Planar Steering Laws and UAV Formations

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Outline

- Applications for meter-scale UAVs
- Planar model based on unit-speed motion with steering control
- Nonlinear analysis of a control law for two vehicles
    - Connection to gyroscopic systems
    - Contrast with synthetic potential methods
- Generalizations to *n* vehicles
- Implementation issues
- Future research directions

Applications

• Coordinating the flight of swarms of small, inexpensive (expendable) UAVs.
- Surveillance.
- Sensing (chemical, biological, etc.).
- Payload delivery (e.g., distributing a ground-based sensor network).

• Possible implications for UGV or UUV swarms, or biological swarming/schooling systems.

Dragon Eye

Photo credit: Jonathan Finer, The Washington Post

Why Formations of Meter-Scale UAVs?

• Advantages of UAVs over ground vehicles:

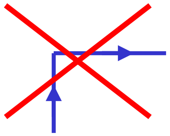
- Fewer obstacles in the air than on the ground (particularly in an adversarial environment).
- Destinations can be reached faster in the air, due to higher speeds and direct paths of travel.
- Better communication channels in the air than on the ground.
- Aerial sensing is often preferred to ground-based sensing.

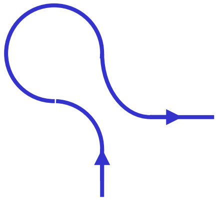
• Meter-scale UAVs are preferred because smaller vehicles are too strongly affected by wind and are too constrained in payload.

• Advantages of UAV formations:

- Redundancy (particularly in adversarial environments).
- Faster or more thorough completion of sensing/surveillance missions.
- Multi-vehicle missions that cannot be accomplished using a single vehicle (e.g., sensing a target simultaneously from multiple directions).

UAV Modeling





• UAV features that our model should capture:

- High speed ⇔ sluggish maneuvering.
- Turning ⇒ significant energy penalty (due to side slip).
- An autopilot takes into account the detailed vehicle kinematics.

• Vehicles modeled as point particles moving at **unit speed** and subject to **steering control**.

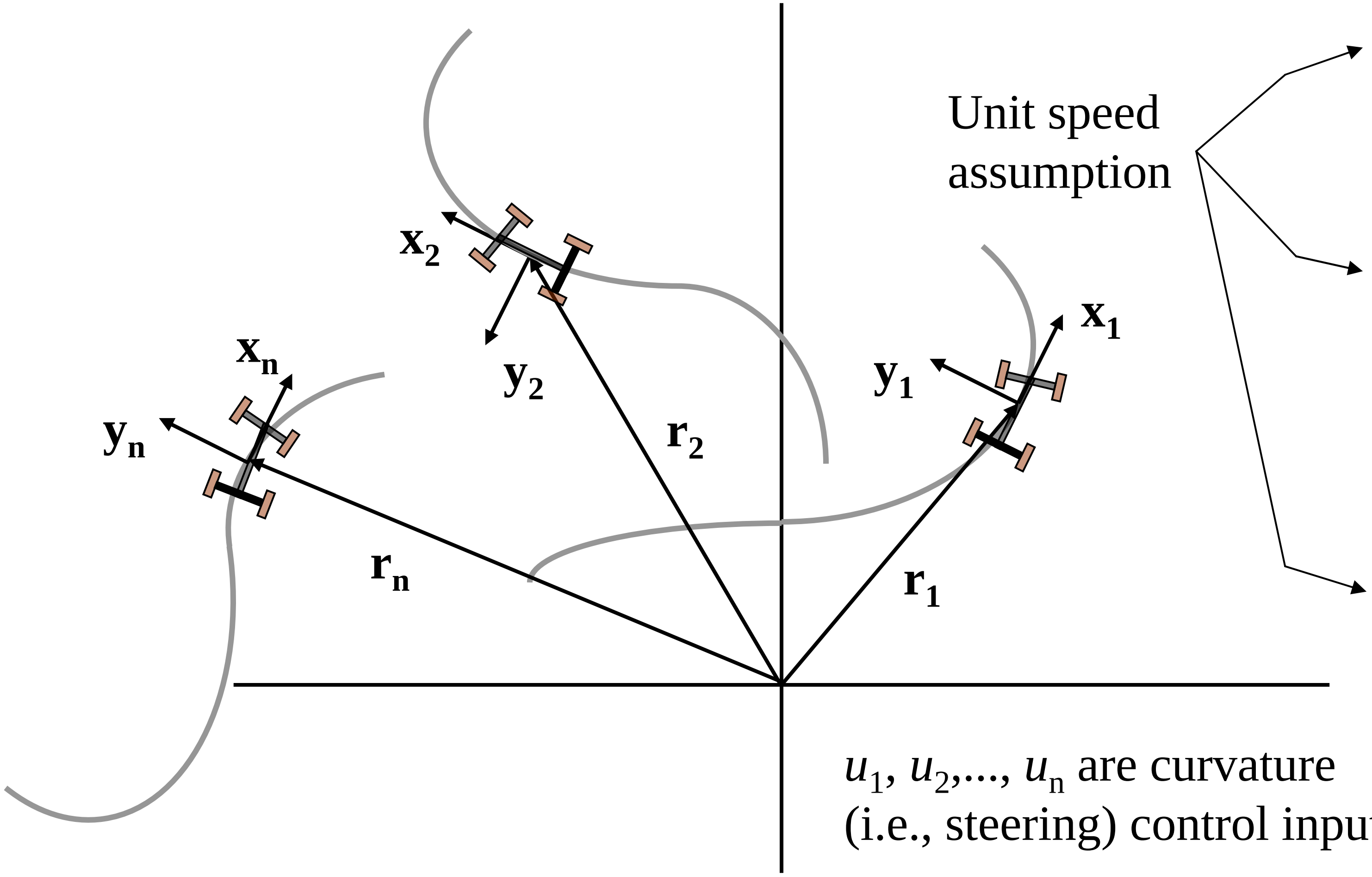
• A **formation control law** is a feedback law which specifies these steering controls.

• This modeling may be appropriate in other settings in which there are high speeds and penalties associated with turning (e.g. loss of dynamic stability).

Dragon Runner

(Photo from U.S. Marine Corps website)

Planar Model (Frenet-Serret Equations)



$$\dot{\mathbf{r}}_1 = \mathbf{x}_1$$
$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1$$
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1$$

$$\dot{\mathbf{r}}_2 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$
$$\dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

$$\vdots$$

$$\dot{\mathbf{r}}_n = \mathbf{x}_n$$
$$\dot{\mathbf{x}}_n = \mathbf{y}_n u_n$$
$$\dot{\mathbf{y}}_n = -\mathbf{x}_n u_n$$

u_1, u_2,..., u_n are curvature (i.e., steering) control inputs.

Specifying u_1, u_2,..., u_n as feedback functions of ($\mathbf{r}_1$, $\mathbf{x}_1$, $\mathbf{y}_1$), ($\mathbf{r}_2$, $\mathbf{x}_2$, $\mathbf{y}_2$),..., ($\mathbf{r}_n$, $\mathbf{x}_n$, $\mathbf{y}_n$) defines a **control law**.

Planar Control Law for Two Vehicles

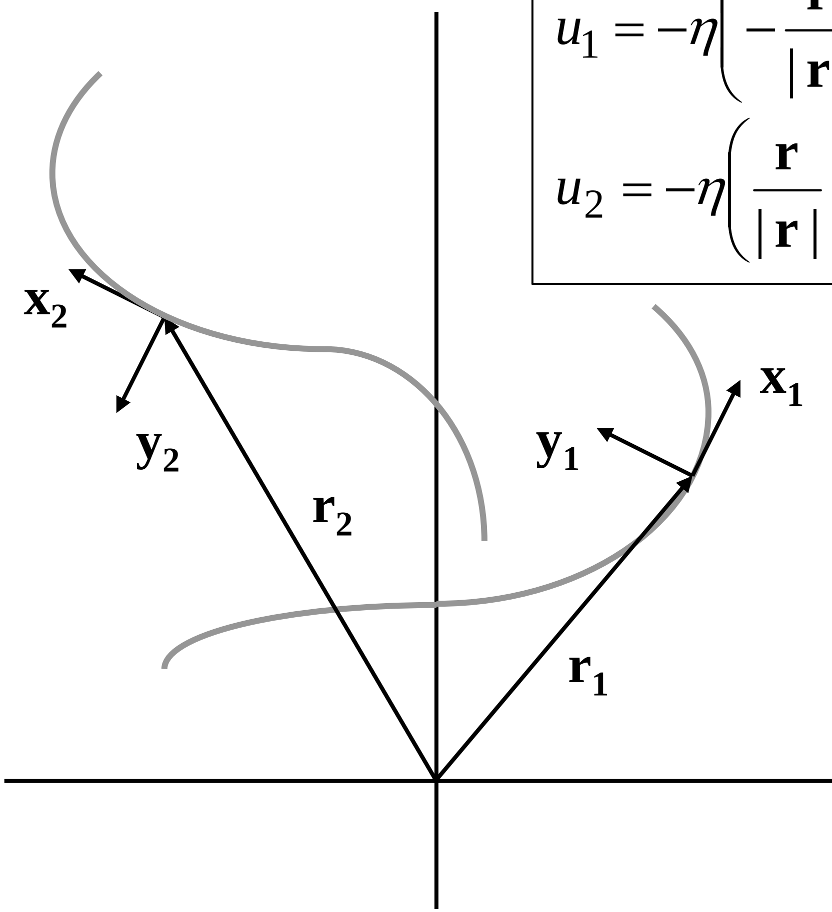
$$\dot{\mathbf{r}}_1 = \mathbf{x}_1 \qquad \dot{\mathbf{r}}_2 = \mathbf{x}_2 \qquad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1 \qquad \dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$

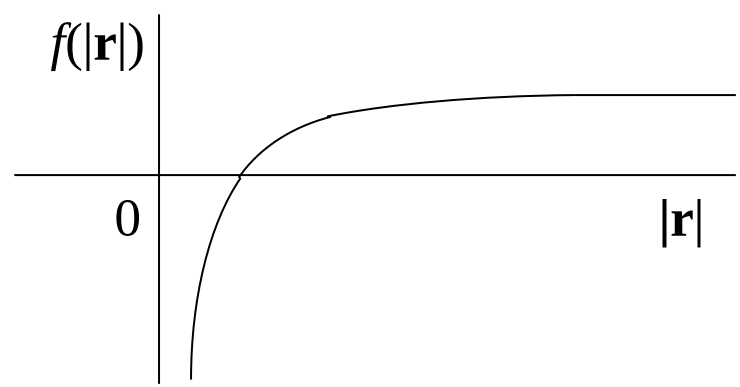
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1 \qquad \dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

$$u_1 = -\eta\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1\right)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) + \mu\mathbf{x}_2\cdot\mathbf{y}_1$$

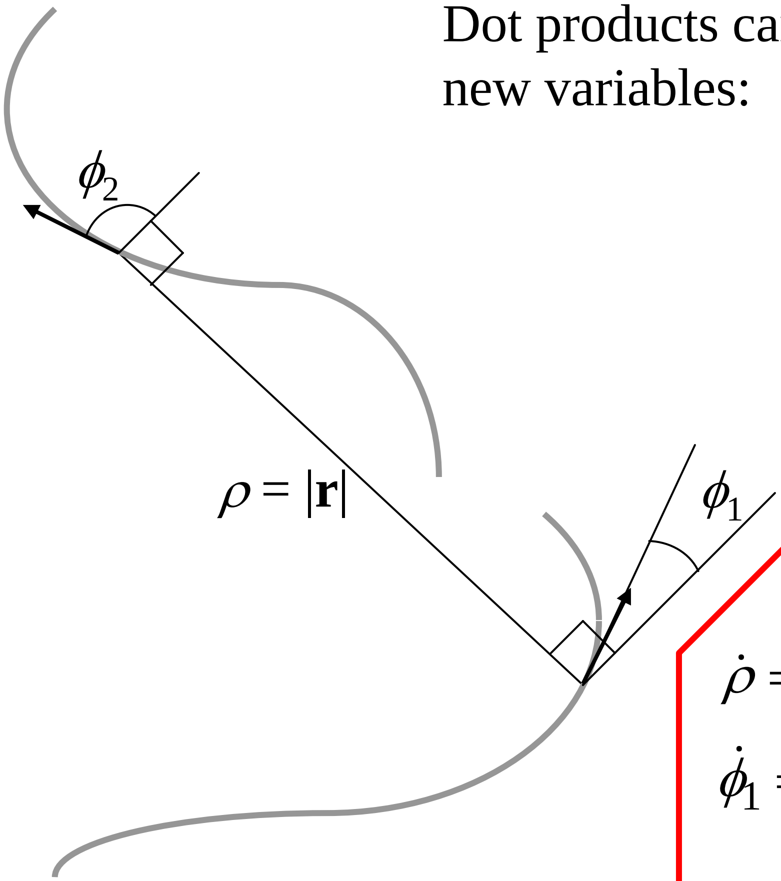
$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$



$$\mu > \frac{\eta}{2} > 0$$



Two-Vehicle Law: Change of Variables



Dot products can be expressed as sines and cosines in the new variables:

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1 = \sin\phi_1 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1 = \cos\phi_1$$

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2 = \sin\phi_2 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2 = \cos\phi_2$$

$$\mathbf{x}_2\cdot\mathbf{y}_1 = \sin(\phi_2-\phi_1)$$

$$\mathbf{x}_1\cdot\mathbf{y}_2 = \sin(\phi_1-\phi_2)$$

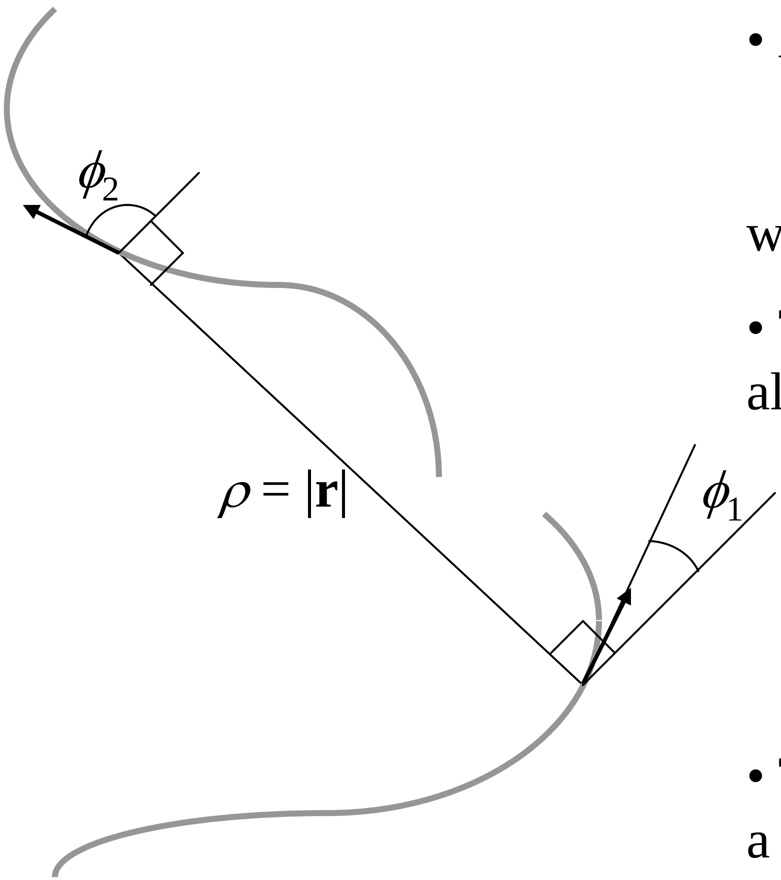
System after change of variables:

$$\dot\rho = \sin\phi_2 - \sin\phi_1$$

$$\dot\phi_1 = -\eta\sin\phi_1\cos\phi_1 + f(\rho)\cos\phi_1 + \mu\sin(\phi_2-\phi_1) + (1/\rho)(\cos\phi_2-\cos\phi_1)$$

$$\dot\phi_2 = -\eta\sin\phi_2\cos\phi_2 - f(\rho)\cos\phi_2 + \mu\sin(\phi_1-\phi_2) + (1/\rho)(\cos\phi_2-\cos\phi_1)$$

Two-Vehicle Law: Lyapunov Function



• A Lyapunov function is

$$V_{pair} = -\ln(\cos(\phi_2 - \phi_1) + 1) + h(\rho)$$

where $f(\rho) = dh/d\rho$.

• The derivative of V_{pair} with respect to time along trajectories of the system is

$$\dot{V}_{pair} = \frac{\partial V_{pair}}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial V_{pair}}{\partial \varphi_2}\dot{\phi}_2 + \frac{\partial V_{pair}}{\partial \rho}\dot{\rho}$$
$$\vdots$$
$$\leq 0.$$

• This Lyapunov function is the key to proving a convergence result for the two-UAV system.

Note: V_{pair} is **not** to be thought of as a synthetic potential (commonly used in robotics for directing motion toward a target or away from obstacles). V_{pair} is a Lyapunov function for the **shape dynamics** of the two-vehicle formation.

Two-Vehicle Law: Convergence Result

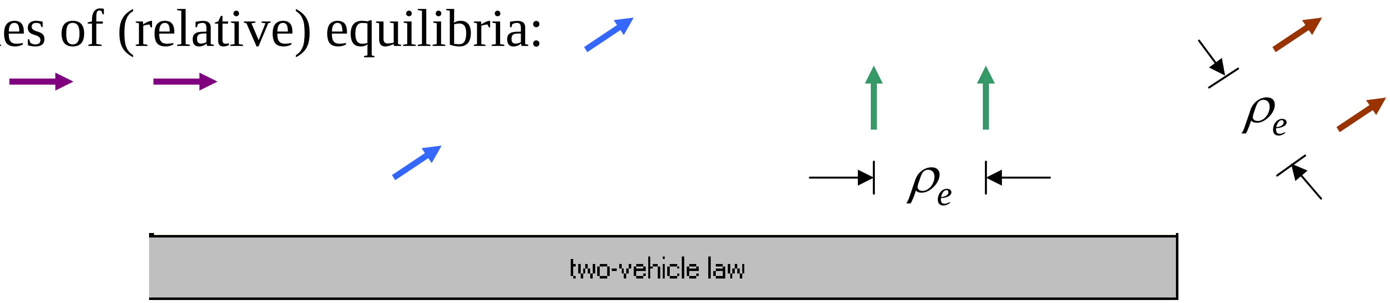
- Our Lyapunov function must be "radially unbounded," meaning $V_{pair} \to \infty$ as $\rho \to 0$ and as $\rho \to \infty$. (Some minor technical assumptions are also needed.)

Proposition (Justh, Krishnaprasad): For any initial condition satisfying $|\phi_2 - \phi_1| \neq \pi$ and $\rho > 0$, the system converges to the set of equilibria, which has the form

$$\left\{\left(\rho, \tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \forall \rho > 0\right\} \cup \left\{\left(\rho, -\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right), \forall \rho > 0\right\} \cup \left\{\left(\rho_e, 0, 0\right) \middle| f(\rho_e) = 0\right\}$$

Proof: Uses LaSalle's Invariance Principle.

- Examples of (relative) equilibria:



Two-Vehicle Law: Intuition

Steering control equation for UAV #2:

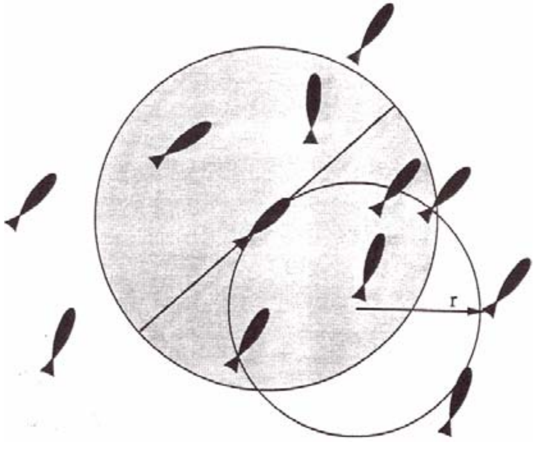
$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$

Align each vehicle perpendicular to the baseline between the vehicles.

Steer toward or away from the other vehicle to maintain appropriate separation.

Align with the other vehicle's heading.

- Biological analogy (swarming, schooling):
  - Decreasing responsiveness at large separation distances.
  - Switch from attraction to repulsion based on separation distance or density.
  - Mechanism for alignment of headings.



D. Grünbaum, "Schooling as a strategy for taxis in a noisy environment," in *Animal Groups in Three Dimensions*, J.K. Parrish and W.M. Hamner, eds., Cambridge University Press, 1997.

Two-Vehicle Law: Key Ideas

- Unit-speed motion with steering control.

  - **Gyroscopic forces** preserve kinetic energy of each particle.

  - In mechanics, gyroscopic forces are associated with **vector potentials**.

- Shape variables: relative distances and angles.
- Lyapunov function $\Rightarrow$ convergence result for the shape dynamics.
- Equilibria of the shape dynamics = relative equilibria of the vehicle dynamics.
- Vehicle re-labeling symmetry.
- Lie group formulation:

  - The dynamics of each particle can be expressed as a left-invariant system evolving on $SE(2)$, the group of rigid motions in the plane.

  - $G=SE(2)$ is a symmetry group for the dynamics: the control law is invariant under rigid motions of the entire formation.

  - V_{pair} is also invariant under G.

  - Therefore, we can consider the reduced system evolving on shape space = $(G \times G)/G = G$.

Gyroscopic Forces and Vector Potentials

Consider the Lagrangian:

Kinetic energy — Scalar potential — Vector potential (linear-in-velocity term)

$$L = \tfrac{1}{2}\dot{x}\cdot M\dot{x} + y(x)\cdot\dot{x} - x\cdot Kx,$$

$$x \in \mathbb{R}^n,\ \ M = M^T > 0,\ \ K = K^T.$$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\big(M\dot{x} + y(x)\big) - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + \left(\frac{\partial y}{\partial x}\right)\dot{x} - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + Q(x)\dot{x} + Kx$$

$$= 0,$$

$$(*)\ \ Q(x) = \left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial y}{\partial x}\right)^T \Rightarrow Q(x) = -Q^T(x).$$

Note: Lagrangian with linear-in-velocity term $\Rightarrow$ skew term in the dynamics, but the converse only holds if $Q(x)$ can be expressed as in $(*)$ for some $y(x)$.

Gyroscopically Interacting Particles

For a single particle:

$$\mathbf{r} = \text{position}, \quad \mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \ddot{\mathbf{r}},$$
$$m = \text{mass} = 1,$$
$$H = \text{kinetic energy} = \tfrac{1}{2}\|\mathbf{v}\|^2,$$
$$m\mathbf{a} = \mathbf{F} \Leftrightarrow \ddot{\mathbf{r}} = \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} \dot{\mathbf{r}}.$$

Note: **F** is a **gyroscopic force** (Recall Lorentz force law for a charged particle in a magnetic field)

$\Longrightarrow$

$$\dot{H} = 0,$$
$$\dot{\mathbf{r}} = \begin{pmatrix} \sqrt{2H}\cos\theta \\ \sqrt{2H}\sin\theta \end{pmatrix},$$
$$\dot{\theta} = u.$$

$\Downarrow$ Restrict to the level-set of H given by H=1/2.

Note that *u* may be a complicated function of time, and may involve feedback. $\rightarrow$

$$\dot{\mathbf{r}} = \mathbf{x}$$
$$\dot{\mathbf{x}} = \mathbf{y}u$$
$$\dot{\mathbf{y}} = -\mathbf{x}u$$

Frenet-Serret equations

For multiple particles, the kinetic energy of **each particle** is conserved, and the particles interact via **gyroscopic forces**.

Shape Space for *n* Vehicles

Frenet-Serret Equations

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$

$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$

$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$

$$j = 1, 2, \ldots, n.$$

Group variables

$$g_j = \left[\begin{array}{cc|c} \mathbf{x}_j & \mathbf{y}_j & \mathbf{r}_j \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$j = 1, 2, \ldots, n.$$

$g_1, g_2, \ldots, g_n \in G = SE(2)$, the group of rigid motions in the plane.

Dynamics

$$\dot{g}_j = g_j\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_j\right)$$

$$= g_j \xi_j, \quad \xi_j \in se(2), \quad j = 1, 2, \ldots, n.$$

Configuration space

$$S = \overbrace{G \times G \times \cdots \times G}^{n \text{ copies}}$$

Assume the controls $u_1, u_2, \ldots, u_n$ are functions of shape variables only.

Shape variables capture **relative** vehicle positions and orientations.

Shape variables

$$\tilde{g}_j = g_1^{-1} g_j, \quad j = 2, \ldots, n.$$

Shape space

$$R = \overbrace{G \times G \times \cdots \times G}^{n-1 \text{ copies}}$$

Two-Vehicle Law: Lie Group Setting

- Dynamics on configuration space $S=G\times G$, where $G=SE(2)$:

$$g_1 = \left[\begin{array}{cc|c} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{r}_1 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_1 & -\sin\theta_1 & \mathbf{r}_1 \\ \sin\theta_1 & \cos\theta_1 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_1 = g_1\xi_1 = g_1(A_0 + A_1 u_1).$$

$$\xi_1, \xi_2 \in se(2)$$

$$g_2 = \left[\begin{array}{cc|c} \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{r}_2 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_2 & -\sin\theta_2 & \mathbf{r}_2 \\ \sin\theta_2 & \cos\theta_2 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_2 = g_2\xi_2 = g_2(A_0 + A_1 u_2).$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Shape variable: $g = g_1^{-1} g_2$
- Dynamics on shape space $R=G$: $\dot{g} = g\xi$,

$$\xi = \xi_2 - g^{-1}\xi_1 g = \xi_2 - Ad_{g^{-1}}\xi_1 \in se(2).$$

- Controls as functions of the shape variable g:

$$u_1(g) = -\eta\left(\frac{g_{13}g_{23}}{r^2}\right) + f(r)\left(\frac{g_{23}}{r}\right) + \mu g_{21}, \quad g = [g_{ij}], \quad r = \sqrt{g_{13}^2 + g_{23}^2},$$

$$u_2(g) = -\eta\left(\frac{g^{13}g^{23}}{r^2}\right) + f(r)\left(\frac{g^{23}}{r}\right) + \mu g^{21}, \quad g^{-1} = [g^{ij}].$$

Lyapunov Function:

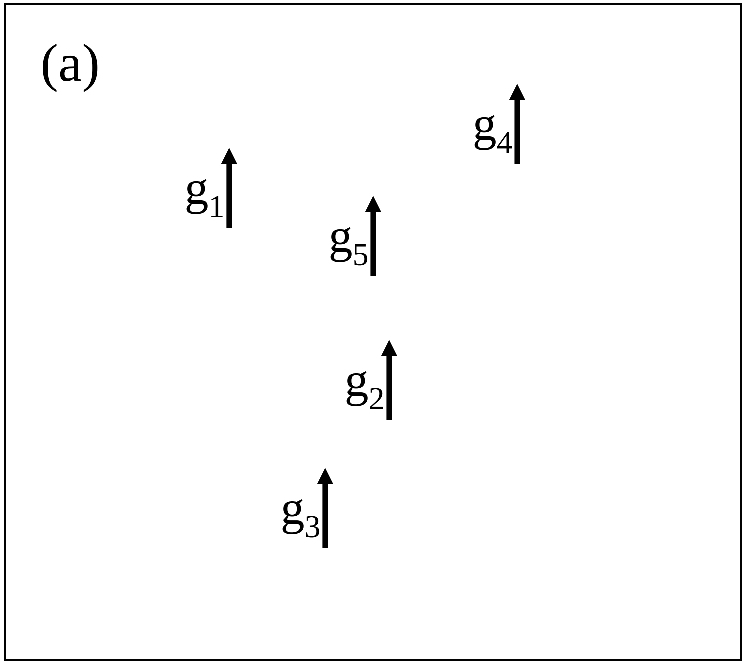
$$V_{pair} = V_{pair}(g).$$

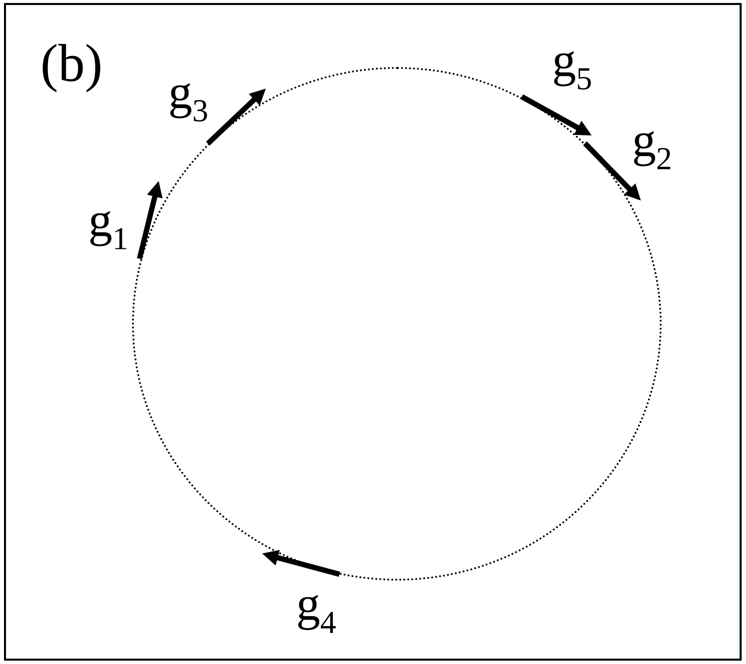
Characterization of Equilibrium Shapes

Proposition (Justh, Krishnaprasad): For equilibrium shapes (i.e., relative equilibria of the dynamics on configuration space), $u_1 = u_2 = \ldots = u_n$, and there are only two possibilities:

(a) $u_1 = u_2 = \ldots = u_n = 0$: all vehicles head in the same direction (with arbitrary relative positions), or

(b) $u_1 = u_2 = \ldots = u_n \neq 0$: all vehicles move on the same circular orbit (with arbitrary chordal distances between them).





Stabilizing Control Laws

(a) Control law for rectilinear motion:

$$u_j = \frac{1}{n}\sum_{k\neq j}\left[-\eta\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_j\right)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)+f(|\mathbf{r}_{jk}|)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)+\mu\mathbf{x}_k\cdot\mathbf{y}_j\right]$$

$$\mathbf{r}_{jk} = \mathbf{r}_k - \mathbf{r}_j, \quad f(|\mathbf{r}_{jk}|) = \alpha\left[1-\left(\frac{r_o}{|\mathbf{r}_{jk}|}\right)^2\right], \quad \mu > \frac{\eta}{2} > 0, \quad \alpha > 0.$$

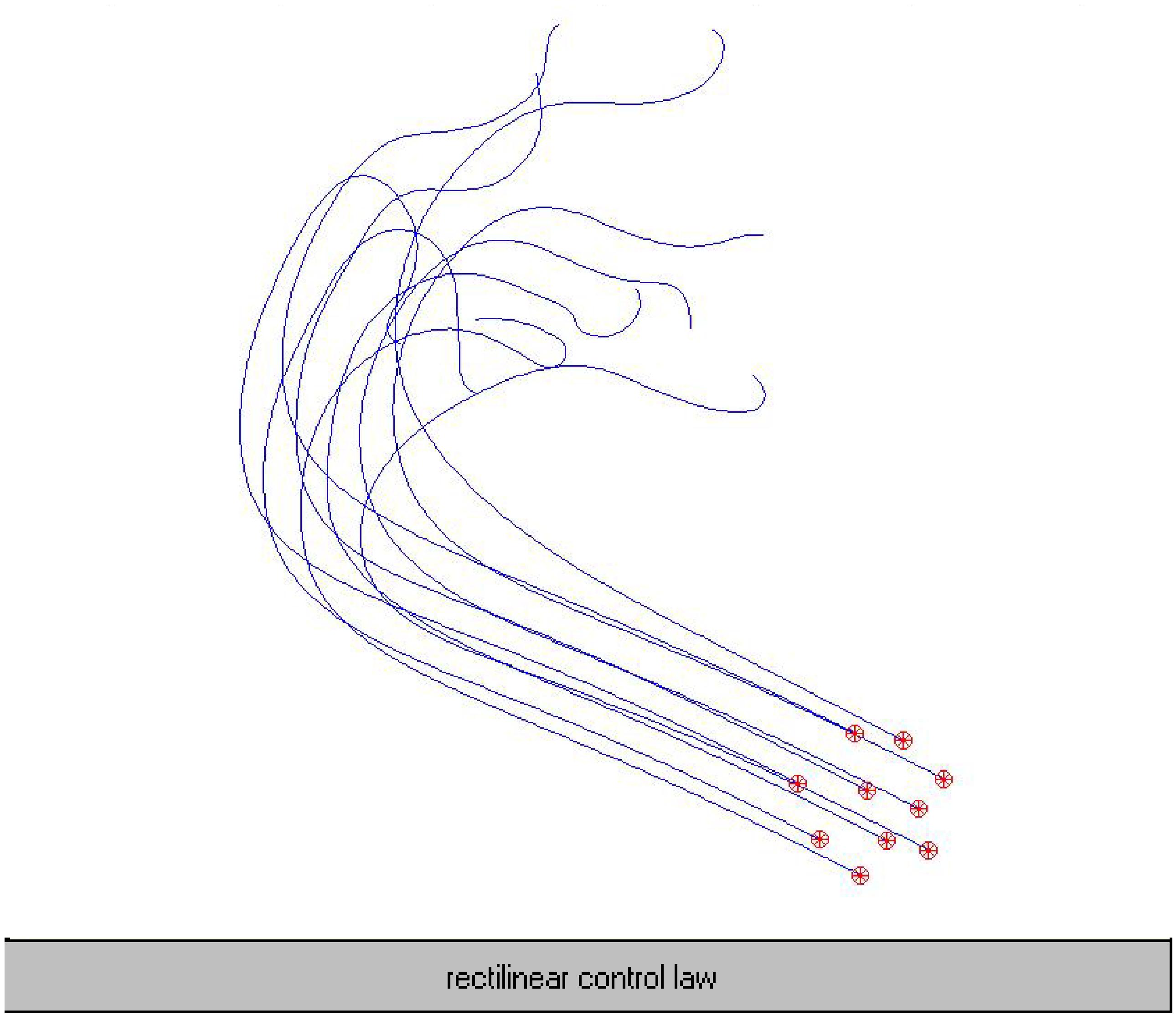
- Generalizes the two-vehicle law to *n* vehicles.
- Convergence result only proved for n=2.

(b) Control law for circular motion:

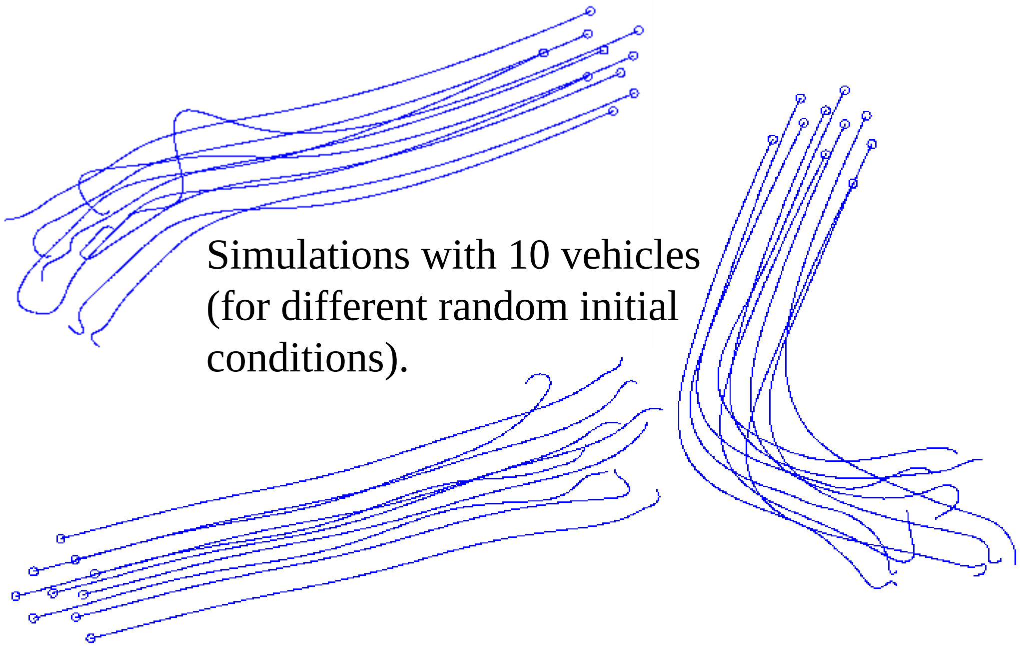
$$u_j = \frac{1}{n}\sum_{k\neq j}\left[\pm\eta\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_j\right)+f(|\mathbf{r}_{jk}|)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)\right]$$

At present, it is **conjectured** (based on simulation results) that these control laws stabilize their respective relative equilibria. However, analytical work is ongoing.

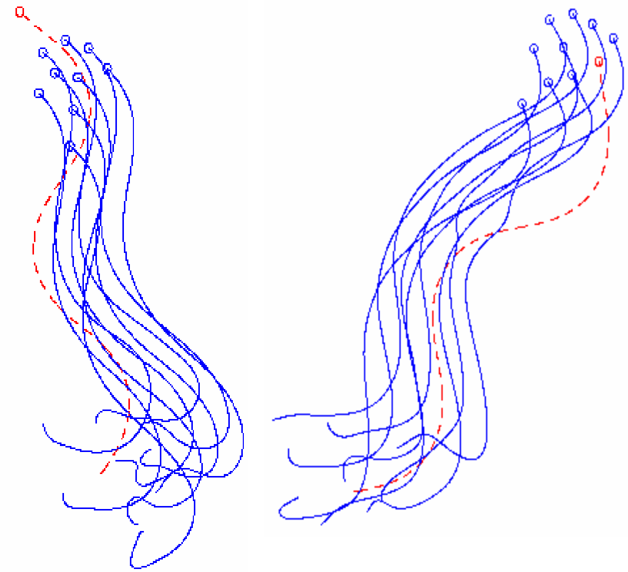
Rectilinear Control Law Simulations



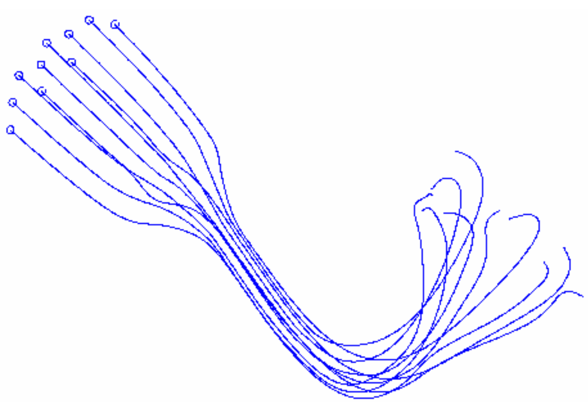
Rectilinear Control Law Simulations



Simulations with 10 vehicles (for different random initial conditions).

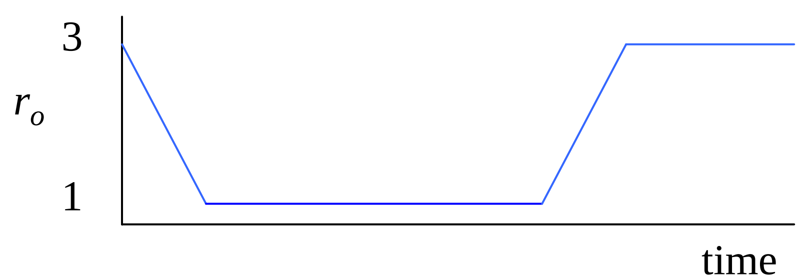


Leader-following behavior: the red vehicle follows a prescribed path (dashed line).

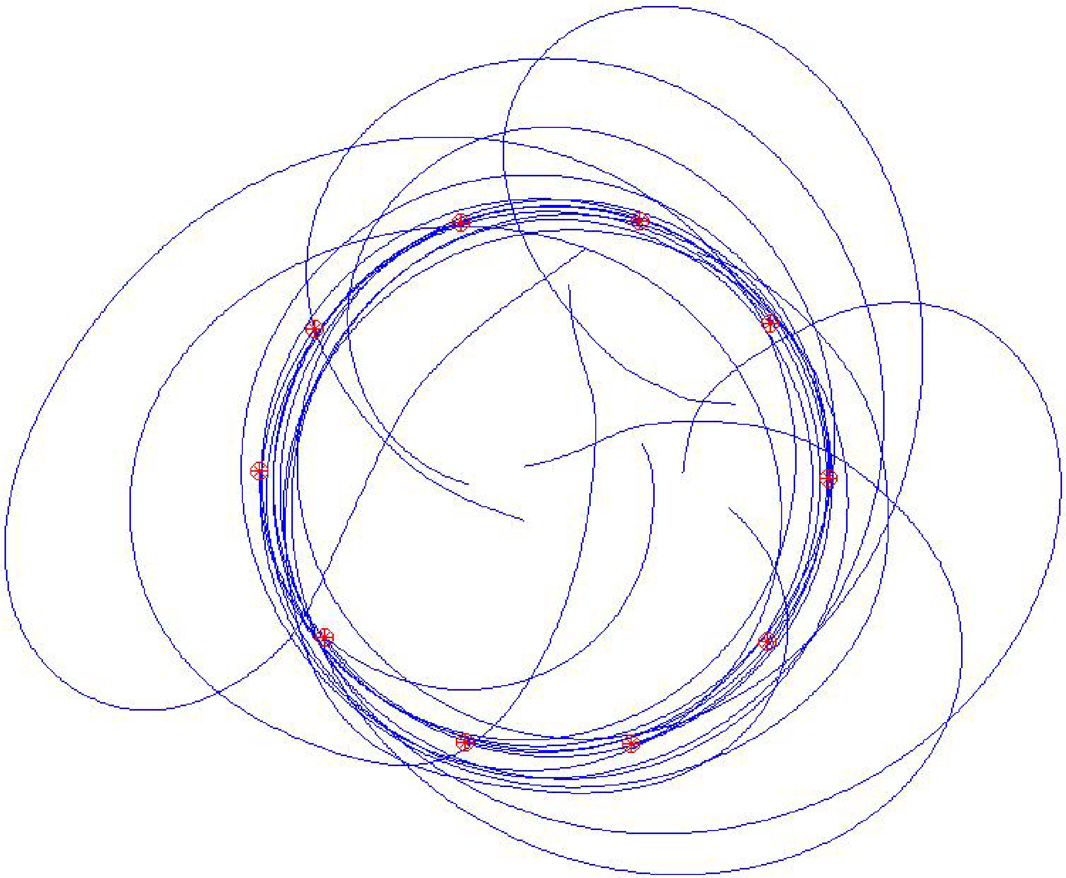


On-the-fly modification of the separation parameter.

Normalized Separation Parameter vs. Time



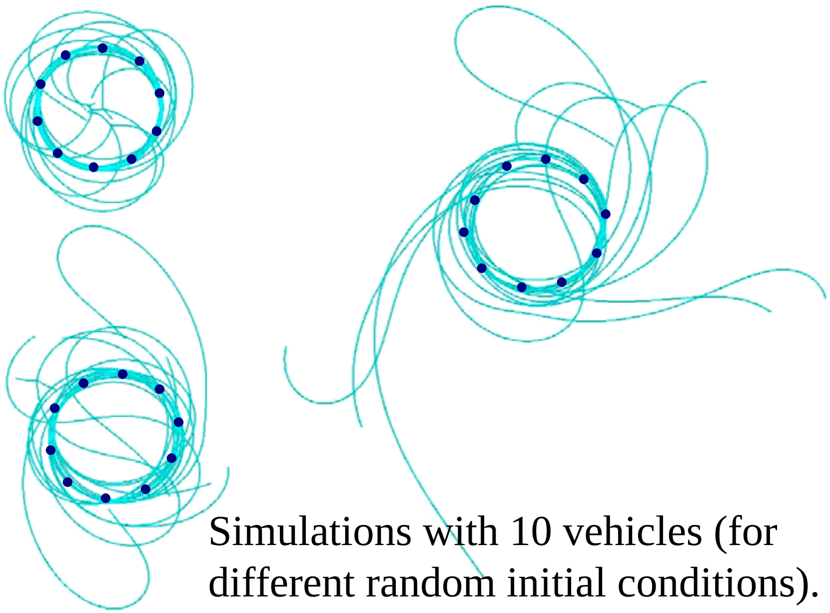
Circular Control Law Simulations



circular control law

Circular Control Law

$$u_j = \frac{1}{n}\sum_{k \neq j}\left[\pm\eta\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_j\right) + f(|\mathbf{r}_{jk}|)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)\right]$$



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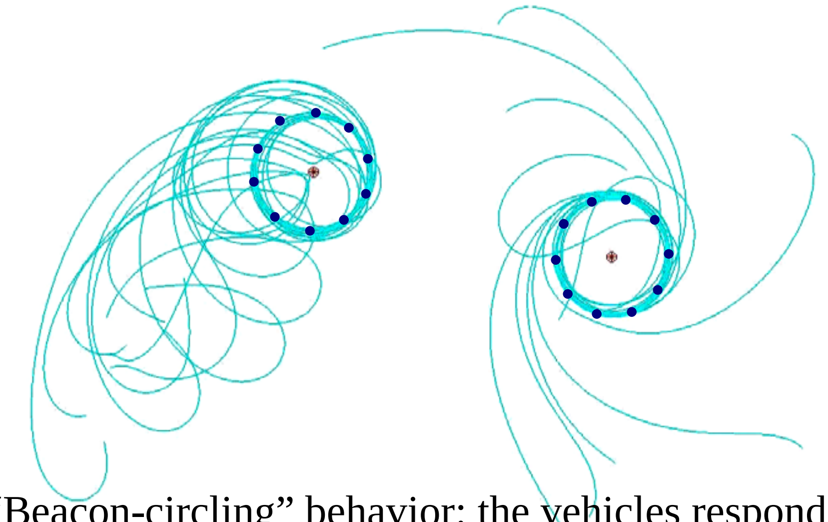
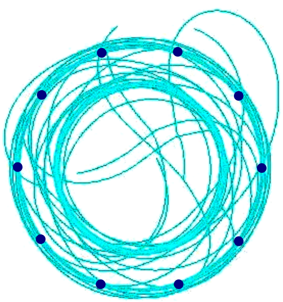
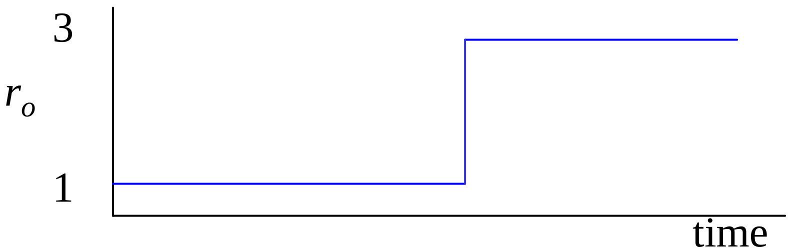


"Beacon-circling" behavior: the vehicles respond to a beacon, as well as to each other.



On-the-fly modification of the separation parameter.

Normalized Separation Parameter vs. Time



Convergence Result for $n > 2$

• We consider rectilinear relative equilibria, and the Lyapunov function

$$V = \sum_{j=1}^{n} \sum_{k<j} \left[-\ln\left(\cos(\theta_j - \theta_k) + 1\right) + h(\| \mathbf{r}_j - \mathbf{r}_k \|) \right]$$

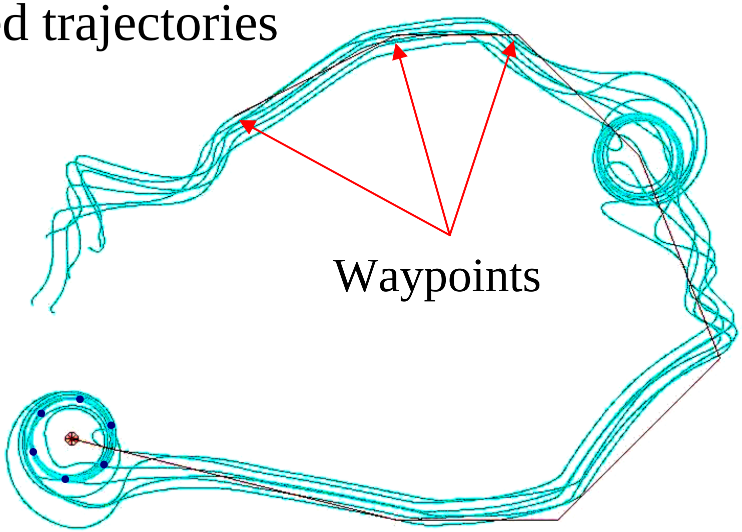
• **Convergence Result** (Justh, Krishnaprasad): There exists a sublevel set Ω of V and a control law (depending only on shape variables) such that $\dot{V} \leq 0$ on Ω.

• With this Lyapunov function, we cannot prove global convergence for $n > 2$.

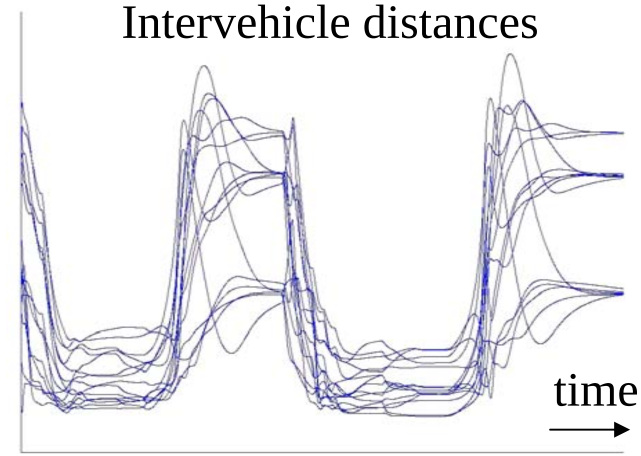
• Although we obtain an explicit formula for the controls u_j, $j=1,\ldots,n$, there is no guarantee that this particular choice of controls will result in convergence to a particular desired equilibrium shape in Ω.

Performance Criteria

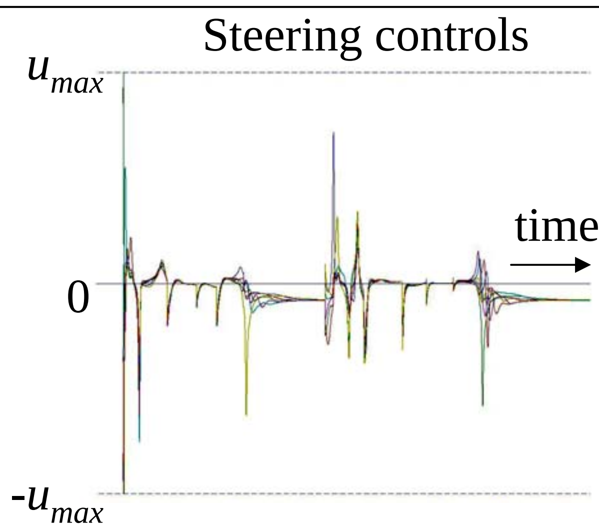
• Faithful following of waypoint-specified trajectories

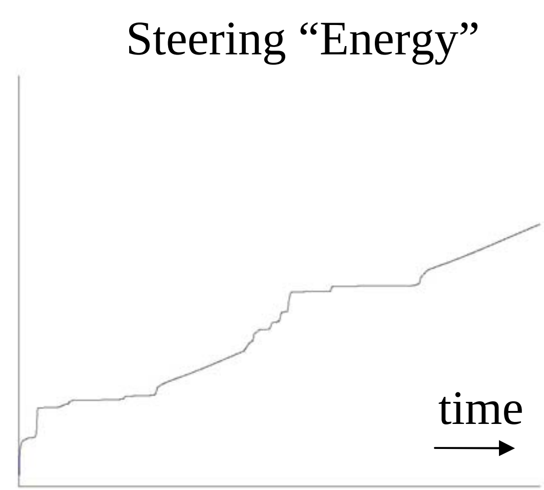


• Sufficient separation between vehicles (to avoid collisions)



• Minimize steering: for UAVs, turning requires considerably more energy than straight, level flight. Maneuverability is also limited.





Implementation Approaches

• The form of the rectilinear and circular control laws was originally motivated by implementation considerations.

- "Simplest" control laws which appear to stabilize the relative equilibria of interest.
- Vehicle re-labeling symmetry.

1. Motion description language approach

- GPS and wireless communication network used to exchange state information.
- Each vehicle simulates a time-discretization of the dynamical system model.
- Constraints on vehicle maneuverability are respected, so each vehicle's autopilot can track the trajectory generated by the dynamical system model.
- "Simplicity" means low computational and communication cost.

2. Sensor-based approach

- Each vehicle uses direct sensor measurements of the range and relative orientation of neighbors to generate steering controls.
- "Simplicity" means minimal hardware required for implementation.

Motion Description Language Approach

• Each vehicle simulates the dynamical system model for the entire formation in real time; i.e., the vehicles all run the same **motion plan**.

- Simulating the dynamical system amounts to estimating the control inputs, positions, and orientations of all the vehicles.

- Disturbances (e.g., wind for UAVs) lead to estimation errors.

- GPS and communication used to reinitialize the estimators.

• The motion plan can be changed on the fly, but all vehicles must make the change simultaneously.

- Interrupts, due to the environment or human intervention, can change the motion plan (e.g., dynamical system parameters).

- The communication protocol must ensure that all vehicles update their motion plans simultaneously.

• This approach is consistent with **motion description language** formalism; see, e.g., V. Manikonda, P.S. Krishnaprasad, and J. Hendler, "Languages, Behaviors, Hybrid Architectures, and Motion Control," in *Mathematical Control Theory*, J. Baillieul and J.C. Willems, eds., Springer, pp. 199-226, 1999.

Time Discretization

• Control laws specify $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ at each time instant t.

• Instead, compute $u_1(t_m)$, $u_2(t_m)$, ..., $u_n(t_m)$, where $t_m = mT$ for $m=1, 2, ...$, and let

$$u_j(t) = u_j(t_m), \quad \forall t \in [t_m, t_{m+1}).$$

• Maximum value of T is determined by the control law.

$T = ½$ seems to be a reasonable choice (for η, μ, $\alpha \approx 1$).

• Piecewise constant controls allow the vehicle positions to be computed using simple formulas:

$$\begin{bmatrix} \mathbf{x}_j(t_{m+1}) & \mathbf{y}_j(t_{m+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_j(t_m) & \mathbf{y}_j(t_m) \end{bmatrix} \begin{bmatrix} \cos\left(u_j(t_m)T\right) & -\sin\left(u_j(t_m)T\right) \\ \sin\left(u_j(t_m)T\right) & \cos\left(u_j(t_m)T\right) \end{bmatrix}$$

$$\mathbf{r}_j(t_{m+1}) = \frac{1}{u_j(t_m)} \begin{bmatrix} \mathbf{x}_j(t_m) & \mathbf{y}_j(t_m) \end{bmatrix} \begin{bmatrix} \sin\left(u_j(t_m)T\right) \\ 1 - \cos\left(u_j(t_m)T\right) \end{bmatrix} + \mathbf{r}_j(t_m)$$

• Length scale determined by r_o. Multiply u_j by speed parameter ν to obtain actual steering controls from unit-speed steering controls.

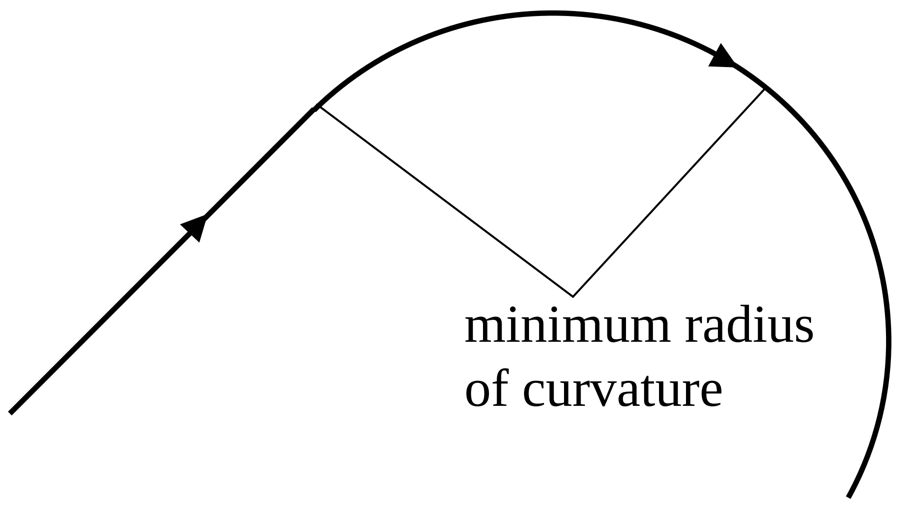
Limited Steering Authority

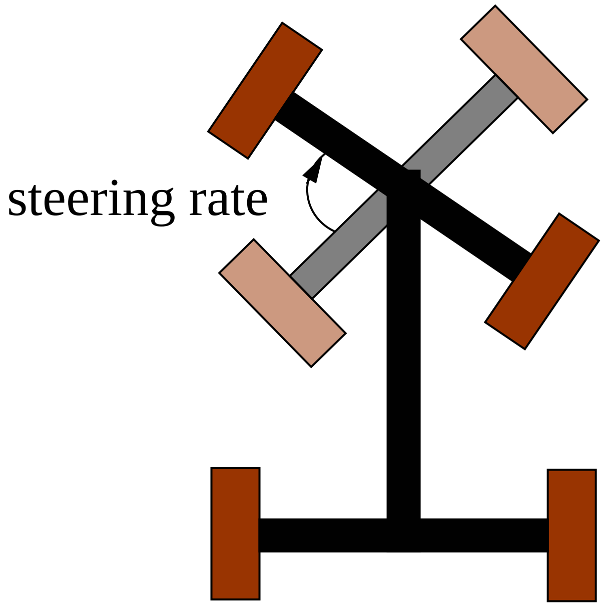
- u_{max} = maximum (absolute) value the steering control is permitted to take.

$$u_j = \begin{cases} -u_{max}, & \text{if } u_j^{ideal} < -u_{max} \\ u_j^{ideal}, & \text{if } -u_{max} < u_j^{ideal} < u_{max} \\ u_{max}, & \text{if } u_j^{ideal} > u_{max} \end{cases}$$

$$j = 1, 2, \ldots, n.$$

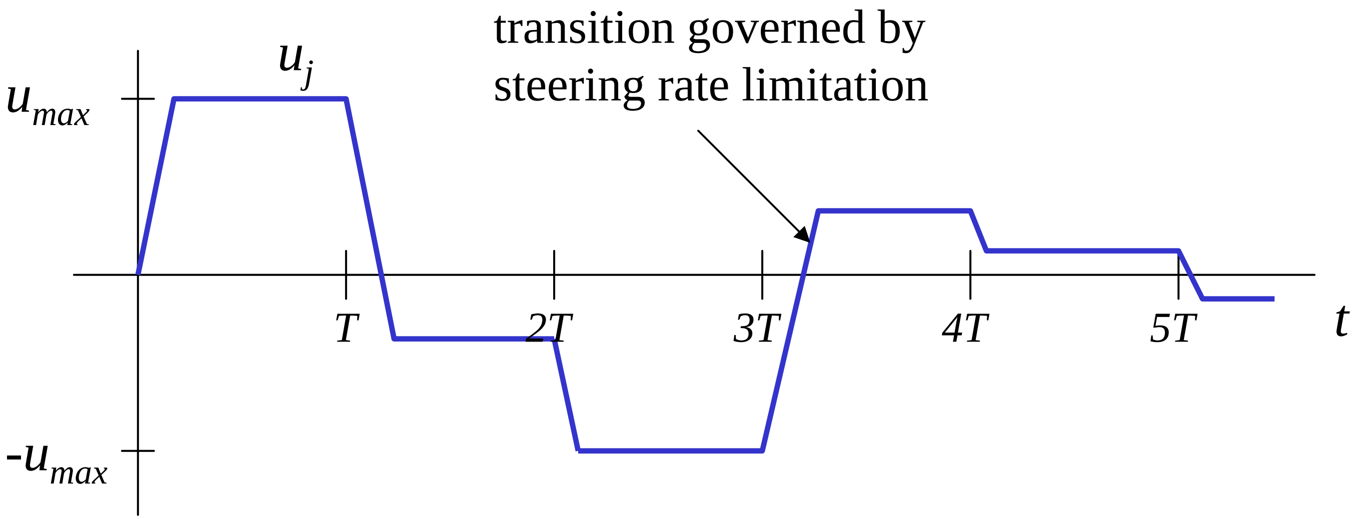
- u_{max} is determined **either** by the minimum radius of curvature or by the **steering rate**.





Finite Steering Rate Effects

• Why steering rate matters:



• The transition time should be a small fraction of the interval *T*.

• If the transition times are not trivial, they can be taken into account in the estimator equations by using Simpson's Rule.

Computational Complexity

• Each vehicle must compute both the controls and the positions and headings for all of the vehicles

- Position and heading calculation for n vehicles (given the controls) ~ n.

- Calculation of the controls for all n vehicles ~ n^2.

• Dealing with computational complexity of the control calculation as n becomes large:

- Mollifiers (i.e., η, μ, and f depend on $|\mathbf{r}_{jk}|$, and go to zero for large $|\mathbf{r}_{jk}|$).

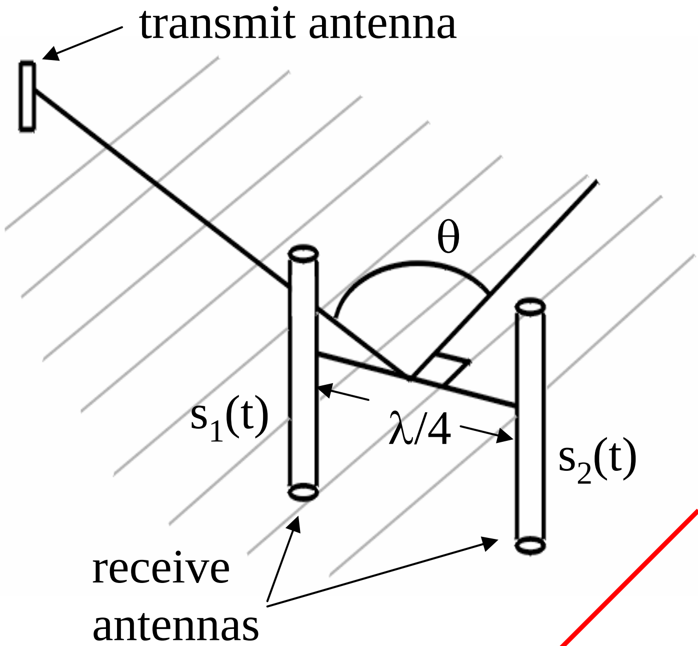
- Low-precision computation of u_j, using the *shape variables*

$$\tilde{g}_j = g_1^{-1} g_j, \quad j = 2, \ldots, n.$$

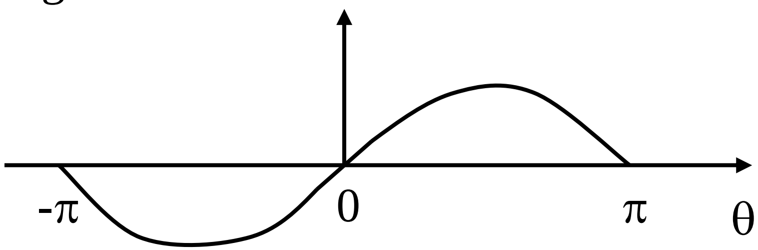
• Low-precision computation of the controls using the (**x**, **y**, **r**) variables **fails**.

• Thus, shape-variable notions and the Lie group formulation have important implications for both analysis and numerics.

Sensor-Based Implementation

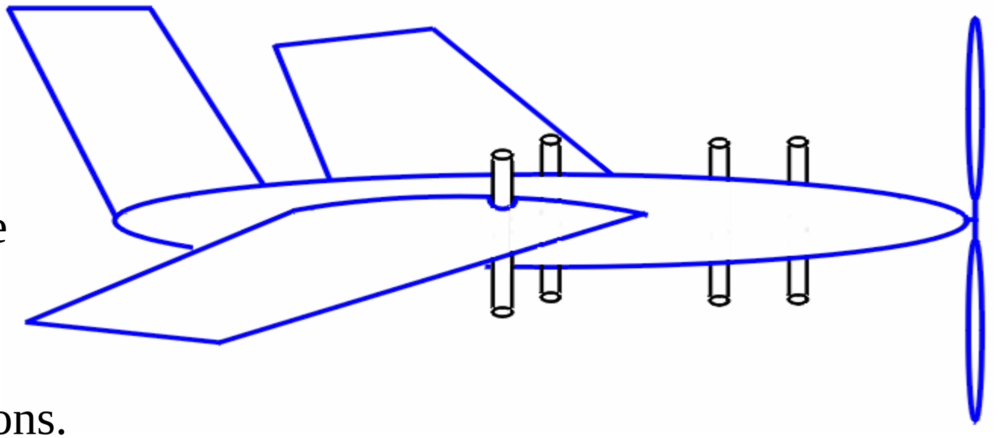


• One pair of antennas gives a sinusoidal function of angle of arrival.



• Range is inversely related to received power.

• Two pairs of antennas, used for both transmitting and receiving, can provide all the terms in the control law.

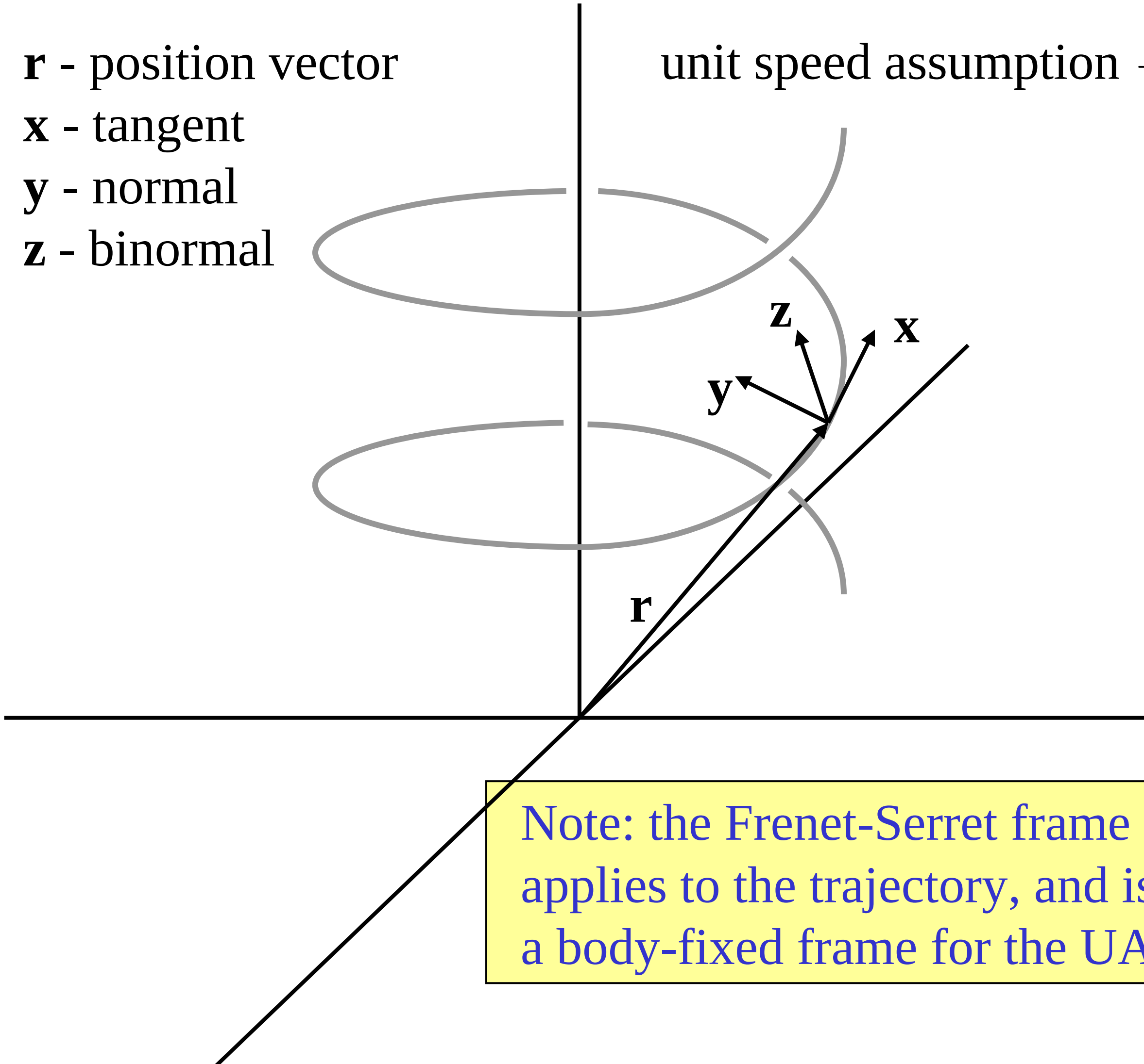


• Antenna separation and transmission frequency are related to UAV dimensions.

• GPS is not required.

3-Dimensional Frenet-Serret Equations

r - position vector
x - tangent
y - normal
z - binormal



unit speed assumption → $\dot{\mathbf{r}} = \mathbf{x}$

$$\dot{\mathbf{x}} = \mathbf{y}u - \mathbf{z}v$$
$$\dot{\mathbf{y}} = -\mathbf{x}u + \mathbf{z}w$$
$$\dot{\mathbf{z}} = \mathbf{x}v - \mathbf{y}w$$

u, *v*, *w* are control inputs (two of which uniquely specify the trajectory)

Frenet-Serret:
- $v = 0$
- u = curvature
- w = torsion

Note: the Frenet-Serret frame applies to the trajectory, and is **not** a body-fixed frame for the UAV

Continuum Model

- Vector field (in polar coordinates):

$$\begin{pmatrix} d\mathbf{r}/dt \\ \\ d\theta/dt \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ u \end{pmatrix}.$$

- Continuity equation (Liouville equation):

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial (u\rho)}{\partial \theta} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \nabla_{\mathbf{r}} \rho \right].$$

- Conservation of matter:

$$\int_G \rho(t, \mathbf{r}, \theta) d\mathbf{r} d\theta = 1, \quad \forall t.$$

- Energy functional:

- This continuum formulation only involves two scalar fields: the density $\rho(t,\mathbf{r},\theta)$ and the steering control $u(t,\mathbf{r},\theta)$.

- However, the underlying space is 3-dimensional (for planar formations).

- Incorporating time and/or spatial derivatives in the equation for u yields a coupled system of PDEs for ρ and u.

$$V_c(t) = \frac{1}{2} \int_G \int_G \left[-\ln\left(\cos(\theta - \tilde{\theta}) + 1 \right) + h(\| \mathbf{r} - \tilde{\mathbf{r}} \|) \right] \rho(t, \mathbf{r}, \theta) \rho(t, \tilde{\mathbf{r}}, \tilde{\theta}) d\mathbf{r} d\theta d\tilde{\mathbf{r}} d\tilde{\theta}.$$

References

E.W. Justh and P.S. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research Technical Report TR 2002-38, 2002 (see http://www.isr.umd.edu). (Abridged version submitted to *Systems and Control Letters*.)

E.W. Justh and P.S. Krishnaprasad, "Steering laws and continuum models for planar formations," submitted to IEEE Conference on Decision and Control, 2003.