=0.1$, $\sigma=0.2828$, $\lambda=2$, $\gamma=0.2$, N = number of exercise dates, "Sim": simulation method, standard error in parentheses, European price ($N=1$): \$8.393.

where a_n is given by the expression for a in (24) with σ^2 replaced by $\nu_n^2=\sigma^2+n\gamma^2/\eta$. Given (28) and the multiplicity of the price process (26), the three assumptions are satisfied and Propositions 7–12 apply; in particular, the secant and tangent algorithms will provide upper and lower bounds, respectively, on the true option prices.

Table 3 shows the results of applying our algorithms to a six-month ($T=0.5$) put option written on the jump-diffusion model without dividends ($\delta=0$), $r=0.1$, $\sigma=0.2828$, $\lambda=2$, $\gamma=0.2$, using 100 interpolating points per early-exercise date for each algorithm and enough terms in the summations (27) and (28) to bound the truncation errors to less than 10^{-4} and 10^{-6}, respectively. For comparison, shown in the row labeled "Sim" (standard errors in parentheses), are (biased low) prices obtained via a simulation method in Grant et al. (1996), indicating that for this example, simulation-based methods require significantly more computation for a comparable level of numerical precision.

The algorithms can also be used to price pure (continuously exercisable) American options by using Richardson extrapolation. Denoting P_i as the price of the option with i exercise opportunities at $\{jT/i\colon j=1,\ldots,i\}$ (P_1 would correspond to the European option), the three-point Richardson extrapolation is given by $P_3+3.5(P_3-P_2)-0.5(P_2-P_1)$, and the four-point Richardson extrapolation is given by $P_4+29/3(P_4-P_3)-23/6(P_3-P_2)+1/6(P_2-P_1)$. Table 4 shows the results for the previous jump-diffusion example, comparing with the lattice method of Amin (1993). The algorithms are quite competitive in terms of accuracy/computation trade-offs. The upper and lower bounds also provide a guideline for determining an appropriate number of interpolation points, whereas determining when the number of discretization time steps is sufficiently large in applying lattice methods is often arbitrary.

Table 4 Estimating ("Pure") American Put Option Price Under Merton Jump Diffusion

	n	Price	CPU
Tan	3	8.83	1.7
	4	8.83	4.5
Sec	3	8.83	1.2
	4	8.86	3.1
Lattice (Amin)	5	8.03	0.02
	10	8.68	0.05
	20	8.78	0.35
	50	8.83	4.9
	100	8.84	43

Notes. $S_0=100$, $K=100$, $T=0.5$, $r=0.1$, $\sigma=0.2828$, $\lambda=2$, $\gamma=0.2$, n: number of time steps in Amin lattice or # points in Richardson extrapolation for Tan/Sec alg.

4. Conclusions

We presented a new approach to pricing American-style derivatives through approximating the value function with linear interpolation functions, which converts the pricing of an American-style derivative to that of pricing a portfolio of European call options (of varying strikes and maturities), and in many cases obtain tight analytical upper and lower bounds for the true price. Implementation is quite modular, in that the same code can be used for essentially any asset price model, simply by substituting the new European option price function where it is called by the interpolation model. To illustrate the approach, we applied our algorithms to American call and put options written on underlying assets following geometric Brownian motion and jump-diffusion processes. Relative to the existing approaches of Amin (1993) and Huang et al. (1996), our methods demonstrate that they are particularly effective in the American-style setting for which they are designed, but also offer a viable alternative for continuously exercisable options when used with Richardson extrapolation. Our methods can also easily handle pure jump processes (Laprise 2002). Laprise (2002) also describes application to American-Asian options, where the path-dependent payoff depends on the current average of the stock price over a specified duration (cf., Ben Ameur et al. 2002, Wu and Fu 2003). However, extensions to derivatives on more than one underlying random process, e.g., multiple assets or stochastic volatility, is an open research problem.

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Appendix

The following is used in the proofs of Propositions 4 and 6:

LEMMA A.1. *Let Z be a random variable.*

(a) *Let $g(\cdot)$ be convex. If either $g(\cdot)$ is nondecreasing and $h(z;\cdot)$ is convex $\forall z$, or $g(\cdot)$ is nonincreasing and $h(z;\cdot)$ is concave $\forall z$, then $E[g(h(Z;\cdot))]$ is a convex function.*

(b) *Let $g(\cdot)$ be concave. If either $g(\cdot)$ is nondecreasing and $h(z;\cdot)$ is concave $\forall z$, or $g(\cdot)$ is nonincreasing and $h(z;\cdot)$ is convex $\forall z$, then $E[g(h(Z;\cdot))]$ is a concave function.*

(*In particular, if both $g(\cdot)$ and $h(z;\cdot)$ are linear, then $E[g(h(Z;\cdot))]$ is also linear.*)

PROOF. We prove (a); the proof of (b) is nearly identical. For $0\le\lambda\le 1$ and any x, y, z,

$$\begin{aligned}\lambda g(h(z;x))+(1-\lambda)g(h(z;y)) &\ge g(\lambda h(z;x)+(1-\lambda)h(z;y))\\ &\ge g(h(z;\lambda x+(1-\lambda)y)),\end{aligned}$$

where the first inequality follows from the convexity of g and the second inequality follows from either $h(z;\cdot)$ convex and $g(\cdot)$ nondecreasing, or $h(z;\cdot)$ concave and $g(\cdot)$ nonincreasing. Combining the result with the linearity of the expectation operator gives (a). □

PROOF OF PROPOSITION 1. We prove the ($\le$) case (the ($\ge$) case is identical) by induction. Let $i=N-2$. $\widehat{V}_{N-1}\le\widetilde{V}_{N-1}=V_{N-1}$ implies

$$\begin{aligned}\widetilde{H}_{N-2}(S) &= e^{-r\tau}E[\widehat{V}_{N-1}(S_{N-1})\mid S_{N-2}=S]\\ &\le e^{-r\tau}E[V_{N-1}(S_{N-1})\mid S_{N-2}=S]=H_{N-2}(S),\end{aligned}$$

so $\widetilde{V}_{N-2}\le V_{N-2}$. By induction, $\widetilde{V}_{i+1}\le V_{i+1}$. As $\widehat{V}_{i+1}\le\widetilde{V}_{i+1}\le V_{i+1}$, we can similarly show $\widetilde{H}_i\le H_i$; hence $\widetilde{V}_i\le V_i$. □

PROOF OF PROPOSITION 2. First we establish the following claim: For $i=0,\ldots,N-2$, if $|\widehat{V}_{i+1}(S)-V_{i+1}(S)|<\alpha$, then $|\widetilde{V}_i(S)-V_i(S)|<\alpha$.

$$\begin{aligned}|\widetilde{V}_i(S)-V_i(S)| &= |\max(L(S),\widetilde{H}_i(S))-\max(L(S),H_i(S))|\\ &\le |\widetilde{H}_i(S)-H_i(S)|\\ &= |e^{-r\tau}E[\widehat{V}_{i+1}(S_{i+1})-V_{i+1}(S_{i+1})\mid S_i=S]|\\ &\le e^{-r\tau}E[|\widehat{V}_{i+1}(S_{i+1})-V_{i+1}(S_{i+1})|\mid S_i=S]\\ &\qquad\qquad\text{(Jensen's inequality)}\\ &< e^{-r\tau}E[\alpha\mid S_i=S]\le\alpha.\end{aligned}$$

Now we prove the proposition by induction. At $i=N-2$, $|\widehat{V}_{N-1}(S)-V_{N-1}(S)|<\epsilon_{N-1}$, which, with the claim, implies $|\widetilde{V}_{N-2}(S)-V_{N-2}(S)|<\epsilon_{N-1}$. Assume the result holds for t_{i+1}; i.e., $|\widetilde{V}_{i+1}(S)-V_{i+1}(S)|<\sum_{j=i+2}^{N-1}\epsilon_j$. Then, with $|\widehat{V}_{i+1}(S)-\widetilde{V}_{i+1}(S)|<\epsilon_{i+1}$, we have

$$\begin{aligned}|\widehat{V}_{i+1}(S)-V_{i+1}(S)| &= |(\widehat{V}_{i+1}(S)-\widetilde{V}_{i+1}(S))+(\widetilde{V}_{i+1}(S)-V_{i+1}(S))|\\ &\le |\widehat{V}_{i+1}(S)-\widetilde{V}_{i+1}(S)|+|\widetilde{V}_{i+1}(S)-V_{i+1}(S)|\\ &< \epsilon_{i+1}+\sum_{j=i+2}^{N-1}\epsilon_j=\sum_{j=i+1}^{N-1}\epsilon_j.\end{aligned}$$

Thus, by the claim, $|\widetilde{V}_i(S)-V_i(S)|<\sum_{j=i+1}^{N-1}\epsilon_j$. □

PROOF OF PROPOSITION 3. For ease of notation, we omit "i" superscripts and subscripts. By (9), for $S\ge x_0$,

$$\widehat{V}(S)=\sum_{j=1}^{n+1}(m_j(S-x_{j-1})+\widetilde{V}(x_{j-1}))\mathbf{1}\{x_{j-1}\le S<x_j\},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. As $\mathbf{1}\{x_{j-1}\le S<x_j\}=\mathbf{1}\{S\ge x_{j-1}\}-\mathbf{1}\{S\ge x_j\}$,

$$\begin{aligned}\widehat{V}(S) &= \sum_{j=1}^{n+1}(m_j(S-x_{j-1})+\widetilde{V}(x_{j-1}))\mathbf{1}\{S\ge x_{j-1}\}\\ &\quad-\sum_{j=1}^{n+1}(m_j(S-x_{j-1})+\widetilde{V}(x_{j-1}))\mathbf{1}\{S\ge x_j\}\\ &= (m_1(S-x_0)+\widetilde{V}(x_0))\mathbf{1}\{S\ge x_0\}\\ &\quad+\sum_{j=2}^{n+1}(m_j(S-x_{j-1})+\widetilde{V}(x_{j-1}))\mathbf{1}\{S\ge x_{j-1}\}\\ &\quad-\Bigg[(m_{n+1}(S-x_n)+\widetilde{V}(x_n))\mathbf{1}\{S\ge x_{n+1}\}\\ &\qquad+\sum_{j=1}^{n}(m_j(S-x_{j-1})+\widetilde{V}(x_{j-1}))\mathbf{1}\{S\ge x_j\}\Bigg]\\ &= ((m_1-m_0)(S-x_0)+\widetilde{V}(x_0))\mathbf{1}\{S\ge x_0\}\\ &\quad-(m_{n+1}(S-x_n)+\widetilde{V}(x_n))\mathbf{1}\{S\ge x_{n+1}\}\\ &\quad+\sum_{j=1}^{n}(m_{j+1}(S-x_j)+\widetilde{V}(x_j))\mathbf{1}\{S\ge x_j\}\\ &\quad-\sum_{j=1}^{n}(m_j(S-x_j)+\widetilde{V}(x_j))\mathbf{1}\{S\ge x_j\}\\ &= \widetilde{V}(0)-(m_{n+1}(S-x_n)+\widetilde{V}(x_n))\mathbf{1}\{S\ge x_{n+1}\}\\ &\quad+\sum_{j=0}^{n}(m_{j+1}-m_j)(S-x_j)^+,\end{aligned}\tag{29}$$

where in the third equality, $m_0=0$ and $m_j(S-x_{j-1})+\widetilde{V}(x_{j-1})=m_j(S-x_j)+\widetilde{V}(x_j)$.

Now for $x_{n+1}<\infty$, i.e., $x_{n+1}=x_n-\widetilde{V}(x_n)/m_{n+1}$ and $m_{n+2}=0$,

$$\begin{aligned}-(m_{n+1}(S-x_n)+\widetilde{V}(x_n)) &= -(m_{n+1}(S-x_{n+1}))\\ &= (m_{n+2}-m_{n+1})(S-x_{n+1}),\end{aligned}$$

and for $x_{n+1}=\infty$, $\mathbf{1}\{S\ge x_{n+1}\}=0$. Thus, by (29),

$$\widehat{V}(S)=\widetilde{V}(0)+\sum_{j=0}^{n+1}(m_{j+1}-m_j)(S-x_j)^+,$$

where, if $x_{n+1}=\infty$, the last term in the summation is zero. □

PROOF OF PROPOSITION 4. First, as $\widetilde{V}_0\equiv\widetilde{H}_0$, $\widetilde{H}_0$ convex implies $\widetilde{V}_0$ convex. Further, for $i=1,\ldots,N-1$, as $\widetilde{V}_i(\cdot)=\max(L(\cdot),\widetilde{H}_i(\cdot))$ and $L(\cdot)$ is convex, $\widetilde{H}_i$ convex implies $\widetilde{V}_i$ convex. Consider $H_{N-1}(x)=e^{-r\tau}E[L(S_N)\mid S_{N-1}=x]=e^{-r\tau}E[L(h(Z;x,\Theta))]$. By Lemma A.1(a), H_{N-1} is convex if either $L(\cdot)$ is nondecreasing and $h(Z;\cdot,\Theta)$ is convex, or $L(\cdot)$ is nonincreasing and $h(Z;\cdot,\Theta)$ is concave. For $i=N-1,\ldots,1$, assume $\widetilde{V}_i$ and $h(Z;\cdot,\Theta)$ are convex, and $m^{(i)}_{n_i+1}\ge m^{(i)}_{n_i}$. By (11), $\widetilde{H}_{i-1}$ is convex if

(a) $m^{(i)}_1E[S_i\mid S_{i-1}=x]$ is convex; and

(b) for $j=1,\ldots,\bar{n}_i$, $(m^{(i)}_{j+1}-m^{(i)}_j)V^E(x,x^{(i)}_j,\tau)$ is convex, where $\bar{n}_i=n_i+1$ if $x^{(i)}_{n_i+1}<\infty$, and $\bar{n}_i=n_i$ otherwise.

We first consider (b). The convexity of $\widetilde{V}_i$ implies that $m_{j+1}^{(i)} - m_j^{(i)} \geq 0$ for $j = 1, \ldots, n_i - 1$, which, with the previous assumption, implies the inequality for $j = 1, \ldots, n_i$. If $x_{n_i+1}^{(i)} < \infty$, then $m_{n_i+1}^{(i)} < 0$ and $m_{n_i+2}^{(i)} = 0$, which implies that $m_{n_i+2}^{(i)} > m_{n_i+1}^{(i)}$. Therefore, $m_{j+1}^{(i)} - m_j^{(i)} \geq 0$ for $j = 1, \ldots, \bar{n}_i$. Next, if we let $g(x) = (x - x_j^{(i)})^+$ we have $V^E(\cdot, x_j^{(i)}, \tau) = e^{-r\tau}E[g(h(Z; \cdot, \Theta))]$. As $g(\cdot)$ is convex and nondecreasing and $h(Z; \cdot, \Theta)$ is convex, by Lemma A.1(a), $V^E(\cdot, x_j^{(i)}, \tau)$, hence $(m_{j+1}^{(i)} - m_j^{(i)})V^E(\cdot, x_j^{(i)}, \tau)$, for $j = 1, \ldots, \bar{n}_i$, is convex.

For (a), if we let $g(x) = x$, then $E[S_i \mid S_{i-1} = x] = E[g(h(Z; x, \Theta))]$. By the convexity of h, Lemma A.1 implies that $E[S_i \mid S_{i-1} = x]$ is convex. However, (a) requires convexity of the product $m_1^{(i)}E[S_i \mid S_{i-1} = x]$, which is achieved if we further require $m_1^{(i)} \geq 0$ or linearity of h, which would make $E[S_i \mid S_{i-1} = x]$ linear, and hence the product linear. □

Proof of Proposition 5. The proof is nearly identical to that for Proposition 3.

Proof of Proposition 6. The proof is nearly identical to that for Proposition 4.

Proof of Proposition 7. Note that $L(S) = (S - K)^+$ is convex and nondecreasing. The proof is via induction, where we also prove $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex, $\widetilde{V}_i(0) = 0$, and $\partial\widetilde{H}_i(S)/\partial S \leq e^{-\delta\tau}$.

For $i = N-1$, the claim is given by (18). Further, $L(\cdot)$ convex and nondecreasing and $h(Z; \cdot, \Theta)$ convex by Assumption 3 imply V_{N-1} and H_{N-1} are convex by Proposition 4. Also,

$$V_{N-1}(0) = H_{N-1}(0) = V^E(0, K, \tau) = 0$$

by Assumption 2, and

$$\frac{\partial}{\partial S}H_{N-1}(S) = \frac{\partial}{\partial S}V^E(S, K, \tau) \leq e^{-\delta\tau}$$

by Assumption 1. By induction, assume

$$\widetilde{V}_{i+1}(S) = \begin{cases} \widetilde{H}_{i+1}(S) & \text{if } S < \tilde{s}_{i+1}^*, \\ S - K & \text{if } S \geq \tilde{s}_{i+1}^*, \end{cases}$$

for some finite $\tilde{s}_{i+1}^*$, where $\widetilde{V}_{i+1} \geq V_{i+1}$ and $\widetilde{H}_{i+1} \geq H_{i+1}$. Further, assume $\widetilde{V}_{i+1}$ and $\widetilde{H}_{i+1}$ are convex, $\widetilde{V}_{i+1}(0) = 0$, and $\partial\widetilde{H}_{i+1}(S)/\partial S \leq e^{-\delta\tau}$.

Let $x_n^{(i+1)} = \tilde{s}_{i+1}^*$ and $m_{n+1}^{(i+1)} = 1.0$. We consider the properties of $\widehat{V}_{i+1}$. First, $\widetilde{V}_{i+1} \geq 0$ and $\widetilde{V}_{i+1}(0) = 0$ imply $m_1^{(i+1)} \geq 0$. Further, $\partial\widetilde{H}_{i+1}(S)/\partial S \leq e^{-\delta\tau}$ implies $m_n^{(i+1)} \leq e^{-\delta\tau}$. Thus, $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$. Hence, by Proposition 4, $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex. Next, as $x_n^{(i+1)} = \tilde{s}_{i+1}^*$ and $m_{n+1}^{(i+1)} = 1.0$, $\widehat{V}_{i+1}(S) = \widetilde{V}_{i+1}(S) = S - K$ for $S > x_n^{(i+1)}$. Thus, by Proposition 1 and the argument in §2.1, $\widetilde{V}_i \geq V_i$ and $\widetilde{H}_i \geq H_i$.

Next, as $x_0^{(i+1)} = 0$, $\widetilde{V}_{i+1}(0) = 0$, and $m_{n+1}^{(i+1)} \geq 0$, by (11),

$$\widetilde{H}_i(S) = \sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big)V^E\big(S, x_j^{(i+1)}, \tau\big).$$

In particular, by Assumption 2, $\widetilde{V}_i(0) = \widetilde{H}_i(0) = 0$. Now, $\widetilde{V}_{i+1}$ convex implies $m_{j+1}^{(i+1)} \geq m_j^{(i+1)}$ for $j = 1, \ldots, n-1$, and as $m_1^{(i+1)} \geq 0 = m_0^{(i+1)}$ and $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$, the coefficients in the summation are all positive. Further, by Assumption 1,

$$\begin{aligned}\frac{\partial}{\partial S}\widetilde{H}_i(S) &= \sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big)\frac{\partial}{\partial S}V^E\big(S, x_j^{(i+1)}, \tau\big) \\ &\leq e^{-\delta\tau}\sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big) = e^{-\delta\tau}.\end{aligned}$$

Next, we note that for $S < K$, $L(S) = 0$ implies $\widetilde{V}_i(S) = \widetilde{H}_i(S)$. Thus, as $\partial\widetilde{H}_i(S)/\partial S$ is bounded above by a value strictly smaller than 1 (because $\delta > 0$) and $\partial(S - K)/\partial S = 1$, there exists a unique, finite $\tilde{s}_i^* > K$ such that $L(\tilde{s}_i^*) = \widetilde{H}_i(\tilde{s}_i^*)$ and $L(S) < (>)\widetilde{H}_i(S)$ for $S < (>)\tilde{s}_i^*$; i.e.,

$$\widetilde{V}_i(S) = \begin{cases} \widetilde{H}_i(S) & \text{if } S < \tilde{s}_i^*, \\ S - K & \text{if } S \geq \tilde{s}_i^*. \end{cases}$$

This completes the proof of the induction step and thus the proposition. □

Proof of Proposition 8. Proposition 7 and the convergence of $\widetilde{H}_i$ to H_i imply that the true optimal policy is a threshold policy. Noting that $H_i(S) \geq S - K$ for $S \leq s_i^*$ and $H_i(s_i^*) = s_i^* - K$, $\widetilde{H}_i(s_i^*) \geq s_i^* - K$ from Proposition 7, which implies $\tilde{s}_i^* \geq s_i^*$. □

Proof of Proposition 9. The proof is via induction where we also prove $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex, $\widetilde{V}_i(0) = 0$, and $\partial\widetilde{H}_i(S)/\partial S \leq e^{-\delta\tau}$.

For $i = N-1$, the claim of the proposition is given by (18), and V_{N-1} and H_{N-1} convex, $V_{N-1}(0) = 0$, and $\partial H_{N-1}(S)/\partial S \leq e^{-\delta\tau}$ are shown in the proof of Proposition 7.

By induction, assume

$$\widetilde{V}_{i+1}(S) = \begin{cases} \widetilde{H}_{i+1}(S) & \text{if } S < \tilde{s}_{i+1}^*, \\ S - K & \text{if } S \geq \tilde{s}_{i+1}^*, \end{cases}$$

for some finite $\tilde{s}_{i+1}^* \leq s_{i+1}^*$, where $\widetilde{V}_{i+1} \leq V_{i+1}$ and $\widetilde{H}_{i+1} \leq H_{i+1}$. Further, assume $\widetilde{V}_{i+1}$ and $\widetilde{H}_{i+1}$ are convex, $\widetilde{V}_{i+1}(0) = 0$, and $\partial\widetilde{H}_{i+1}(S)/\partial S \leq e^{-\delta\tau}$.

Let $x_n^{(i+1)} = \tilde{s}_{i+1}^*$, $m_n^{(i+1)} = \partial\widetilde{H}_{i+1}(S)/\partial S|_{S=x_n^{(i+1)}}$, and $m_{n+1}^{(i+1)} = 1.0$. We consider the properties of $\widehat{V}_{i+1}(\cdot)$. First, $\partial\widetilde{H}_{i+1}(S)/\partial S \leq e^{-\delta\tau}$ implies $m_n^{(i+1)} \leq e^{-\delta\tau}$. Thus, $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$. As $m_0^{(i+1)} = 0$ and $h(Z; \cdot, \Theta)$ is convex by Assumption 3, $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex by Proposition 6. Next, as $x_n^{(i+1)} = \tilde{s}_{i+1}^*$ and $m_{n+1}^{(i+1)} = 1.0$, $\widehat{V}_{i+1}(S) = \widetilde{V}_{i+1}(S)$ for $S > x_n^{(i+1)}$. Thus, by Proposition 1 and the argument in §2.2, $\widetilde{V}_i \leq V_i$ and $\widetilde{H}_i \leq H_i$.

We now consider $\widetilde{H}_i$ given by (13). First, as $m_0^{(i+1)} = 0$ and $m_{n+1}^{(i+1)} \geq 0$, the first and last terms of the summation in (13) are zero. Thus, as $x_0^{(i+1)} = 0$ and $\widetilde{V}_{i+1}(0) = 0$, we have

$$\widetilde{H}_i(S) = \sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big)V^E\big(S, y_j^{(i+1)}, \tau\big).$$

In particular, by Assumption 2, $\widetilde{V}_i(0) = \widetilde{H}_i(0) = 0$. Now, $\widetilde{H}_{i+1}$ convex implies $m_{j+1}^{(i+1)} \geq m_j^{(i+1)}$ for $j = 0, \ldots, n-1$, and as $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$, the coefficients in the summation are all positive. Further, by Assumption 1,

$$\begin{aligned}\frac{\partial}{\partial S}\widetilde{H}_i(S) &= \sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big)\frac{\partial}{\partial S}V^E\big(S, y_j^{(i+1)}, \tau\big) \\ &\leq e^{-\delta\tau}\sum_{j=0}^{n}\big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big) = e^{-\delta\tau}.\end{aligned}$$

Next, we note that for $S < K$, $L(S) = 0$ implies $\widetilde{V}_i(S) = \widetilde{H}_i(S)$. Therefore, as $\partial\widetilde{H}_i(S)/\partial S$ is bounded above by a value strictly smaller than 1 and $\partial(S - K)/\partial S = 1$, there exists a unique, finite $\tilde{s}_i^* > K$ such that $L(\tilde{s}_i^*) = \widetilde{H}_i(\tilde{s}_i^*)$ and $L(S) < (>)\widetilde{H}_i(S)$

for $S < (>)\tilde{s}_i^*$; i.e.,

$$\widetilde{V}_i(S) = \begin{cases} \widetilde{H}_i(S) & \text{if } S < \tilde{s}_i^*, \\ S - K & \text{if } S \geq \tilde{s}_i^*. \end{cases}$$

Now consider s_i^*. $H_i(S) \geq S - K$ for $S \leq s_i^*$ and $H_i(s_i^*) = s_i^* - K$. Thus, $\widetilde{H}_i(s_i^*) \leq s_i^* - K$, implying $\tilde{s}_i^* \leq s_i^*$. This completes the proof of the induction step. $\square$

Proof of Proposition 10. We first note that $L(S) = (K-S)^+$ is convex and nonincreasing, and $h(Z; \cdot, \Theta)$ is both convex and concave. The proof is via induction where we also prove $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex, $\widetilde{V}_i(0) = K$, and $-e^{-\delta\tau} \leq \partial \widetilde{H}_i(S)/\partial S \leq 0$.

For $i = N-1$, the proposition claim is given by (20). Further, $L(\cdot)$ convex and nonincreasing and $h(Z; \cdot, \Theta)$ concave imply V_{N-1} and H_{N-1} convex by Proposition 4. By (19),

$$\frac{\partial}{\partial S} H_{N-1}(S) = -e^{-\delta\tau} + \frac{\partial}{\partial S} V^E(S, K, \tau).$$

Thus, by Assumption 1, $-e^{-\delta\tau} \leq \partial H_{N-1}(S)/\partial S \leq 0$. Finally, by (20), $V_{N-1}(0) = L(0) = K$.

By induction, assume

$$\widetilde{V}_{i+1}(S) = \begin{cases} K - S & \text{if } S < \tilde{s}_{i+1}^*, \\ \widetilde{H}_{i+1}(S) & \text{if } S \geq \tilde{s}_{i+1}^*, \end{cases}$$

for some $\tilde{s}_{i+1}^*$, where $\widetilde{V}_{i+1} \geq V_{i+1}$ and $\widetilde{H}_{i+1} \geq H_{i+1}$. Further, assume $\widetilde{V}_{i+1}$ and $\widetilde{H}_{i+1}$ are convex, $\widetilde{V}_{i+1}(0) = K$, and $-e^{-\delta\tau} \leq \partial \widetilde{H}_{i+1}(S)/\partial S \leq 0$.

Let $x_1^{(i+1)} = \tilde{s}_{i+1}^*$, $m_1^{(i+1)} = -1.0$, and $m_{n+1}^{(i+1)} = 0$. We consider the properties of $\widehat{V}_{i+1}$. First, $\partial \widetilde{H}_{i+1}(S)/\partial S \leq 0$ implies $m_j^{(i+1)} \leq 0$ for $j = 2, \ldots, n$. Thus, $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$. Hence, as $h(Z; \cdot, \Theta)$ is linear, $\widetilde{V}_i$ and $\widetilde{H}_i$ are convex by Proposition 4. Next, for $S > x_n^{(i+1)}$, $\widetilde{V}_{i+1}(S) = \widetilde{H}_{i+1}(S)$ implies $\partial \widetilde{V}_{i+1}(S)/\partial S \leq 0$. Thus, as $m_{n+1}^{(i+1)} = 0$, $\widehat{V}_{i+1}(S) \geq \widetilde{V}_{i+1}(S)$ for $S > x_n^{(i+1)}$. Thus, by Proposition 1 and the argument in §2.1, $\widetilde{V}_i \geq V_i$ and $\widetilde{H}_i \geq H_i$.

As $x_0^{(i+1)} = 0$, $\widetilde{V}_{i+1}(0) = K$, $m_{n+1}^{(i+1)} = 0$, $m_1^{(i+1)} = -1.0$, $m_0^{(i+1)} = 0$, $V^E(S, 0, \tau) = Se^{-\delta\tau}$,

$$\widetilde{H}_i(S) = Ke^{-r\tau} - Se^{-\delta\tau} + \sum_{j=1}^{n} \big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big) V^E\big(S, x_j^{(i+1)}, \tau\big)$$

by (11). In particular, by Assumption 2, $\widetilde{H}_i(0) = Ke^{-r\tau} \leq K = L(0)$, which implies that $\widetilde{V}_i(0) = K$. Now, $\widetilde{V}_{i+1}$ convex implies $m_{j+1}^{(i+1)} \geq m_j^{(i+1)}$ for $j = 1, \ldots, n-1$, and as $m_{n+1}^{(i+1)} \geq m_n^{(i+1)}$, the coefficients in the summation are all positive. Further,

$$\frac{\partial}{\partial S} \widetilde{H}_i(S) = -e^{-\delta\tau} + \sum_{j=1}^{n} \big(m_{j+1}^{(i+1)} - m_j^{(i+1)}\big) \frac{\partial}{\partial S} V^E\big(S, x_j^{(i+1)}, \tau\big).$$

Noting that $m_{n+1}^{(i+1)} = 0$ and $m_1^{(i+1)} = -1.0$, Assumption 1 implies that $-e^{-\delta\tau} \leq \partial \widetilde{H}_i(S)/\partial S \leq 0$.

Next, for $S \geq K$, $\widetilde{H}_i(S) > L(S) = 0$; i.e., $\widetilde{V}_i(S) = \widetilde{H}_i(S)$. Thus, as $\widetilde{H}_i(0) \leq L(0)$ and $\partial \widetilde{H}_i(S)/\partial S \geq -1$, there exists an $\tilde{s}_i^* < K$ such that $\widetilde{H}_i(S) < (\geq) L(S)$ for $S < (\geq) \tilde{s}_i^*$; i.e.,

$$\widetilde{V}_i(S) = \begin{cases} K - S & \text{if } S < \tilde{s}_i^*, \\ \widetilde{H}_i(S) & \text{if } S \geq \tilde{s}_i^*, \end{cases}$$

where $\tilde{s}_i^* > 0$ unless $r = 0$. This completes the proof of the induction step. $\square$

Proof of Proposition 11. Proposition 10 and the convergence of $\widetilde{H}_i$ to H_i imply that the true optimal policy is a threshold policy. Noting that $H_i(S) \geq K - S$ for $S \geq s_i^*$ and $H_i(s_i^*) = K - s_i^*$, $\widetilde{H}_i(s_i^*) \geq K - s_i^*$ from Proposition 10, which implies $\tilde{s}_i^* \leq s_i^*$. $\square$

Proof of Proposition 12. The proof is very similar to that of Propositions 9 and 10.

References

Amin, K. I. 1993. Jump diffusion option valuation in discrete time. *J. Finance* **48**(5) 1833–1863.

Ben Ameur, H., M. Breton, P. L'Ecuyer. 2002. A numerical procedure for pricing American-style Asian options. *Management Sci.* **48**(5) 625–643.

Broadie, M., J. Detemple. 1996. American option valuation: New bounds, approximations, and a comparison of existing methods. *Rev. Financial Stud.* **9**(4) 1211–1250.

Broadie, M., P. Glasserman. 1997a. Pricing American-style securities using simulation. *J. Econom. Dynam. Control* **21**(8/9) 1323–1352.

Broadie, M., P. Glasserman. 1997b. Monte Carlo methods for pricing high-dimensional American options: An overview. *Net Exposure* **3** 15–37.

Carr, P., R. Jarrow, R. Mynemi. 1992. Alternative characterizations of American put options. *Math. Finance* **2**(2) 87–106.

Carriere, J. F. 1996. Valuation of the early-exercise price for derivative securities using simulations and splines. *Insurance: Math. Econom.* **19** 19–30.

Das, S. 1997. Discrete-time bond and option pricing for jump-diffusion processes. *Rev. Derivatives Res.* **1**(3) 211–244.

Fu, M. C., J. Q. Hu. 1995. Sensitivity analysis for Monte Carlo simulation of option pricing. *Probab. Engrg. Inform. Sci.* **9** 417–446.

Fu, M. C., S. B. Laprise, D. B. Madan, Y. Su, R. Wu. 2001. Pricing American options: A comparison of Monte Carlo simulation approaches. *J. Comput. Finance* **4**(3) 39–88.

Grant, D., G. Vora, D. Weeks. 1996. Simulation and the early-exercise option problem. *J. Financial Engrg.* **5**(3) 211–227.

Huang, J., M. Subrahmanyam, G. Yu. 1996. Pricing and hedging American options: A recursive integration method. *Rev. Financial Stud.* **9**(1) 277–300.

Ju, N. 1998. Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Rev. Financial Stud.* **11**(3) 627–646.

Kim, I. J. 1990. The analytical valuation of American options. *Rev. Financial Stud.* **3**(4) 547–572.

Laprise, S. B. 2002. Stochastic dynamic programming: Simulation based approaches and applications to finance. Doctoral thesis, University of Maryland at College Park, College Park, MD.

Laprise, S. B., M. C. Fu, A. E. B. Lim, S. I. Marcus. 2001. A new approach to pricing American-style derivatives. *Proc. 2001 Winter Simulation Conf.* 329–337.

Longstaff, F. A., E. S. Schwartz. 2001. Valuing American options by simulation: A simple least-squares approach. *Rev. Financial Stud.* **14** 113–147.

Merton, R. C. 1976. Option pricing when underlying stock returns are discontinuous. *J. Financial Econom.* **3** 125–144.

Omberg, E. 1987. The valuation of American puts with exponential exercise policies. *Adv. Futures Options Res.* **2** 117–142.

Tilley, J. 1993. Valuing American options in a path simulation model. *Trans. Soc. Actuaries* **45** 83–104.

Tsitsiklis, J. N., B. Van Roy. 2001. Regression methods for pricing complex American-style options. *IEEE Trans. Neural Networks* **14**(4) 694–703.

Wu, R., M. C. Fu. 2003. Optimal exercise policies and simulation-based valuation for American-Asian options. *Oper. Res.* **51**(1) 52–66.

Zhang, X. 1997. Numerical analysis of American option pricing in a jump-diffusion model. *Math. Oper. Res.* **22**(3) 668–690.