The dynamics of coupled planar rigid bodies. Part I: Reduction, equilibria and stability

N. Sreenath  
Department of Electrical Engineering and the Systems Research Center, University of Maryland, College Park, Maryland 20742, USA

Y. G. Oh  
Department of Mathematics, University of California, Berkeley, California 94720, USA

P. S. Krishnaprasad  
Department of Electrical Engineering and the Systems Research Center, University of Maryland

and J. E. Marsden  
Department of Mathematics, University of California, Berkeley

Abstract

This paper studies the dynamics of coupled planar rigid bodies, concentrating on the case of two or three bodies coupled with a hinge joint. The Hamiltonian structure is non-canonical and is obtained using the methods of reduction, starting from canonical brackets on the cotangent bundle of the configuration space in material representation. The dynamics on the reduced space for two bodies occurs on cylinders in $\mathbb{R}^4$; stability of the equilibria is studied using the energy-Casimir method and is confirmed numerically. The phase space of the two bodies contains a homoclinic orbit which produces chaotic solutions when the system is perturbed by a third body. This and a study of periodic orbits are discussed in part II. The number and stability of equilibria and their bifurcations for three bodies as system parameters are varied are studied here; in particular, it is found that there are always four or six equilibria.

1. Introduction

The techniques of reduction of Hamiltonian systems with symmetry and the attendant energy-Casimir method have proved to be useful in a wide variety of problems, including fluid and plasma stability (Holm, Marsden, Ratiu and Weinstein, 1985), rigid-body dynamics with attachments and internal rotors (Holmes and Marsden, 1983; Koiller, 1985; Krishnaprasad, 1985; Krishnaprasad and Marsden, 1987), and bifurcations of liquid drops (Lewis, Marsden and Ratiu, 1986a,b). In this paper we shall apply these techniques to the case of planar rigid bodies coupled by a hinge joint. Many of the results for the two and three bodies...
generalize to multibody structures and other modifications, such as the inclusion of hinge torques. In subsequent papers we shall be studying this as well as the problem of coupled three-dimensional rigid bodies (for example, with a ball-in-socket or hinge joint). We also expect that the non-canonical Hamiltonian methods that are useful here will be useful in related problems of control (see (Van der Schaft, 1984; Sanchez de Alvarez, 1986).

The reduction technique used here goes back to Arnold (1966), Meyer (1973), and Marsden and Weinstein (1974), amongst others. It involves starting with a Poisson manifold $P$ and a Lie group $G$ acting on $P$ by canonical transformations. The reduced phase space $P/G$ (assume it has no singularities) has a natural Poisson structure whose symplectic leaves are the Marsden–Weinstein–Meyer spaces $J^{-1}(\mu)/G_{\mu} = J^{-1}(\mathcal{O})/G$, where $\mu \in g^*$, the dual of the Lie algebra of $G$, $J^1$ is an equivariant momentum map for the action of $G$ on $P$, $G_{\mu}$ is the isotropy group of $\mu$ (relative to the coadjoint action) and $\mathcal{O}$ is the coadjoint orbit through $\mu$. If $P = T^*G$ and $G$ acts by left translations, then $P/G$ is identifiable with $g^*$ equipped with the (-) Lie–Poisson bracket:

$$\{F, H\}(\mu) = -\left\langle \mu, \left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right] \right\rangle.$$  \hfill (1.1)

The symplectic leaves in this case are just the coadjoint orbits. For $G = SO(3)$ we get the (Pauli–Martin) bracket for rigid body dynamics:

$$\{F, H\}(I) = -I \cdot (\nabla F \times \nabla H).$$  \hfill (1.2)

Here $I \in SO(3)^*$ is identified with a vector in $\mathbb{R}^3$ and represents the angular momentum of the rigid body in a body-fixed frame. If $I$ is the moment-of-inertia tensor so that $I = I\omega$; where $\omega$ is the body angular velocity, then Euler’s equations

$$\frac{dI}{dt} = I \times \omega$$  \hfill (1.3)

are equivalent to Hamilton’s equations

$$\dot{F} = \{F, H\},$$  \hfill (1.4)

where $H(I) = \frac{1}{2}(I, \omega) = \frac{1}{2}(I^\top l, l)$.

Notice that (1.2) is a non-canonical bracket; that is, the usual $(q, p)$ Poisson-bracket formalism has disappeared through the reduction process. One of our first goals in the paper will be to develop a similar bracket for the dynamics of two coupled planar rigid bodies. We start with the canonical bracket on the cotangent bundle of configuration space just as one starts with $T^*SO(3)$ (parametrized by Euler angles $(\theta, \varphi, \psi)$ and their conjugate momenta $(\psi, p_\psi)$) in rigid-body dynamics.

When these procedures are carried out for coupled rigid-body dynamics (§§2 to 4) we find that concepts akin to the ‘augmented body’ (cf. (Wittenburg, 1977)) come out in a natural way. The reduced Poisson structure obtained is a Poisson structure in $\mathbb{R}^3$ (not of Lie–Poisson type, however) whose symplectic leaves are cylinders. The reduced dynamics on one of these cylinders for specific rigid-body parameters† is shown in Fig. 1. Here $\mu_1$ and $\mu_2$ are the momenta of the two bodies and $\theta$ is the angle between the axes.

Being two-dimensional and Hamiltonian, the phase space is integrable. Notice that there are two saddle points. This is confirmed by a linearization of the energy-Casimir analysis for the stable manifold of the saddle point and a threshold analysis of the unstable manifold of the saddle point corresponding to the two bodies (cf. van der Schaft, 1984; Guckenheimer and Holmes, 1983) for information on instability and period doubling.

Another benefit of doing the reduction procedure is that the generalization to three-body motion can be made using similar ideas. Further discussion of this is given in §§6 and the three-dimensional case.

We now summarize one of the two main results of this paper. The Hamiltonian form for the dynamics of this structure are given in §§2 to 4:

\begin{align*}
d_i & \text{ distance from the hinge to body } i \\
\omega_i & \text{ angular velocity of body } i \\
\theta & \text{ joint angle from body 1 to body 2} \\
\lambda(\theta) & \text{ d}_1 \text{d}_2 \cos \theta \\
m_i & \text{ mass of body } i = 1, 2 \\
\varepsilon & \text{ moment of inertia of body } i = 1, 2 \\
\mu_1 + \mu_2 = 50.
\end{align*}

† The parameters chosen, in the notation used here, are $m_1/m_2 = 1/2$. 

Fig. 1. Phase portrait of the reduced system.
modifications, such as the inclusion shall be studying this as well as the bodies (for example, with a ball-in that the non-canonical Hamiltonian in related problems of control (see 1986).

Back to Arnold (1966), Meyer (1973), not others. It involves starting with a on P by canonical transformations. has no singularities) has a natural are the Marsden–Weinstein–Meyer the dual of the Lie algebra of G, or the action of G on P, \( G \mu \) is the (action) and \( \mathcal{O} \) is the coadjoint orbit translations, then \( P/G \) is identifiable bracket:

\[
\left[ \frac{\delta F}{\delta \mu}, \frac{\delta H}{\delta \mu} \right].
\]

(1.1)

coadjoint orbits. For \( G = SO(3) \) we dynamics:

\[
F = \mathbf{F} \times \nabla H.
\]

(1.2)
in \( \mathbb{R}^3 \) and represents the angular frame. If \( I \) is the moment-of-inertia body angular velocity, then Euler's

\[
\dot{\mathbf{F}} = \frac{\partial H}{\partial \mathbf{F}}.
\]

(1.3)

\[
\dot{\gamma} = \frac{\partial H}{\partial \gamma}.
\]

(1.4)

bracket; that is, the usual \((q,p)\) through the reduction process. One of similar bracket for the dynamics of with the canonical bracket on the just as one starts with \( T^*SO(3) \) and their conjugate momenta coupled rigid-body dynamics (§§2 to body' (cf. Wittenburg, 1977))

\[
\lambda(\theta) \quad d_1, d_2 \cos \theta
\]

\[
m_i \quad \text{mass of body } i = 1, 2
\]

\[
\epsilon \quad m_1 m_2 / (m_1 + m_2) = \text{reduced mass}
\]

†The parameters chosen, in the notation of §§2 to 4 are \( \tilde{I}_1 = 105.55, \tilde{I}_2 = 70, \epsilon = 55.55, \) and \( \mu_1 + \mu_2 = 50. \)

Fig. 1. Phase portrait of a planar two-body system
The dynamics of the system is described by the following Euler–Lagrange equations for $\theta$, $\omega_1$, $\omega_2$:

$$
\begin{align*}
\dot{\theta} &= \omega_2 - \omega_1, \\
\dot{\omega}_1 &= -\gamma(\tilde{I}_1\omega_2^2 + \varepsilon\lambda\omega_1^2), \\
\dot{\omega}_2 &= \gamma(\tilde{I}_1\omega_1^2 + \varepsilon\lambda\omega_2^2).
\end{align*}
$$

(1.5)

For the Hamiltonian structure it is convenient to introduce the momenta

$$
\mu_1 = \tilde{I}_1\omega_1 + \varepsilon\lambda\omega_2, \quad \mu_2 = \tilde{I}_2\omega_2 + \varepsilon\lambda\omega_1,
$$

(1.6a)

that is,

$$
\begin{pmatrix}
\mu_1 \\
\mu_2 
\end{pmatrix} = J
\begin{pmatrix}
\omega_1 \\
\omega_2 
\end{pmatrix}, \quad J = \begin{pmatrix} I_1 & \varepsilon\lambda \\ 0 & I_2 \end{pmatrix}
$$

(1.6b)

(this is done via the Legendre transform in §4). The evolution equations for $\mu_i$ are obtained by solving (1.6) for $\omega_1$, $\omega_2$ and substituting into (1.5). The Hamiltonian is

$$
H = \frac{1}{2}(\omega_1, \omega_2)J(\omega_1) \omega_2,
$$

(1.7a)
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that is,

\[ H = \frac{1}{2}(\mu_1, \mu_2)J^{-1}(\mu_1, \mu_2), \]  \hspace{1cm} (1.7b)

which is the total kinetic energy for the two bodies. The Poisson structure on the
\((\theta, \mu_1, \mu_2)\)-space (called \(P\) in §3) is

\[ \{F, H\} = \{F, H\}_2 - \{F, H\}_1, \]  \hspace{1cm} (1.8)

where

\[ \{F, H\}_i = \frac{\partial F}{\partial \mu_i} \frac{\partial H}{\partial \theta} - \frac{\partial H}{\partial \mu_i} \frac{\partial F}{\partial \theta}. \]

The evolution equations (1.5) are then equivalent to Hamilton’s equations
\(\dot{\theta} = \{F, H\}\). Casimirs for the bracket (1.8) are readily checked to be

\[ C = \Phi(\mu_1 + \mu_2) \]  \hspace{1cm} (1.9)

for \(\Phi\) any function of one variable; that is, \(\{F, C\} = 0\) for any \(F\). One can also verify directly from (1.5) that, correspondingly, \(d\mu/dt = 0\), where \(\mu = \mu_1 + \mu_2\) is
the system angular momentum.

The symplectic leaves of (1.8) are described by the variables \(\nu = (\mu_2 - \mu_1)/2\), \(\theta\) which parametrize the cylinder shown in Fig. 1. The bracket in terms of \((\theta, \nu)\) is
the canonical one on \(T^*S^1\):

\[ \{F, H\} = \frac{\partial F}{\partial \theta} \frac{\partial H}{\partial \nu} - \frac{\partial H}{\partial \theta} \frac{\partial F}{\partial \nu}. \]  \hspace{1cm} (1.10)

As we shall see, this canonical structure on \(T^*S^1\) is consistent with the
Satzers-Marsden-Kummer cotangent bundle reduction theorem (Abraham and

2. Kinematical preliminaries (for two coupled planar rigid bodies)

In this section we set up the phase space for the dynamics of our problem. Refer
to Fig. 4 and define the following quantities.

- \(d_{12}\) the vector from the centre of mass of body 1 to the hinge point in a
  reference configuration (fixed)
- \(d_{21}\) the vector from the centre of mass of body 2 to the hinge point in a
  reference configuration (fixed)
- \(R(\theta_i)\) the rotation through angle \(\theta_i\) giving the current
  orientation of body \(i\) (written as a matrix relative to the fixed standard
  inertial frame)
- \(r_i\) current position of the centre of mass of body \(i\)
- \(r\) current position of the system centre of mass
- \(v_i\) the vector from the system centre of mass to the centre of mass of
  body \(i\)
- \(\theta\) \(\theta_2 - \theta_1\) joint angle
- \(R(\theta)\) joint rotation, \(R(\theta_2) \cdot R(-\theta_1)\)
The basic configuration space we start with is $Q$, the subset of $\text{SE}(2) \times \text{SE}(2)$ (two copies of the special Euclidean group of the plane) consisting of pairs $((R(\theta_1), r_1), (R(\theta_2), r_2))$ satisfying the hinge constraint

$$r_2 = r_1 + R(\theta_1)d_{12} - R(\theta_2)d_{21}. \quad (2.1)$$

Notice that $Q$ is of dimension 4 and is parametrized by $\theta_1$, $\theta_2$ and, say $r_1$; that is, $Q = S^1 \times S^1 \times \mathbb{R}^2$. We form the velocity phase space $TQ$ and momentum phase space $T^*Q$.

The Lagrangian on $TQ$ is just the kinetic energy (relative to the inertial frame) given by summing the kinetic energies of each body. For convenience, we recall how this proceeds: let $X_i$ denote a position vector in body 1 relative to the centre of mass of body 1, and let $\rho_i(X_i)$ denote the mass density of body 1. Then the current position of the point with material label $X_i$ is

$$x_i = R(\theta_i)X_i + r_i. \quad (2.2)$$

Thus

$$\dot{x}_i = \dot{R}(\theta_i)X_i + \dot{r}_i,$n and so the kinetic energy of body 1 is

$$K_1 = \frac{1}{2} \int \rho_1(x_1) ||\dot{x}_1||^2 \, d^2x_1$$

$$= \frac{1}{2} \int \rho_1(x_1) (\dot{R}X_1 + \dot{r}_1) \cdot \dot{R}X_1 + 2\dot{R}X_1 \cdot \dot{r}_1 + ||\dot{r}_1||^2 \, d^2x_1.$$ (2.3)

But

$$\langle \dot{R}X_1, \dot{R}X_1 \rangle = \text{tr} (\dot{R}X_1(\dot{R}X_1)^T) = \text{tr} (\dot{R}X_1^T \dot{R})$$

and

$$\int \rho_1(X_1) \langle \dot{R}X_1, \dot{r}_1 \rangle \, d^2x_1 = \left\langle \dot{R} \int \rho_1(X_1)X_1 \, d^2x_1, \dot{r}_1 \right\rangle = 0 \quad (2.5)$$

since $X_1$ is the vector relative to the inertial frame and (2.5) into (2.3) and defining the transformation matrix $T' = R(\theta_1)R(\theta_2)^T$ we get

$$K_1 = \frac{1}{2} \text{tr} (\dot{T}'T'_{\theta_1} + \dot{T}'T'_{\theta_2})$$

with a similar expression for $K_2$, we have

$$L : TQ \rightarrow \mathbb{R}$$

The equations of motion then are $\dot{\mathbf{r}} = \mathbf{L}$. Equivalently, they are Hamiltonian.

For later convenience, we shall choose $\omega_2 = \theta_2$, $r_1^0$ and $r_2^0$. To do this note

$$m_1 \dot{r}_1^0 = \dot{r}_1^0 + m_2 \dot{r}_2^0$$

and, subtracting $r$ from both sides

$$r_2^0 = r_1^0 + m_2 \dot{r}_2^0.$$

From (2.10) and (2.11) we find that

$$m_1 \dot{r}_2^0 = \dot{r}_2^0 \quad (2.10)$$

and

$$m_2 \dot{r}_2^0 = \dot{r}_2^0 \quad (2.11)$$

Now we substitute

$$r_1 = r + \dot{r}_1$$

and

$$r_2 = r + \dot{r}_2$$

into (2.8) to give

$$L = \frac{1}{2} \text{tr} (\dot{R}(\theta_1)T'_{\theta_1} + \dot{R}(\theta_2)T'_{\theta_2})$$

But $m_1 \dot{r}_1 + m_2 \dot{r}_2 = 0$ since

$$L = \frac{1}{2} \text{tr} (\dot{R}(\theta_1)T'_{\theta_1} + \dot{R}(\theta_2)T'_{\theta_2})$$

simplifies to

$$L = \frac{1}{2} \text{tr} (\dot{R}(\theta_1)T'_{\theta_1}) + \frac{p^2}{2m}$$

where $p = m ||\dot{r}||$ is the magnitude of the angular momentum.
since $X_1$ is the vector relative to the center of mass of body 1. Substituting (2.4) and (2.5) into (2.3) and defining the matrix

$$ \mathbf{I}^1 = \int \rho(X_1)X_1X_1^T \, d^2X_1 $$

we get

$$ K_1 = \frac{1}{2} \text{tr} \left( \dot{R}(\theta_1)^T \dot{R}(\theta_1) \right) + \frac{1}{2} m_1 \left\| \dot{r}_1 \right\|^2; $$

with a similar expression for $K_2$ we let

$$ L : TQ \to \mathbb{R} \quad \text{be} \quad L = K_1 + K_2. \quad (2.8) $$

The equations of motion then are the Euler–Lagrange equations for this $L$ on $TQ$. Equivalently, they are Hamilton’s equations for the corresponding Hamiltonian.

For later convenience, we shall rewrite the energy (2.8) in terms of $\omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_2$, $\mathbf{r}_1^0$ and $\mathbf{r}_2^0$. To do this note that, by definition,

$$ m \mathbf{r} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2, \quad (2.9) $$

where $m = m_1 + m_2$, and so, as $\mathbf{r}_1 = \mathbf{r} + \mathbf{r}_1^0$,

$$ 0 = m_1 \mathbf{r}_1^0 + m_2 \mathbf{r}_2^0 \quad (2.10) $$

and, subtracting $\mathbf{r}$ from both sides of (2.1),

$$ \mathbf{r}_2^0 = \mathbf{r}_1^0 + R(\theta_1) \mathbf{d}_{12} - R(\theta_2) \mathbf{d}_{21}. \quad (2.11) $$

From (2.10) and (2.11) we find that

$$ \mathbf{r}_2^0 = \frac{m_1}{m} (R(\theta_1) \mathbf{d}_{12} - R(\theta_2) \mathbf{d}_{21}) \quad (2.12a) $$

and

$$ \mathbf{r}_1^0 = -\frac{m_2}{m} (R(\theta_1) \mathbf{d}_{12} - R(\theta_2) \mathbf{d}_{21}). \quad (2.12b) $$

Now we substitute

$$ \mathbf{r}_1 = \mathbf{r} + \mathbf{r}_1^0 \quad \text{so} \quad \dot{\mathbf{r}}_1 = \dot{\mathbf{r}} + \dot{\mathbf{r}}_1^0 \quad (2.13a) $$

and

$$ \mathbf{r}_2 = \mathbf{r} + \mathbf{r}_2^0 \quad \text{so} \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{r}} + \dot{\mathbf{r}}_2^0 \quad (2.13b) $$

into (2.8) to give

$$ L = \frac{1}{2} \text{tr} \left( \dot{R}(\theta_1)^T \dot{R}(\theta_1) + \dot{R}(\theta_2)^T \dot{R}(\theta_2) \right) + \frac{1}{2} \left[ m_1 \left\| \dot{\mathbf{r}} + \dot{\mathbf{r}}_1^0 \right\|^2 + m_2 \left\| \dot{\mathbf{r}} + \dot{\mathbf{r}}_2^0 \right\|^2 \right]. \quad (2.14) $$

But $m_1 \dot{\mathbf{r}}_1^0 + m_2 \dot{\mathbf{r}}_2^0 = 0$ since $m_1 \mathbf{r}_1^0 + m_2 \mathbf{r}_2^0 = 0$ from (2.10). Thus (2.14) simplifies to

$$ L = \frac{1}{2} \text{tr} \left( \dot{R}(\theta_1)^T \dot{R}(\theta_1) + \dot{R}(\theta_2)^T \dot{R}(\theta_2) \right) + \left( \frac{p^2}{2m} \right) + \frac{1}{2} m_1 \left\| \dot{\mathbf{r}}_1^0 \right\|^2 + \frac{1}{2} m_2 \left\| \dot{\mathbf{r}}_2^0 \right\|^2, \quad (2.15) $$

where $p = m \left\| \dot{\mathbf{r}}_1 \right\|$ is the magnitude of the system momentum.
Now write

\[ R(\theta) = \frac{d}{dt} \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix} \]

\[ = \begin{pmatrix} -\sin \theta_1 & -\cos \theta_1 \\ \cos \theta_1 & -\sin \theta_1 \end{pmatrix} \omega_1 \omega_1 = R(\theta) \begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix} = R(\theta) \omega_1, \]

so that (2.12) gives

\[ \dot{\mathbf{r}}_2 = \frac{m_1}{m} (R(\theta_1) \dot{\omega}_1 \mathbf{d}_{12} - R(\theta_2) \dot{\omega}_2 \mathbf{d}_{21}), \quad \dot{\mathbf{r}}_1 = -\frac{m_2}{m} (R(\theta_1) \dot{\omega}_1 \mathbf{d}_{12} - R(\theta_2) \dot{\omega}_2 \mathbf{d}_{21}). \]

Substituting (2.17) and (2.16) into (2.15) gives

\[ L = \frac{1}{2} \text{tr} \left( \dot{\omega}_1^T \dot{\omega}_1^T + \dot{\omega}_2^T \dot{\omega}_2^T \right) + \frac{p^2}{2m} + \frac{m_1 m_2}{m} \| \dot{\omega}_1 \mathbf{d}_{12} - R(\theta_2 - \theta_1) \dot{\omega}_2 \mathbf{d}_{21} \|^2. \]

Finally we note that

\[ \frac{1}{2} \text{tr} \left( \dot{\omega}_1^T \dot{\omega}_1^T \right) = \frac{1}{2} \text{tr} \left( \dot{\omega}_2^T \dot{\omega}_2^T \right) = \frac{1}{2} \text{tr} \left( \begin{pmatrix} \omega_1^T & 0 \\ 0 & \omega_2^T \end{pmatrix} \right) = \omega_1^T \dot{\mathbf{r}}_1 = \omega_2^T \dot{\mathbf{r}}_2, \]

(2.19.1)

where

\[ I_1 = \int \rho(X_1, Y_1)(X_1^2 + Y_1^2) dX_1 dY_1 \]

is the usual moment of inertia of body 1 about its center of mass. One similarly derives (2.19.2) where 1 is replaced by 2 throughout. The final term in (2.18) is manipulated as follows:

\[ \| \dot{\omega}_1 \mathbf{d}_{12} - R(\theta) \dot{\omega}_2 \mathbf{d}_{21} \|^2 = \| \dot{\omega}_1 \| \| \mathbf{d}_{12} \|^2 - 2 \text{tr} (\dot{\omega}_1 \mathbf{d}_{12}, R(\theta) \dot{\omega}_2 \mathbf{d}_{21}) + \| \dot{\omega}_2 \mathbf{d}_{21} \|^2 \]

\[ = \omega_1^2 \mathbf{d}_{12} + \omega_2^2 \mathbf{d}_{21} - 2 \text{tr} (\dot{\omega}_1 \mathbf{d}_{12}, R(\theta) \dot{\omega}_2 \mathbf{d}_{21}) \]

\[ = \omega_1^2 \mathbf{d}_{12} + \omega_2^2 \mathbf{d}_{21} - 2 \omega_1 \omega_2 \mathbf{d}_{12}, R(\theta) \mathbf{d}_{21}. \]

Substituting (2.19.1), (2.19.2) and (2.20) into (2.18) gives

\[ L = \frac{1}{2} (\omega_1^2 \dot{\mathbf{r}}_1 + \omega_2^2 \dot{\mathbf{r}}_2) + 2 \omega_1 \omega_2 \mathbf{e} \lambda(\theta) + \frac{p^2}{2m}, \]

(2.21)

where

\[ \lambda(\theta) = -\langle \mathbf{d}_{12}, R(\theta) \mathbf{d}_{21} \rangle = -[\mathbf{d}_{12} \cdot \mathbf{d}_{21} \cos \theta - (\mathbf{d}_{12} \times \mathbf{d}_{21}) \cdot \dot{\mathbf{z}} \sin \theta]. \]

(2.22)

Remarks 1. If \( \mathbf{d}_{12} \) and \( \mathbf{d}_{21} \) are parallel (that is, the reference configuration is chosen with \( \mathbf{d}_{12} \) and \( \mathbf{d}_{21} \) aligned), then (2.22) gives \( \lambda(\theta) = d_1 d_2 \cos \theta \), as in §1.

2. The quantities \( \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \) are the moments of inertia of 'augmented' bodies as defined in §1; for example \( \dot{\mathbf{r}}_1 \) is the moment of inertia of body 1 augmented by putting a mass \( \varepsilon \) at the hinge point.

3. Reduction to the center of mass frame

In this section we reduce the dynamics by the action of the translation group \( \mathbb{R}^2 \). This group acts on the original configuration space \( Q \) by

\[ \mathbf{v} \cdot ((\mathbf{R}(\theta_1), \mathbf{r}_1), (\mathbf{R}(\theta_2), \mathbf{r}_2)) = ((\mathbf{R}(\theta_1), \mathbf{r}_1 + \mathbf{v}), (\mathbf{R}(\theta_2), \mathbf{r}_2 + \mathbf{v})). \]

(3.1)

4. Reduction by rotations

To complete the reduction, we reduce the configuration space \( S^1 \times S^1 \) that was action obviously given by

\[ J((\theta_1, \mu_1), \mathbf{r}, \omega). \]
This is well defined since the hinge constraint (2.1) is preserved by this action. The induced momentum map on \( TQ \) is calculated by the standard formula

\[
J_\xi = \frac{\partial L}{\partial \dot{q}}, \tag{3.2a}
\]

or on \( T^*Q \) by

\[
J_\xi = p_\xi \xi^t_Q(q), \tag{3.2b}
\]

where \( \xi^t_Q \) is the infinitesimal generator of the action on \( Q \) (see (Abraham and Marsden, 1978)). To implement (3.2) we parametrize \( Q \) by \( \theta_1, \theta_2 \) and \( r \) with \( r_1 \) and \( r_2 \) determined by (2.12) and (2.13). From (2.15) we see that the momentum conjugate to \( r \) is

\[
p = \frac{\partial L}{\partial \dot{r}} = m\dot{r}
\]

(3.3)

and so (3.2) gives

\[
J_\xi = (p, \xi), \quad \xi \in \mathbb{R}^2. \tag{3.4}
\]

Thus \( J = p \) is conserved since \( H \) is cyclic in \( r \) and so \( H \) is translation invariant. The corresponding reduced space is obtained by fixing \( p = p_0 \) and letting

\[ P_{p_0} = J^{-1}(p_0)/\mathbb{R}^2 \]

(see (Abraham and Marsden, 1978, Chapter 4)). But \( P_{p_0} \) is clearly isomorphic to \( T^*(S^1 \times S^1) \), that is, to the space of \( \theta_1, \theta_2 \) and their conjugate momenta. The reduced Hamiltonian is simply the Hamiltonian corresponding to (2.21) with \( p \) regarded as a constant.

Note that in this case the reduced symplectic manifold is a cotangent bundle, in agreement with the cotangent-bundle reduction theorem (Abraham and Marsden, 1978; Kummer, 1981). The reduced phase space has the canonical symplectic form; one can also check this directly here.

In (2.21) we can adjust \( L \) by a constant and thus assume that \( p = 0 \); this obviously does not affect the equations of motion.

Let us observe that the reduced system is given by geodesic flow on \( S^1 \times S^1 \) since (2.21) is quadratic in the velocities. Indeed the metric tensor is just the matrix \( J \) given by (1.6), so the conjugate momenta are \( \mu_1, \mu_2 \) given by (1.6).

We remark, finally, that the reduction to centre-of-mass coordinates here is somewhat simpler and more symmetric than the Jacobi–Haretu reduction to centre-of-mass coordinates for \( n \) point masses. (Just taking the positions relative to the centre of mass does not achieve this since this does not reduce the dimension at all!) What is different here is that the two bodies are hinged, and so by (2.12), \( r_1^0 \) and \( r_2^0 \) are determined by the other data.

4. Reduction by rotations

To complete the reduction, we reduce by the diagonal action of \( S^1 \) on the configuration space \( S^1 \times S^1 \) that was obtained in §3. The momentum map for this action is obviously given by

\[
J((\theta_1, \mu_1), (\theta_2, \mu_2)) = \mu_1 + \mu_2. \tag{4.1}
\]
For purposes of later stability calculations, we shall find it convenient to form the Poisson reduced space

\[ P := T^*(S^1 \times S^1)/S^1 \]  

(4.2)

whose symplectic leaves are the reduced symplectic manifolds

\[ P_\mu = J^{-1}(\mu)/S^1 \subset P. \]

We coordinatize \( P \) by \( \theta = \theta_2 - \theta_1, \mu_1 \) and \( \mu_2 \); topologically, \( P = S^1 \times \mathbb{R}^2 \). The Poisson structure on \( P \) is computed in the standard way: take two functions \( F(\theta, \mu_1, \mu_2) \) and \( H(\theta, \mu_1, \mu_2) \). Regard them as functions of \( \theta_1, \theta_2, \mu_1, \mu_2 \) by substituting \( \theta = \theta_2 - \theta_1 \) and compute the canonical bracket. It is clear that the asserted bracket (1.8) is what results. The Casimirs on \( P \) are obtained by composing \( J \) with Casimirs on the dual of the Lie algebra of \( S^1 \); that is, with arbitrary functions of one variable; thus (1.9) results. This can of course be checked directly.

If we parametrize \( P_\mu \) by \( \theta \) and \( \nu = \frac{1}{2}(\mu_2 - \mu_1) \), then the Poisson bracket on \( P_\mu \) becomes the canonical one. This, again, is consistent with the cotangent-bundle reduction theorem which asserts in this case that the reduction of \( T^*(S^1 \times S^1) \) by \( S^1 \) is symplectically diffeomorphic to \( T^*(S^1 \times S^1)/S^1 \equiv T^*S^1 \). There are no 'magnetic' terms since the reduced configuration space \( S^1 \) is one-dimensional, and hence has no non-zero two-forms.

The realization of \( P_\mu \) as \( T^*S^1 \) is not unique. For example we can parametrize \( P_\mu \) by \( (\theta_2, \mu_2) \) or by \( (\theta_1, \mu_1) \), each of which also gives the canonical bracket. (In the general theory there can be more than one one-form \( \alpha_\mu \) by which one embeds \( P_\mu \) into \( T^*S^1 \), as well as more than one way to identify \( (S^1 \times S^1)/S^1 \equiv S^1 \). The three listed above correspond to three such choices of \( \alpha_\mu \).)

Remark The reduced bracket on \( T^*(S^1 \times S^1)/S^1 \) can also be obtained from the general formula for the bracket on \( (P \times T^*G)/G \equiv P \times g^* \) found in (Krishnaprasad and Marsden, 1987); it produces one of the variants above, depending on whether we take \( G \) to be parametrized by \( \theta_1 \) or \( \theta_2 \), or \( \theta_2 - \theta_1 \).

The reduced Hamiltonian on \( P \) is just (1.7b) regarded as a function of \( \mu_1, \mu_2 \) and \( \theta \). We therefore know that the Euler–Lagrange equations (1.5) are equivalent to \( \dot{F} = (F, H) \) for the reduced bracket (1.8).

We can also obtain a Hamiltonian system on the leaves, parametrized by say \( (\theta, \nu) \). We simply take (1.7b), namely

\[ H = \frac{1}{2\Delta}(\mu_1, \mu_2)\left( \begin{array}{cc} \bar{I}_1 & -\nu \lambda \\ -\nu \lambda & I_1 \end{array} \right)(\mu_2), \]

(4.3)

where \( \Delta = \bar{I}_1 \bar{I}_2 - \nu^2 \lambda^2 \), and substitute \( \mu_1 = \frac{1}{2} \nu - \nu, \mu_2 = \nu + \frac{1}{2} \mu \) producing

\[ H = \frac{1}{2\Delta} \left( \bar{I}_1 + \bar{I}_2 + 2\nu \lambda \right)\nu^2 + \frac{1}{2\Delta} \left[ (\bar{I}_1 - \bar{I}_2)\nu + \frac{1}{2\Delta}(\mu^2(\bar{I}_1 + \bar{I}_2 - 2\nu \lambda)) \right]. \]

(4.4)

The presence of the linear term in \( \nu \) can be eliminated by completion of squares: it is not there in the general theory (Abraham and Marsden, 1978; Smale, 1970) because reduced coordinates adapted to the metric of the kinetic energy are used; these are produced by the completion of squares. Notice that the Hamiltonian now is the form of kinetic plus potential energy but that the metric now on \( S^1 \) is \( \theta \)-dependent and, unless \( d_1 \) or \( d_2 \) is zero, potential piece is usually referred to.

We summarize as follows.

**Theorem 1** The reduced phase space \( \tilde{P} \) is the three-dimensional Poisson manifold whose symplectic leaves are the cylinders with equation \( \theta = \theta_2 - \theta_1 \) and \( \mu_2 = \mu_1 \).

The reduced dynamics are given by

\[ \dot{\theta} = \frac{\partial H}{\partial \mu_2} - \frac{\partial H}{\partial \mu_1}, \]

where \( H \) is given by (1.7b). The equations

\[ \frac{\partial \theta}{\partial \nu} = \frac{\partial H}{\partial \nu}, \]

where \( H \) is given by (4.4).

**5. Equilibria and stability by variational methods**

We now use Arnold's energy-Casimir method (1985; Krishnaprasad and Marsden, 1987) for the study of their stability. An equivalent alternate point of \( H \) given by (4.4) in \( (\theta, \nu) \) is the equilibrium.

To search for equilibria we look for a point \( (\theta, \nu) \) at which the bracket (1.8) and \( \dot{F} = (F, H) \), vanishes (1.7b). The conditions \( \dot{\mu}_1 = \dot{\mu}_2 = 0 \) give

\[ -\frac{1}{2}(\mu_1, \mu_2) \]

that is,

\[ -\frac{1}{2}(\mu_1, \mu_2). \]

Clearly

\[ \frac{d\mu_1}{d\theta} = \dot{\mu}_1 = 0, \]

from (1.6), so (5.2) becomes

\[ -\frac{1}{2}(\omega_1, \omega_2), \]

that is,

\[ -\frac{1}{2}(\omega_1, \omega_2). \]

The equilibrium condition \( \dot{\theta} = 0 \) implies \( \omega_1 = \omega_2 \).
We shall find it convenient to form the

\[ S^1 / S^1 \]  \hspace{1cm} (4.2)

complex manifolds

\[ S^3 \subset P, \]

\[ S^1 \subset \mu_2, \]  \hspace{1cm} (4.3)

topologically, \( P = S^1 \times \mathbb{R} \). The

lie standard way: take two functions \( \alpha_1, \alpha_2, \mu_1, \mu_2 \) by

the Lie bracket algebra of \( S^1 \); that is, with \( (\alpha_1, \alpha_2) \). This can of course be

(1, 1), then the Poisson bracket on \( P \)

consistent with the cotangent-bundle

that the reduction of \( T^*(S^1 \times S^1) \) by

\[(S^1 \times S^1) / S^1 \equiv T^*S^1. \]  \hspace{1cm} (4.4)

There are no

manifolds \( S^1 \) is one-dimensional, and

one-form \( \alpha_\mu \) gives the canonical bracket. (In the

one-form \( \alpha_\mu \) embeds

identify \((S^1 \times S^1) / S^1 = S^1. \) The

choices of \( \alpha_\mu \))

\( S^1 \) can also be obtained from the

\( T^*G \times G = \mathbb{R} \times \mathbb{R} \) found in (Kris-

one of the variants above, depending

by \( \theta_1 \) or \( \theta_2 \), or \( \theta_2 - \theta_1. \)

(1.7b) regarded as a function of \( \mu_1, \mu_2, \)

holer–Lagrange equations (1.5) are

\[ T^*G \times G = \mathbb{R} \times \mathbb{R} \]  \hspace{1cm} (4.5)

on the leaves, parametrized by say

\[ \frac{-1}{2\Delta}(\dot{I}_1 + \dot{I}_2 - 2\varepsilon\lambda) \]  \hspace{1cm} (4.6)

eliminated by completion of squares:

\[ \varepsilon = \frac{1}{2} \Delta \]

metric of the kinetic energy are used;

squares. Notice that the Hamiltonian

energy but that the metric now on \( S^1 \)

\[ \theta \]  \hspace{1cm} (4.8)

is \( \theta \)-dependent and, unless \( d_1 \) or \( d_2 \) vanishes, it is a non-trivial dependence. The

potential piece is usually referred to as the amended potential.

We summarize as follows.

\textbf{Theorem 1} The reduced phase space for two coupled planar rigid bodies is the

three-dimensional Poisson manifold \( P = S^1 \times \mathbb{R} \) with the bracket (1.8); its symplectic

leaves are the cylinders with canonical variables \( (\theta, \nu) \). Casimirs are given by (1.9).

The reduced dynamics are given by \( \dot{\theta} = \{ F, H \} \) or, equivalently,

\[ \dot{\theta} = \frac{\partial H}{\partial \theta}, \quad \dot{\mu}_1 = \frac{\partial H}{\partial \mu_1}, \quad \dot{\mu}_2 = \frac{\partial H}{\partial \mu_2}, \]  \hspace{1cm} (4.5)

where \( H \) is given by (1.7b). The equivalent dynamics on the leaves is given by

\[ \frac{\partial \theta}{\partial t} = \frac{\partial H}{\partial \nu}, \quad \frac{\partial \nu}{\partial t} = -\frac{\partial H}{\partial \theta}, \]  \hspace{1cm} (4.6)

where \( H \) is given by (1.4).

\section{5. Equilibria and stability by the energy-Casimir method}

We now use Arnold's energy-Casimir method, as summarized in (Holm \textit{et al.}, 1985; Krishnaprasad and Marsden, 1987) to determine the equilibrium points and their stability. An equivalent alternative to this method is to look for critical points of \( H \) given by (4.4) in \( (\theta, \nu) \)-space and test \( d^2H \) for definiteness at these equilibria.

To search for equilibrium we look directly at Hamilton's equations on \( P \). Using

the bracket (1.8) and \( \dot{\theta} = \{ F, H \} \), we obtain equations (4.5), where \( H \) is given by

(1.7b). The conditions \( \dot{\mu}_1 = \dot{\mu}_2 = 0 \) become

\[ \frac{\partial H}{\partial \theta} = 0; \]  \hspace{1cm} (5.1a)

that is,

\[ -\frac{1}{2}\mu_1 \dot{\omega}_1 - \frac{1}{2}\mu_2 \dot{\omega}_2 \]  \hspace{1cm} (5.1b)

Clearly

\[ \frac{d \omega_1}{d \theta} = \begin{pmatrix} 0 & \varepsilon \lambda' \\ \varepsilon \lambda' & 0 \end{pmatrix} \]  \hspace{1cm} (5.2)

from (1.6), so (5.2) becomes

\[ \frac{1}{2}(\omega_1, \omega_2)(0, \varepsilon \lambda') = 0; \]  \hspace{1cm} (5.3)

that is,

\[ -\varepsilon \lambda' \omega_1 \omega_2 = 0. \]  \hspace{1cm} (5.4)

The equilibrium condition \( \dot{\theta} = 0 \) becomes \( \dot{\theta}_1 \mu_1 - \varepsilon \lambda \mu_2 = \dot{\theta}_2 \mu_2 - \varepsilon \lambda \mu_1 \) or, equivalently, \( \omega_1 = \omega_2. \)
Thus, the equilibria are given by
(i) \( \omega_1 = \omega_2 = 0 \), or
(ii) \( \omega_1 = \omega_2 \neq 0, \lambda' = 0 \).
Let us, for simplicity, choose our reference configuration so that \( \mathbf{d}_{12} \) and \( \mathbf{d}_{21} \) are parallel. Then
\[
\lambda'(\theta) = \mathbf{d}_{12} \cdot \mathbf{d}_{21} \sin \theta
\]
so the equilibria in case (ii) occur when
(ii)' either (a) \( \mathbf{d}_{12} = 0 \) or \( \mathbf{d}_{21} = 0 \), or (b) \( \theta = 0 \) or \( \pi \).
The case in which \( \theta = \pi \) corresponds to the case of folded bodies, while \( \theta = 0 \) corresponds to extended bodies.

The first step in the energy-Casimir method is to realize the equilibria as critical points of \( H + C \), where \( H \) is given by (1.7b) and \( C = \Phi(\mu_1, + \mu_2) \).
One calculates from (5.2) and (1.7) that
\[
\begin{align*}
\frac{\partial H}{\partial \theta} &= \epsilon \lambda' \omega_1 \omega_2, \\
\frac{\partial H}{\partial \mu_1} &= \omega_1, \\
\frac{\partial H}{\partial \mu_2} &= \omega_2,
\end{align*}
\]
(5.5)
where
\[
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix} = \mathbf{J}^{-1}
\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix}
\tilde{l}_1 \mu_1 - \epsilon \lambda \mu_2 \\
\tilde{l}_2 \mu_2 - \epsilon \lambda \mu_1
\end{pmatrix}.
\]
The first variation is
\[
d(H + C) = \frac{\partial H}{\partial \theta} d\theta + \left( \frac{\partial H}{\partial \mu_1} + \Phi' \right) d\mu_1 + \left( \frac{\partial H}{\partial \mu_2} + \Phi' \right) d\mu_2,
\]
(5.6)
from which it is clear that critical points of \( H + C \) correspond to equilibria of (4.5) with
\[
\Phi'(\mu_e) = - \left( \frac{\partial H}{\partial \mu_1} \right)_e = - \left( \frac{\partial H}{\partial \mu_2} \right)_e,
\]
(5.7)
where the subscript \( e \) means evaluation at the equilibrium. As in other examples (the rigid body and heavy top in (Holm et al., 1985)), \( \Phi'(\mu_e) \) is arbitrary.

The matrix of the second variation is
\[
\delta^2(H + C) = \begin{pmatrix}
\frac{\partial^2 H}{\partial \theta^2} & \frac{\partial^2 H}{\partial \theta \partial \mu_1} & \frac{\partial^2 H}{\partial \theta \partial \mu_2} \\
\frac{\partial^2 H}{\partial \theta \partial \mu_1} & \frac{\partial^2 H}{\partial \mu_1^2} + \Phi' & \frac{\partial^2 H}{\partial \mu_1 \partial \mu_2} + \Phi'' \\
\frac{\partial^2 H}{\partial \theta \partial \mu_2} & \frac{\partial^2 H}{\partial \mu_1 \partial \mu_2} + \Phi' & \frac{\partial^2 H}{\partial \mu_2^2} + \Phi''
\end{pmatrix},
\]
(5.8)
where
\[
\frac{\partial^2 H}{\partial \theta \partial \mu_1} = - \frac{\epsilon \lambda'}{\Delta} \left( \tilde{l}_2 \omega_2 - \epsilon \lambda \omega_1 \right),
\]
and
\[
\frac{\partial^2 H}{\partial \theta \partial \mu_2} = - \frac{\epsilon \lambda'}{\Delta} \left( \tilde{l}_1 \omega_1 - \epsilon \lambda \omega_2 \right).
\]
At equilibrium, \( \lambda = \pm d_{12} d_2 \) ( + if \( \theta = 0 \) or \( \pi \), and
\[
\mathbf{J}^{-1} = \frac{1}{(\tilde{l}_1 \tilde{l}_2 - \epsilon^2 \lambda^2)} \frac{\partial^2 H}{\partial \mu_1 \partial \mu_2}
\]
and
\[
\frac{\partial^2 H}{\partial \theta \partial \mu_1} = - \epsilon \lambda
\]
where \( \omega_e = \omega_1 = \omega_2 \neq 0 \) at equilibrium.

This matrix is clearly positive definite \( \Phi'(\mu_e) \geq 0 \) and is indefinite for any other \( \mu_e \).

Another way to do the stability analysis is to consider the kinetic plus potential energy with effect of the magnetic terms.

Minima of \( V \) are then the stable equilibria.

For three or more bodies, this matrix will not work because the symplectic structure is not conserved.
configuration so that $d_{12}$ and $d_{21}$ are

$$d_{21} \sin \theta = 0 \text{ or } \pi.$$

The case of folded bodies, while $\theta = 0$

is to realize the equilibria as critical

and $C = \Phi(\mu_1 + \mu_2)$.

$$\frac{\partial H}{\partial \mu_1} = \omega_1, \frac{\partial H}{\partial \mu_2} = \omega_2,$$

$$\frac{\partial H}{\partial \mu_1} = -\frac{\epsilon \lambda'}{\Delta^2} (\tilde{I}_2 \omega_2 - \epsilon \lambda \omega_1), \quad \frac{\partial H}{\partial \mu_2} = -\frac{\epsilon \lambda'}{\Delta^2} (-\epsilon \lambda \omega_2 + \tilde{I}_1 \omega_1),$$

and

$$\frac{\partial^2 H}{\partial \theta \partial \mu_1} = -\epsilon \lambda' \frac{\partial H}{\partial \mu_1}, \quad \frac{\partial^2 H}{\partial \theta \partial \mu_2} = -\epsilon \lambda' \frac{\partial H}{\partial \mu_2}.$$

At equilibrium, $\lambda = \pm d_1 d_2$ (if $\theta = 0$, $-\epsilon d_1 d_2$ if $\theta = \pi$) so

$$J^{-1} = \frac{1}{(\tilde{I}_1 \tilde{I}_2 - \epsilon^2 d_1 d_2)} \begin{pmatrix} \tilde{I}_2 & +\epsilon d_1 d_2 \\ 0 & \tilde{I}_1 \end{pmatrix},$$

and

$$\frac{\partial^2 H}{\partial \theta^2} = -\epsilon \lambda' \omega^2 = \pm \epsilon d_1 d_2 \omega^2_z,$$

where $\omega_z = \omega_1 = \omega_2 \neq 0$ at equilibrium. Thus (5.8) becomes

$$\delta^2(H + C) = \begin{pmatrix} \pm \epsilon \lambda d_1 d_2 \omega^2 & 0 \\ 0 & J^{-1} + \Phi'(1 \ 1) \end{pmatrix}. \tag{5.9}$$

This matrix is clearly positive definite if $d_1 \neq 0$, $d_2 \neq 0$ if $\theta = 0$ (+ sign) and $\Phi'(\mu_2) \geq 0$ and is indefinite for any choice of $\Phi'(\mu_2)$ if $\theta = \pi$.

Another way to do the stability analysis is to use the reduced Hamiltonian on $T^*S^1$ given by equation (4.4). After completing squares, $H$ will have the form of kinetic plus potential energy with effective potential given by

$$V(\theta) = \frac{1}{2\Delta} \left[ \frac{1}{2} \mu^2(\tilde{I}_1 + \tilde{I}_2 - 2\epsilon \lambda) + \frac{(\tilde{I}_1 - \tilde{I}_2)^2 \mu^2}{4(\tilde{I}_1 + \tilde{I}_2 + 2\epsilon \lambda)} \right]. \tag{5.10}$$

Minima of $V$ are then the stable equilibria while maxima are unstable.

For three or more bodies, this method of looking for minima of the potential will not work because the symplectic structures on the symplectic leaves will have magnetic terms.

**Theorem 2** The dynamics of the 2-body problem is completely integrable and contains one stable relative equilibrium solution ($\theta = 0$—the stretched-out case) and one unstable relative equilibrium solution ($\theta = \pi$—the folded-over case). The dynamics contain a homoclinic orbit, as in Fig. 1.
6. Multibody problems

We have proved that the Hamiltonian formulation of the previous sections extends in a natural way to systems of \( N \) planar rigid bodies connected to form a tree structure (Fig. 5). Since the general statement of this result requires significant additional notation and the explicit introduction of the notion of nested bodies, we limit ourselves to the special case of a chain of \( N \) bodies (Fig. 6).

**Theorem 3** The total kinetic energy (Hamiltonian) for an open chain of \( N \) planar rigid bodies connected together by hinge joints takes the form

\[
H = \mathbf{\mu}^T \cdot \mathbf{J}^{-1} \cdot \mathbf{\mu}
\]

where \( \mathbf{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)^T \) is the momentum vector and \( \mathbf{J} \) is the corresponding \( N \times N \) pseudo-inertia matrix which is a function of the set of relative (or joint) angles between adjacent bodies. The Hamiltonian is given by

\[
\dot{\mathbf{\mu}}_i = \frac{\partial H}{\partial \theta_{i,i+1}},
\]

where \( \theta_{i,i+1} \) is the joint angle between bodies \( i \) and \( i+1 \).

The associated Poisson structure is

\[
\{f, g\} = \sum_{i=1}^{N-1} \left( \frac{\partial f}{\partial \mu_i} \cdot \frac{\partial g}{\partial \mu_{i+1}} - \frac{\partial f}{\partial \mu_{i+1}} \cdot \frac{\partial g}{\partial \mu_i} \right)
\]

This is proved in a way similar to that of Krishnaprasad and Marsden, 1987)

The structure of equilibria and the mix of complex and interesting as the number of bodies increases. A mixture of topological and geometric properties emerges in the phase portraits.

In the remainder of this section, we consider multibody problems by giving an example of a system of three planar rigid bodies.

**Fig. 5.** Planar multi-body system—tree case

**Fig. 6.** Planar multi-body system—chain case
angles between adjacent bodies. The reduced dynamics takes the form

$$
\begin{align*}
\dot{\mu}_1 &= \frac{\partial H}{\partial \theta_{2,1}}, \\
\dot{\mu}_2 &= \frac{\partial H}{\partial \theta_{3,2}} - \frac{\partial H}{\partial \theta_{2,1}}, \\
&\vdots \\
\dot{\mu}_i &= \frac{\partial H}{\partial \theta_{i+1,i}} - \frac{\partial H}{\partial \theta_{i,i-1}}, \\
&\vdots \\
\dot{\mu}_N &= -\frac{\partial H}{\partial \theta_{N,N-1}},
\end{align*}
$$

(6.2)

where $\theta_{i+1,i}$ is the joint angle between body $i+1$ and body $i$.

The associated Poisson structure is given by

$$
(f, g) = \sum_{i=1}^{N-1} \left( \frac{\partial f}{\partial \mu_i} - \frac{\partial f}{\partial \mu_{i+1}} \right) \frac{\partial g}{\partial \theta_{i+1,i}} - \frac{\partial f}{\partial \mu_{i+1}} \left( \frac{\partial g}{\partial \mu_i} - \frac{\partial g}{\partial \mu_{i+1}} \right).
$$

(6.3)

This is proved in a way similar to the two-body case (see Sreenath, Krishnaprasad and Marsden, 1987)).

The structure of equilibria and the associated stability analysis become quite complex and interesting as the number of interconnected bodies increases. A mixture of topological and geometric methods may be necessary to extract useful information on the phase portraits.

In the remainder of this section, we illustrate some of the complexities of multibody problems by giving an analysis of the equilibria and stability for a system of three planar rigid bodies connected by hinge joints (see Fig. 7).
6.1 Three-body problem

The Hamiltonian of the planar three-body problem is given by equation (6.1) with the momentum vector \( \mathbf{\mu} \) and the coefficient of inertia matrix \( \mathbf{J} \) being defined as below:

\[
\mathbf{\mu} = (\mu_1, \mu_2, \mu_3)^T,
\]

\[
\mathbf{J} = \begin{pmatrix}
I_1 & \tilde{\lambda}_{12}(\theta_{2,1}) & \tilde{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) \\
\tilde{\lambda}_{12}(\theta_{2,1}) & I_2 & \tilde{\lambda}_{23}(\theta_{3,2}) \\
\tilde{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) & \tilde{\lambda}_{23}(\theta_{3,2}) & I_3
\end{pmatrix},
\]

(6.4)

where the \( \tilde{I} \) and \( \tilde{\lambda} \) are defined later. Here \( \theta_{2,1} \) and \( \theta_{3,2} \) are the relative angles between bodies 2 and 1, and bodies 3 and 2, respectively.

The dynamics of a three-body system of planar, rigid bodies in the Hamiltonian setting is given by:

\[
\begin{align*}
\dot{\mu}_1 &= \frac{\partial H}{\partial \theta_{2,1}}, \\
\dot{\mu}_2 &= -\frac{\partial H}{\partial \theta_{2,1}} + \frac{\partial H}{\partial \theta_{3,2}}, \\
\dot{\mu}_3 &= -\frac{\partial H}{\partial \theta_{3,2}}, \\
\dot{\theta}_{2,1} &= \frac{\partial H}{\partial \mu_1} - \frac{\partial H}{\partial \mu_2}, \\
\dot{\theta}_{3,2} &= \frac{\partial H}{\partial \mu_3} - \frac{\partial H}{\partial \mu_2}.
\end{align*}
\]

(6.5)

Remark The sum \( (\mu_1 + \mu_2 + \mu_3) \) of momentum variables is a constant.

Remark The coefficients of inertia \( \tilde{I}_i \) and \( \tilde{\lambda}_{ij} \) are given by

\[
\begin{align*}
\tilde{I}_1 &= [I_1 + (e_{12} + e_{31})(d_{12}, d_{12})], \\
\tilde{I}_2 &= [I_2 + e_{12}(d_{21}, d_{21}) + e_{23}(d_{23}, d_{23}) + e_{31}(d_{23} - d_{21}, d_{23} - d_{21})] \\
\tilde{I}_3 &= [I_3 + (e_{23} + e_{31})(d_{32}, d_{32})], \\
\tilde{\lambda}_{12}(\theta_{2,1}) &= [e_{12}\lambda(-d_{21}, d_{21})(\theta_{2,1}) + e_{31}\lambda(-d_{31}, d_{31})(\theta_{2,1})], \\
\tilde{\lambda}_{23}(\theta_{3,2}) &= [e_{23}\lambda(-d_{23}, d_{23})(\theta_{3,2}) + e_{31}\lambda(-d_{32}, d_{32})(\theta_{3,2})], \\
\tilde{\lambda}_{31}(\theta_{2,1} + \theta_{3,2}) &= e_{31}\lambda(-d_{31}, d_{31})(\theta_{2,1} + \theta_{3,2}), \\
\end{align*}
\]

where \( e_i = \frac{m_i m_j}{m_i + m_j + m_3} \), \( i \neq j \) and \( i, j = 1, 2, 3 \),

\[
\lambda(x, y)(\alpha) = x \cdot y \cos(\alpha) + [x \times y] \sin(\alpha),
\]

where the \( m_i \) and \( I_i \) are the mass and inertia respectively of the body \( i \), and the \( d_{ij} \) are defined as in Fig. 7.

6.2 Three-body problem: equilibria

Refer to Fig. 7. Let the centres of mass of body 1 and body 2 be the origins of the local frames of body 1 and body 2, and let \( O_{23} \) be the coordinate system for body 1 chosen to be parallel to (a) the line joining \( O_1 \) and \( O_{23} \). Similarly, the coordinate system for body 2 is chosen to be parallel to (a) the line joining \( O_2 \) and \( O_{23} \). Define the local coordinate systems to be

\[
\mathbf{d}_{12} = [c_{11}, 0], \quad \mathbf{d}_{21} = [-b_{11}, 0].
\]

The equilibria for the three-body problem in equations (6.5) to be zero. This results in

\[
\frac{\partial H}{\partial \theta_{2,1}} = \frac{\partial H}{\partial \theta_{3,2}} = 0, \quad \dot{\theta}_{2,1} = \dot{\theta}_{3,2} = 0
\]

From the above equations it can be shown that

\[
\omega_1 = \omega_2 = \omega_3.
\]

The system angular momentum \( \mathbf{\mu} \), and

\[
\mu = \omega_0 \left[ \sum_{i=1}^{3} \tilde{I}_i + 2(\tilde{\lambda}_{12}(\theta_{2,1}) + \tilde{\lambda}_{23}(\theta_{3,2})) \right]
\]

or

\[
\omega_0 = \frac{1}{2} \omega_0 \left[ \sum_{i=1}^{3} \tilde{I}_i + 2(\tilde{\lambda}_{12}(\theta_{2,1}) + \tilde{\lambda}_{23}(\theta_{3,2})) \right]
\]

It is a consequence of Theorem 3 and

\[
\frac{\partial H}{\partial \theta_{2,1}} = \frac{1}{2} \frac{\partial}{\partial \theta_{2,1}} (\mathbf{\mu}, \mathbf{\mu}^T) = \frac{1}{2} \frac{\partial}{\partial \theta_{2,1}} [A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \sin(\theta_{3,2})]
\]

or, for the non-degenerate case \( \omega_0 \)

\[
A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \sin(\theta_{3,2}) = 0
\]

where

\[
A_1 = m_1 m_2 m_3, \\
B_1 = [m_3(b_{11} - b_{21})], \\
C_1 = m_1 m_2 m_3
\]

Similarly, for \( \partial H/\partial \theta_{3,2} \) we get

\[
\frac{\partial H}{\partial \theta_{3,2}} = \frac{\omega_0}{2(m_1 + m_2 + m_3)} [A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \sin(\theta_{3,2})]
\]
6.3 Equilibrium of the three-body system

Refer to Fig. 7. Let the centres of mass of the bodies $O_1$, $O_2$, and $O_3$ respectively be the origins of the local frames of references also. Let $O_{12}$ be the joint between body 1 and body 2, and let $O_{23}$ be the joint between body 2 and body 3. The local coordinate system for body 1 is chosen such that the $x$-axis is parallel to the line joining $O_1$ and $O_{12}$. Similarly, the coordinate systems for body 2 and body 3 are chosen to be parallel to (a) the line joining $O_2$ and $O_{12}$, and (b) the line joining $O_3$ and $O_{23}$, respectively. Define the vectors $\mathbf{d}_{12}$, $\mathbf{d}_{21}$, $\mathbf{d}_{23}$, $\mathbf{d}_{32}$, in their respective local coordinate systems to be

\[
\mathbf{d}_{12} = [c_1, 0], \quad \mathbf{d}_{21} = [-b_1, 0], \quad \mathbf{d}_{23} = [e_1, e_2], \quad \mathbf{d}_{32} = [-d_1, 0].
\]

The equilibria for the three-body system can be found by setting the dynamical equations in (6.5) to be zero. This results in the following equations:

\[
\frac{\partial H}{\partial \theta_{2,1}} = \frac{\partial H}{\partial \theta_{3,2}} = 0, \quad \dot{\theta}_{2,1} = \omega_2 - \omega_1 = 0, \quad \dot{\theta}_{3,2} = \omega_3 - \omega_2 = 0. \tag{6.6}
\]

From the above equations it can be seen that

\[
\omega_1 = \omega_2 = \omega_3 = \omega_0 \text{ (constant)}. \tag{6.7}
\]

The system angular momentum $\mu$ and the Hamiltonian $H$ are given by

\[
\mu_i = \omega_0 \left[ \sum_{i=1}^{3} \tilde{I} + 2(\tilde{\lambda}_{12}(\theta_{2,1}) + \tilde{\lambda}_{23} + \tilde{\lambda}_{31}(\theta_{2,1} + \theta_{3,2})) \right]. \tag{6.8}
\]

\[
H = \frac{1}{2} \omega_0 \left[ \sum_{i=1}^{3} \tilde{I} + 2(\tilde{\lambda}_{12}(\theta_{2,1}) + \tilde{\lambda}_{23}(\theta_{2,3}) + \tilde{\lambda}_{31}(\theta_{2,1} + \theta_{3,2})) \right] = \frac{1}{2} \omega_0 \mu, \tag{6.9}
\]

or

\[
\omega_0 = 2H/\mu. \tag{6.10}
\]

It is a consequence of Theorem 3 and (6.6) that,

\[
\left[ \frac{\partial H}{\partial \theta_{2,1}} \right]_{e} = \frac{1}{2} \frac{\partial}{\partial \theta_{2,1}} \left( \langle \mathbf{J}, \mathbf{J} \rangle \right)_{e} = \frac{1}{2} \left( \left[ \mathbf{J} \right]_{e} \frac{\partial \mathbf{J}}{\partial \theta_{2,1}} \mathbf{J} \right)_{e} = \frac{1}{2} \omega_0 \left[ \frac{\partial \mathbf{J}}{\partial \theta_{2,1}} \mathbf{J} \right]_{e} = 2(m_1 + m_2 + m_3) \left[ A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \cos(\theta_{2,1}) \right] = 0
\]

or, for the non-degenerate case ($\omega_0 \neq 0$),

\[
A_1 \sin(\theta_{2,1} + \theta_{3,2}) + B_1 \sin(\theta_{2,1}) + C_1 \cos(\theta_{2,1}) = 0, \tag{6.11}
\]

where

\[
A_1 = m_1 m_3 c_1 d_1, \tag{6.12}
B_1 = [m_3 (b_1 + e_1) + m_2 b_1] m_1 c_1, \tag{6.13}
C_1 = m_1 m_3 c_1 e_2. \tag{6.14}
\]

Similarly, for $\partial H/\partial \theta_{3,2}$ we get

\[
\frac{\partial H}{\partial \theta_{3,2}} = \frac{\omega_0^2}{2(m_1 + m_2 + m_3)} \left[ A_2 \sin(\theta_{2,1} + \theta_{3,2}) + B_2 \sin(\theta_{3,2}) + C_2 \cos(\theta_{3,2}) \right] = 0. \tag{6.15}
\]
where

\[ B_2 = [m_1(b_1 + e_1) + m_2 e_2] m_3 d_1, \]  
\[ C_2 = (m_1 + m_2) m_3 d_1 e_2. \]

We assemble the final equilibrium equations from equations (6.11) and (6.15):

\[
\begin{align*}
A_1 \sin (\theta_{2,1} + \theta_{3,2}) + B_1 \sin (\theta_{2,1}) + C_1 \cos (\theta_{2,1}) &= 0, \\
A_1 \sin (\theta_{2,1} + \theta_{3,2}) + B_2 \sin (\theta_{3,2}) + C_2 \cos (\theta_{3,2}) &= 0.
\end{align*}
\]

(6.18)

### 6.3 Three-body system: special kinematic case

We consider here a case of the three-body system with a special kinematic structure where the centres of mass of the bodies are aligned with the joints in a straight line when the bodies are in a stretched-out position. In this case, we shall prove that equations (6.18) have four or six solutions. For this situation \( e = [e_1, e_2]^T = [e_1, 0]^T \), and so from (6.14) and (6.17)

\[ e_2 = 0 \quad \text{implies that} \quad C_1 = C_2 = 0. \]

Thus (6.18) reduces to

\[
\begin{align*}
A_1 \sin (\theta_{2,1} + \theta_{3,2}) + B_1 \sin (\theta_{2,1}) &= 0, \\
A_1 \sin (\theta_{2,1} + \theta_{3,2}) + B_2 \sin (\theta_{3,2}) &= 0,
\end{align*}
\]

(6.19)  (6.20)

with

\[
\begin{align*}
A_1 &= c_1 d_1 m_1 m_3, \\
B_1 &= [(b_1 + e_1) m_3 + b_1 d_1] c_1 m_1, \\
B_2 &= [(b_1 + e_1) m_1 + e_1 d_1] m_3.
\end{align*}
\]

(6.21)  (6.22)  (6.23)

Subtracting (6.19) from (6.20) we get

\[ \sin (\theta_{3,2}) = \kappa \sin (\theta_{2,1}), \]

(6.24)

where

\[ \kappa = B_1 / B_2. \]

(6.25)

Expanding (6.19) and substituting (6.24), we get

\[ A_1 \sin (\theta_{2,1}) \cos (\theta_{3,2}) + \kappa \cos (\theta_{2,1}) + \tau = 0, \]

(6.26)

where

\[ \tau = B_1 / A_1. \]

(6.27)

Consequently from (6.24) and (6.26) we have

\[ \sin (\theta_{2,1}) = 0 \quad \text{and} \quad \sin (\theta_{3,2}) = 0 \]

(6.28)

or

\[
\begin{align*}
\sin (\theta_{3,2}) &= \kappa \sin (\theta_{2,1}), \\
\cos (\theta_{3,2}) + \kappa \cos (\theta_{2,1}) + \tau &= 0.
\end{align*}
\]

(6.29)  (6.30)

It is obvious from considering (6.28) that the following four roots of the \( \{\theta_{2,1}, \theta_{3,2}\} \) pair can be readily identified:

\[ \{0, 0\}, \quad \{0, \pi\}, \quad \{\pi, 0\}, \quad \{\pi, \pi\}. \]

(6.31)
\[
\begin{align*}
\frac{m_1 d_1}{d_2} & = m_2 e_1, \\
\frac{m_3 d_1}{d_2} & = m_2 e_2.
\end{align*}
\]

From equations (6.11) and (6.15):
\[
\begin{align*}
\frac{\kappa}{C_1} + C_1 \cos (\theta_{2,1}) & = 0, \\
\frac{\kappa}{C_2} + C_2 \cos (\theta_{3,2}) & = 0.
\end{align*}
\]  

Case

A body system with a special kinematic properties are aligned with the joints in a need-out position. In this case we shall have six solutions. For this situation and (6.17)
\[
C_1 = C_2 = 0.
\]

\[
\begin{align*}
\beta_1 \sin (\theta_{2,1}) & = 0, \\
\beta_2 \sin (\theta_{3,2}) & = 0,
\end{align*}
\]

\[
\begin{align*}
\sin (\theta_{2,1}) & = 0, \\
\sin (\theta_{3,2}) & = 0
\end{align*}
\]

We label these equilibria as the fundamental equilibria. A stick-figure representation (Fig. 8) helps in bringing out the symmetrical way in which these equilibria occur.

The remaining equilibria for this system are computed as the solutions to (6.29) and (6.30). Since the equilibrium equations are nonlinear and parameter dependent, one needs to exercise care while solving them. The parameter dependence of the equilibrium solutions can be summarized by two sets of constraints—parameter-sign and parameter-value constraints respectively. It was found that two extra equilibria (other than the fundamental equilibria) can exist at a time, subject to the existence of suitable values of \( \kappa \) and \( \tau \) satisfying these constraints. The maximum number of equilibria for a general three-body system (special kinematic case) is thus six. For some values of \( \kappa \) and \( \tau \) not satisfying these constraints and for the cases with \( \kappa \) and/or \( \tau \) being zero these extra equilibria merge with the fundamental equilibria to give a total of four equilibria.

6.3.1 Parameter-sign constraints

This constraint set restricts the existence of values of the pair \( \{ \theta_{2,1}, \theta_{3,2} \} \)

\[
\sin (\theta_{2,1} + \theta_{3,2}) = -\tau \sin (\theta_{2,1}).
\]

Taking into account the signs of \( \kappa \) and \( \tau \), from (6.29) and (6.32) we get Fig. 9, which illustrates the feasible regions of the solution pair \( \{ \theta_{2,1}, \theta_{3,2} \} \) to form the parameter-sign constraints.

6.3.2 Parameter-value constraints

The existence of solutions of (6.29) and (6.30) is also dependent on the actual values of \( \kappa \) and \( \tau \) (which are constants for a given three-body system). The parameter-value dependence of the solutions can be formulated by squaring and adding (6.29) and (6.30), and simplifying to get

\[
\begin{align*}
\cos (\theta_{2,1}) & = \frac{1 - \kappa^2 - \tau^2}{2\kappa \tau}, \\
\cos (\theta_{3,2}) & = \frac{\kappa^2 - \tau^2 - 1}{2\tau}.
\end{align*}
\]
so that

\[-1 < \frac{1 - k^2 - \tau^2}{2k\tau} < 1,\]  \hspace{1cm} (6.35)

\[-1 < \frac{k^2 - \tau^2 - 1}{2\tau} < 1.\]  \hspace{1cm} (6.36)

These equations could be represented in the form of a graph as in Fig. 10. The graph has been drawn for $k' > 0$ and $\tau' > 0$, where

\[k' = |k|, \quad \tau' = |\tau|.

6.3.3 Local frames of reference

It is necessary to choose a local frame of reference to parametrize the system and study the choice of the local frames of reference. Let $c = [c_1, 0]^T$ and $d = [d_1, 0]^T$, where the local frame of reference of body $e = [e_1, e_2]^T = [e_1, 0]^T$, where $e_1$ is positive. Parameter $b_1$ could be either negative or positive, the cases when the centre of mass of the hinges $O_{12}$ and $O_{23}$, and (b) outer link $b_1$ or $d_1$ is equal to zero then the three-body problem and a one-body problem.

6.3.4 Parameter-dependent equilibrium

We now delve into particular cases to establish the solutions to the equilibrium equations (6.27) while formulating the necessary conditions.

In all the cases we consider, we have the realizable values of the kinematic parameters. The equilibria are given by $k = \cos(\theta_{2,3})$ (see (6.33) and (6.34)) as the parameter-dependent equilibrium solutions. The results are the same as in each case. The graphs under the conditions read with $\theta_{2,1}$ as the $X$-axis and $\theta_{3,2}$ as the $Y$-axis.
COUPLED PLANAR RIGID BODIES I

6.3.3 Local frames of reference

It is necessary to choose a local frame of reference for each of the bodies in order to parametrize the system and study the system equilibria; refer to Fig. 11. Proper choice of the local frames of reference for bodies 1 and 3 results in the vectors \( c = [c_1, 0]^T \) and \( d = [d_1, 0]^T \), where both \( c_1 \) and \( d_1 \) are positive. In general, the local frame of reference of body 2 could be chosen in such a way that \( e = [e_1, e_2]^T = [e_1, 0]^T \), where \( e_1 \) is positive. Note that if \( b = [b_1, 0]^T \), the kinematic parameter \( b_1 \) could be either negative or positive. The two signs of \( b_1 \) represent cases when the centre of mass of body 2 is (a) inside the line segment joining the hinges \( O_{12} \) and \( O_{23} \), and (b) outside it. If any of the kinematic parameters \( c_1 \) or \( d_1 \) is equal to zero then the three-body problem decomposes into a two-body problem and a one-body problem. It is also important to observe that with this choice of local frames of reference, \( A_1 \) is positive (see (6.21)).

6.3.4 Parameter-dependent equilibria

We now delve into particular cases of the signs of parameters \( \kappa \) and \( \tau \) and establish the solutions to the equilibrium equations. We constantly refer to (6.21) to (6.27) while formulating the necessary conditions.

In all the cases we consider, we first ascertain that there exist physically realizable values of the kinematic parameters \( c_1, b_1, e_1 \) and \( d_1 \) before finding the actual solutions. The equilibria are evaluated based on the signs of \( \cos (\theta_{2,1}) \) and \( \cos (\theta_{3,2}) \) (see (6.33) and (6.34)), and according to the parameter-sign and parameter-value constraints. The results are presented in the form of a table for each case. The graphs under the column parameter-sign constraints have to be read with \( \theta_{2,1} \) as the X-axis and \( \theta_{3,2} \) as the Y-axis. The shaded regions represent
the valid regions of existence of the \( \{ \theta_{2,1}, \theta_{3,2} \} \) pair. In the column of the parameter-value constraints, the regions referred to are the regions of Fig. 10.

Given values of \( \kappa \) and \( \tau \), one can identify the corresponding table depending on the signs of these parameters, and determine which region they belong to with regard to Fig. 10. The two extra equilibria, if any, could then be read off from the table.

**Case 1, in which \( \kappa > 0, \tau > 0 \).** For \( \kappa \) and \( \tau \) to be greater than zero, \( A_1, B_1 \) and \( B_2 \) should be greater than zero. By choice of the local frames of reference we have from (6.22), (6.23) that

\[
(b_1 + e_1)m_3 + b_1m_2 > 0 \quad \text{so} \quad e_1 > - \left(1 + \frac{m_2}{m_3}\right)b_1,
\]

\[
(b_1 + e_1)m_1 + e_1m_2 > 0 \quad \text{so} \quad e_1 > - \left(\frac{m_1}{m_1 + m_2}\right)b_1,
\]

that is,

\[
e_1 > - \left(1 + \frac{m_2}{m_3}\right)b_1. \tag{6.37}
\]

This is automatically satisfied if \( b_1 > 0 \).

The equilibrium solutions are given in a compact form in Table 1.

**Case 2, in which \( \kappa < 0, \tau < 0 \).** This case can be realized if and only if \( B_1 < 0 \) and \( B_2 > 0 \) (since \( A_1 > 0 \) always). Simplifying, from (6.22) and (6.23) we have

\[
- \left(1 + \frac{m_2}{m_3}\right)b_1 > e_1 > - \left(\frac{m_1}{m_1 + m_2}\right)b_1. \tag{6.38}
\]

### Table 1. \( \kappa > 0, \tau > 0 \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( \cos(\theta_{12}) )</th>
<th>( \cos(\theta_{23}) )</th>
<th>Parameter-sign constraints</th>
<th>Parameter-value constraints</th>
<th>Equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td></td>
<td>not satisfied</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td></td>
<td>region 2</td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>&lt;0</td>
<td>&gt;0</td>
<td></td>
<td>region 1</td>
<td></td>
</tr>
<tr>
<td>1.4</td>
<td>&gt;0</td>
<td>&lt;0</td>
<td></td>
<td>region 3</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2. \( \kappa < 0, \tau < 0 \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( \cos(\theta_{12}) )</th>
<th>( \cos(\theta_{23}) )</th>
<th>Parameter-value constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td>&lt;0</td>
<td>&gt;0</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>&gt;0</td>
<td>&lt;0</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3. \( \kappa > 0, \tau < 0 \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( \cos(\theta_{12}) )</th>
<th>( \cos(\theta_{23}) )</th>
<th>Parameter-value constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>&lt;0</td>
<td>&lt;0</td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td>&lt;0</td>
<td>&gt;0</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>&gt;0</td>
<td>&lt;0</td>
<td></td>
</tr>
</tbody>
</table>
Table 2. $\kappa < 0$, $\tau < 0$

<table>
<thead>
<tr>
<th>Case</th>
<th>$\cos(\theta_{12})$</th>
<th>$\cos(\theta_{23})$</th>
<th>Parameter-sign constraints</th>
<th>Parameter-value constraints</th>
<th>Equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td></td>
<td>region 3</td>
<td></td>
</tr>
<tr>
<td>2.2</td>
<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td></td>
<td>region 1</td>
<td></td>
</tr>
<tr>
<td>2.3</td>
<td>$&lt;0$</td>
<td>$&gt;0$</td>
<td></td>
<td>region 2</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td></td>
<td>not satisfied</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. $\kappa > 0$, $\tau < 0$

<table>
<thead>
<tr>
<th>Case</th>
<th>$\cos(\theta_{12})$</th>
<th>$\cos(\theta_{23})$</th>
<th>Parameter-sign constraints</th>
<th>Parameter-value constraints</th>
<th>Equilibria</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td></td>
<td>region 2</td>
<td></td>
</tr>
<tr>
<td>3.2</td>
<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td></td>
<td>not satisfied</td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td>$&lt;0$</td>
<td>$&gt;0$</td>
<td></td>
<td>region 3</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td></td>
<td>region 1</td>
<td></td>
</tr>
</tbody>
</table>
Naturally, equation (6.38) indicates that this case is possible only if $b_1$ is negative (since $e_1 > 0$).

Table 2 gives the equilibria associated with this case if (6.38) is satisfied.

Case 3, in which $\kappa > 0$, $\tau < 0$. For this case since $A_1 > 0$ we have to have $B_1$, $B_2 < 0$, that is,

$$e_1 < - \left( 1 + \frac{m_2}{m_1} \right) b_1 < - \left( \frac{m_1}{m_1 + m_2} \right) b_1.$$

With the choice of local frames of reference, $e_1 > 0$ and so this case is possible only if $b_1$ is negative and

$$e_1 < - \left( \frac{m_1}{m_1 + m_2} \right) b_1. \tag{6.39}$$

The equilibria are as given in Table 3.

Case 4, in which $\kappa < 0$, $\tau > 0$. The necessary condition for this case is

$$-b_1 \left( 1 + \frac{m_2}{m_3} \right) < e_1 < -b_1 \left( \frac{m_1}{m_1 + m_2} \right). \tag{6.40}$$

But $e_1 > 0$, and so $b_1$ has to be negative. Then from (6.40) $e_1/|b_1|$ is greater than 1 but less than a fraction—which is impossible.

So kinematic parameters satisfying $\kappa < 0$ and $\tau > 0$ can never exist.

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