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Jointly Optimal Quantization, Estimation, and Control of Hidden Markov Chains

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Jointly Optimal Quantization, Estimation, and Control of Hidden Markov Chains*

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Abstract

It is of interest to understand the tradeoff between the communication resource comsumption and the achievable system performance in networked control systems. In this paper we explore a general framework for trade-off analysis and decision making in such systems by studying joint quantization, estimation, and control of a hidden Markov chain. We first formulate the joint quantization and estimation problem, where vector quantization with variable-block length is considered. Dynamic programming (DP) is used to find the optimal quantization scheme that minimizes a weighted combination of the estimation error, the communication cost, and the delay due to block coding. The DP equation is solved numerically and simulation shows that this approach is able to capture the tradeoffs among competing objectives by adjusting the cost weights. We then study the joint quantization and control problem. An example problem is solved analytically, which provides interesting insight into the approach. In both the joint quantization/estimation problem and the joint quantization/control problem, we show that the separation principle holds. The approaches to solving these two problems share the same spirit, and can be combined and extended to accomodate more objectives.

1 Introduction

Networked control systems have applications or potential applications in defense, transportation, scientific exploration, and industry, with examples ranging from automated highway systems to unmanned aerial vehicles to MEMS sensor and actuator networks. Communication in networked control systems is often limited due to the large number of subsystems involved, limited battery life and power, and constraints imposed by environmental conditions. Hence an important concern in the development of networked control systems is how to deploy and allocate the communication resources. Proper understanding of the tradeoff between the communication resource consumption and the system performance will help to make such decisions. A great deal of effort has been put into the studies of control systems when communication constraints are present. In particular, stabilization of linear systems with quantized state/output/input has received much attention (see e.g., [1, 2, 3] and the references therein). Other problems, such as the state estimation problem and LQG control under communication constraints, have also been studied [4, 5].

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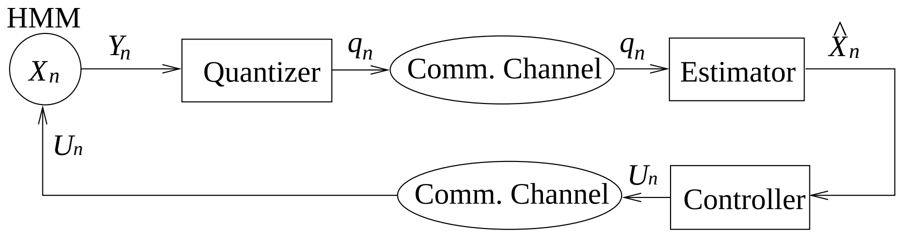


Figure 1: The setup for joint quantization, estimation, and control of an HMM.

In this paper we explore a general framework for trade-off analysis and decision making in networked control systems, by studying jointly optimal quantization, estimation, and control of a Hidden Markov chain. Hidden Markov chains form an important family of Hidden Markov Models (HMMs) [6], and have been widely used in speech processing, computer vision, computational biology, telecommunications, etc. Another reason for us to choose a hidden Markov chain is that numerical and even analytical solutions can be obtained relatively easily, which provide insight into the approach.

Figure 1 illustrates the problem setup. X_n is the state of a homogeneous, controlled, hidden Markov chain taking values in $\mathcal{X} = \{x_1, \cdots, x_S\}$ for some $S \geq 1$. The output Y_n takes values in $\mathcal{Y} = \{y_1, \cdots, y_M\}$ for some $M \geq 1$. Quantized information q_n of the output is sent over a communication channel to a remote processor, where state estimation and control computation are performed (later on we shall justify the "separation" of estimation from control shown in Figure 1). The control U_n is then sent back through a communication channel to the HMM. To highlight the main ideas and simplify the analysis, we assume that communication channel is noise free. The quantizer is allowed to be time-varying; however, the variation of the quantization scheme should depend only on the information available to the receiver.

The control U_n takes values in $\mathcal{U} = \{u_1, \cdots, u_K\}$ for some $K \geq 1$. We use the notation Z_i^j to denote the sequence of random variables $\{Z_i, Z_{i+1}, \cdots, Z_j\}$. For $u \in \mathcal{U}$, $1 \leq i, j \leq S$, we let

$$a_{ij}(u) \stackrel{\triangle}{=} Prob[X_{n+1} = x_i | X_n = x_j, U_n = u] = Prob[X_{n+1} = x_i | X_n = x_j, U_n = u, X_0^{n-1}, U_0^{n-1}].$$

We assume that the output Y_n is dependent on X_n only. For $1 \leq i \leq S$, $1 \leq j \leq M$, we write

$$c_{ij} \stackrel{\triangle}{=} Prob[Y_n = y_j | X_n = x_i] = Prob[Y_n = y_j | X_n = x_i, X_0^{n-1}, U_0^n, Y_0^{n-1}].$$

This paper is divided into two parts. In the first part we are concerned only with joint quantization and estimation, i.e., the loop in Figure 1 is not closed. Sequential vector quantization of Markov sources was considered in [7], where a weighted combination of the entropy rate of the quantized process and the compression error was minimized. We note that such a "Lagrangian distortion measure" appeared earlier in [8]. A similar approach for combined classification and compression was proposed in [9]. We extend the work in [7] to the case of vector quantization with variable block length, and seek the optimal quantization scheme to minimize a weighted combination of the estimation error, the conditional entropy of the quantized output, and the delay due to block coding. The problem is recast as a stochastic control problem and the corresponding value function satisfies a Dynamic Programming (DP) equation of a novel form. We further investigate numerically solving the DP equation and study the effects of weighting coefficients on optimal quantization schemes through simulation.

In the second part of the paper we discuss the problem of joint quantization and control. Following the same spirit as in joint quantization and estimation, we seek the optimal quantization and control scheme to minimize a weighted sum of the communication cost and the cost relating to the system performance. Here we restrict ourselves to sequential vector quantization. For illustrative purposes a simple example problem is solved analytically, which provides interesting insight into the approach.

In both the joint quantization/estimation problem and the joint quantization/control problem, we show that the separation principle [10] holds. Either problem is decomposed into an estimation problem, and a decision (quantization/control) problem based on the state estimation.

The structure of the paper is as follows. In Section 2 we formulate and solve the joint quantization and estimation problem. The resulting DP equation is numerically solved in Section 3. In Section 4 we study the joint quantization and control problem. We finally conclude in Section 5.

2 Joint Quantization and Estimation

2.1 Problem formulation

We consider vector quantization with variable block length. Let $B \geq 1$ be the maximum block length. Given the initial condition for X_0, we decide the first data block $Y_1^{n_1}$ with $n_1 \leq B$ for quantization as well as the actual quantization scheme we will use for $Y_1^{n_1}$. Then at time n_1, we send the quantized $Y_1^{n_1}$, and decide the next data block $Y_{n_1+1}^{n_2}$ $(n_1 + 1 \leq n_2 \leq n_1 + B)$ and the associated quantization scheme based only on the information available to the receiver (i.e., the initial condition of X_0 and the quantization of $Y_1^{n_1}$). This process goes on until the final time $N \geq 1$ is reached. Time instants (e.g., 0, n_1 in the previous discussion) that one makes decisions are called *decision times*. We assume that each transmission is completed instantly and the delay due to communication is zero.

We now formulate the problem precisely. Let Ω_1 be the space of admissible quantization decisions for Y_1^N. Here by a *quantization decision*, we mean a scheme for both division of Y_1^N into (variable-length) blocks and quantization of these blocks. A quantization decision is *admissible* if at each decision time, the length of the next data block and the corresponding quantization scheme are decided solely based on the information available to the remote processor by that time. This makes the sender's decision transparent to the receiver, and eliminates the need to transmit the quantization scheme separately. On the other hand this imposes the requirement of certain computation capability on the sender side.

Let $\Pi_0 = (\pi_0(x_1), \cdots, \pi_0(x_S))$ be the *a priori* PMF (probability mass function) for X_0, where $\pi_0(x_i) = Prob[X_0 = x_i], 1 \leq i \leq S$. Given Π_0 and $\omega \in \Omega_1$, we define the cost

$$J(\Pi_0, \omega) = E[\sum_{n=1}^{N} J^h(n) + \lambda_d J^d(n) + \lambda_e J^e(n)]. \tag{1}$$

Here $\lambda_d, \lambda_e \geq 0$ are weighting coefficients, and $J^h(n)$, $J^d(n)$, $J^e(n)$ are costs relating to the communication needs, the delay due to block coding, and the estimation error at time n, respectively. To be specific,

- $J^h(n)$ is the communication cost at time n. In this section we assume that entropy coding [11] is used, so the (expected) number of bits required to transmit a random vector Z is bounded by

$H[Z]+1$, where $H[Z]$ denotes the entropy of Z. Hence

$$J^h(n) = \begin{cases} 0, & \text{if no transmission at time } n \\ H[Q_n|R_n]+1, & \text{otherwise} \end{cases}, \tag{2}$$

where Q_n is the information transmitted at time n, R_n represents information *sent* before time n, and $H[\cdot|\cdot]$ denotes the conditional entropy.

- $J^d(n)$ is the delay cost *evaluated* at time n. For simplicity, we assume that $J^d(n)$ is equal to the number of samples being delayed at time n. For instance, if one decided to quantize Y_{i-1}^{i+1} as a block, then $J^d(i-1)=1$ (since information about Y_{i-1} has not been transmitted at time $i-1$), $J^d(i)=2$ (since information about both Y_{i-1} and Y_i has not been transmitted at time i), $J^d(i+1)=0$ (no backlog at time $i+1$).

- $J^e(n)$ is the estimation error for X_n. Assume that information of Y_n is contained in the quantized block $Y_{n-i_1}^{n+i_2}$ denoted by Q_{n+i_2} for $i_1, i_2 \geq 0$. Denote the information sent before time $n-i_1$ by R_{n-i_1}. Let $\hat{\Pi}_n$ be the conditional PMF of X_n given $\{\Pi_0, R_{n-i_1}, Q_{n+i_2}\}$, and let $\tilde{\Pi}_n$ be the conditional PMF of X_n given $\{\Pi_0, R_{n-i_1}, Y_{n-i_1}^{n+i_2}\}$. Then we define

  $$J^e(n) = \rho(\hat{\Pi}_n, \tilde{\Pi}_n),$$

  where $\rho(\cdot,\cdot)$ is some metric on the space of probabilities on $\mathcal{X}$. In this paper, we take ρ to be the l_1 metric on $\mathbb{R}^S$. Other metrics such as the Kullback-Liebler divergence can also be used. Note that for ease of presentation we have suppressed the dependence of $\hat{\Pi}_n$ and $\tilde{\Pi}_n$ on appropriate variables in the notation, but one should always keep such dependence in mind.

The joint quantization and estimation problem is to find $\omega^* \in \Omega_1$, such that

$$J(\Pi_0, \omega^*) = \min_{\omega \in \Omega_1} J(\Pi_0, \omega) := V(\Pi_0). \tag{3}$$

2.2 The dynamic programming equation

The joint quantization and estimation problem formulated in Subsection 2.1 can be recast as a stochastic control problem and be solved using dynamic programming. As we shall see, the conditional PMF $\hat{\Pi}_n$ is the information state for the new stochastic control problem while the quantization decision ω is the "control".

As standard in dynamic programming, we first define a sequence of joint quantization and estimation problems. For $1 \leq i \leq N$, we let

$$J_i(\Pi_{i-1}, \omega_i) = E[\sum_{n=i}^{N} J^h(n) + \lambda_d J^d(n) + \lambda_e J^e(n)], \tag{4}$$

and

$$V_i(\Pi_{i-1}) = \min_{\omega_i \in \Omega_i} J_i(\Pi_{i-1}, \omega_i), \tag{5}$$

where Ω_i is the space of admissible quantization decisions for the time period $[i, N]$, and Π_{i-1} is the *a priori* PMF for X_{i-1}, i.e., the initial condition for the i-th problem. Clearly for $i=1$, we recover the original problem formulated in the previous subsection.

We denote by Θ^j the space of quantization (encoding) schemes for a data block of length j, say, Y_k^{k+j-1} for $k \geq 1$. There are M^j possible outcomes for Y_k^{k+j-1}, so each $\mathcal{Q}^j \in \Theta^j$ partitions these M^j outcomes into groups and the group index will carry (compressed) information about Y_k^{k+j-1}. In this paper we are concerned with estimation of X_n and not reconstruction of Y_n; however, considerations of decoding (to the space $\mathcal{Y}$) and the associated compression error can be easily accomodated in the current framework once an appropriate metric is defined on the discrete set $\mathcal{Y}$.

A recursive formula exists for $\hat{\Pi}_n$. Assume that for $i \geq 1$, the data block Y_i^{i+j-1} of length j is quantized with $\mathcal{Q}^j \in \Theta^j$ and transmitted at time $i+j-1$. Let R_i represent the transmitted information up to time $i-1$, and $\hat{\Pi}_{i-1} = (\hat{\pi}_{i-1}(x_1), \cdots, \hat{\pi}_{i-1}(x_S))$ be the conditional PMF of X_{i-1} given R_i. Then the estimate of X_i^{i+j-1} given R_i and $\mathcal{Q}^j(Y_i^{i+j-1})$ can be written in terms of $\hat{\Pi}_{i-1}$ and $\mathcal{Q}^j(Y_i^{i+j-1})$. Indeed, by the Bayes rule,

$$Prob[X_i^{i+j-1}|R_i, \mathcal{Q}^j(Y_i^{i+j-1})] = \frac{Prob[X_i^{i+j-1}|R_i]Prob[\mathcal{Q}^j(Y_i^{i+j-1})|X_i^{i+j-1}, R_i]}{Prob[\mathcal{Q}^j(Y_i^{i+j-1})|R_i]},$$

and the numerator (the denominator is just a normalizing factor) equals

$$\begin{aligned}
& \sum_{s=1}^{S} Prob[X_i^{i+j-1}, X_{i-1} = x_s|R_i]Prob[\mathcal{Q}^j(Y_i^{i+j-1})|X_i^{i+j-1}, R_i] \\
= & \sum_{s=1}^{S} Prob[X_{i-1} = x_s|R_i]Prob[X_i^{i+j-1}|X_{i-1} = x_s, R_i]]Prob[\mathcal{Q}^j(Y_i^{i+j-1})|X_i^{i+j-1}, R_i] \\
= & \sum_{s=1}^{S} \hat{\pi}_{i-1}(x_s)Prob[X_i^{i+j-1}|X_{i-1} = x_s]Prob[\mathcal{Q}^j(Y_i^{i+j-1})|X_i^{i+j-1}],
\end{aligned}$$

by the Markovian property.

Therefore we can write the conditional (marginal) PMFs of X_i^{i+j-1} as

$$\begin{pmatrix} \hat{\Pi}_i \\ \vdots \\ \hat{\Pi}_{i+j-1} \end{pmatrix} = \begin{pmatrix} f_j^1(\hat{\Pi}_{i-1}, \mathcal{Q}^j(Y_i^{i+j-1}) \\ \vdots \\ f_j^j(\hat{\Pi}_{i-1}, \mathcal{Q}^j(Y_i^{i+j-1}) \end{pmatrix}, \tag{6}$$

for some functions $\{f_j^1, \cdots, f_j^j\} =: f_j$. We now work out f_j in detail for $j = 1, 2$.

Example 2.1 *For $j = 1$, we have*

$$\begin{aligned}
& \hat{\pi}_i(x_l) \stackrel{\triangle}{=} Prob[X_i = x_l|\mathcal{Q}^1(Y_i), \hat{\Pi}_{i-1}] \\
= & \frac{(\sum_{s=1}^{S} \hat{\pi}_{i-1}(x_s)a_{ls})(\sum_{m=1}^{M} \mathbf{1}(\mathcal{Q}^1(y_m) = \mathcal{Q}^1(Y_i))c_{lm})}{\sum_{t=1}^{S}(\sum_{s=1}^{S} \hat{\pi}_{i-1}(x_s)a_{ts})(\sum_{m=1}^{M} \mathbf{1}(\mathcal{Q}^1(y_m) = \mathcal{Q}^1(Y_i))c_{tm})},
\end{aligned} \tag{7}$$

where $1 \leq l \leq S$, and $\mathbf{1}(\cdot)$ is the indicator function.

Example 2.2 *For $j = 2$, we first derive*

$$\begin{aligned}
& \hat{\pi}_{i,i+1}(x_{l_1}, x_{l_2}) \stackrel{\triangle}{=} Prob[X_i = x_{l_1}, X_{i+1} = x_{l_2}|\mathcal{Q}^2(Y_i^{i+1}), \hat{\Pi}_{i-1}] \\
= & \frac{(\sum_{s=1}^{S} \hat{\pi}_{i-1}(x_s)a_{l_1 s}a_{l_2 l_1})(\sum_{m_1,m_2=1}^{M} \mathbf{1}(\mathcal{Q}^2(y_{m_1}, y_{m_2}) = \mathcal{Q}^2(Y_i^{i+1}))c_{l_1 m_1}c_{l_2 m_2})}{\sum_{t_1,t_2=1}^{S}(\sum_{s=1}^{S} \hat{\pi}_{i-1}(x_s)a_{t_1 s}a_{t_2 t_1})(\sum_{m_1,m_2=1}^{M} \mathbf{1}(\mathcal{Q}^2(y_{m_1}, y_{m_2}) = \mathcal{Q}^2(Y_i^{i+1}))c_{t_1 m_1}c_{t_2 m_2})},
\end{aligned} \tag{8}$$

where $1 \le l_1, l_2 \le S$. *Then we have*

$$\hat{\pi}_i(x_{l_1}) = \sum_{l_2=1}^{S} \hat{\pi}_{i,i+1}(x_{l_1}, x_{l_2}), \tag{9}$$

$$\hat{\pi}_{i+1}(x_{l_2}) = \sum_{l_1=1}^{S} \hat{\pi}_{i,i+1}(x_{l_1}, x_{l_2}). \tag{10}$$

Note that in Examples 2.1 and 2.2 the transition probability matrix (a_{ij}) is independent u since the control is not involved here.

Proposition 2.1 *The value functions* $\{V_i\}_{i=1}^N$ *satisfy:*

$$V_N(\Pi_{N-1}) = 1 + \min_{\mathcal{Q}^1 \in \Theta^1} H[\mathcal{Q}^1(Y_N)] + \lambda_e E[\rho(f_1^1(\Pi_{N-1}, \mathcal{Q}^1(Y_N)), \tilde{\Pi}_N)], \tag{11}$$

and for $1 \le i \le N-1$,

$$\begin{aligned} V_i(\Pi_{i-1}) = 1 + &\min_{j \in \{1,2,\cdots,\min(B,N-i+1)\}} \{\frac{j(j-1)\lambda_d}{2} + \min_{\mathcal{Q}^j \in \Theta^j} \{H[\mathcal{Q}^j(Y_i^{i+j-1})] \\ &+\lambda_e E[\sum_{n=i}^{i+j-1} \rho(f_j^{n-i+1}(\Pi_{i-1}, \mathcal{Q}^j(Y_i^{i+j-1})), \tilde{\Pi}_n) + V_{i+j}(f_j^j(\Pi_{i-1}, \mathcal{Q}^j(Y_i^{i+j-1})))]\}\}, \end{aligned} \tag{12}$$

where $V_{N+1}(\cdot) \equiv 0$.

Sketch of proof. For $i = N$, no delay is possible and one quantizes Y_N only, which leads to (11).

For $i = N-1$, one has the choice to (a) quantize Y_{N-1} alone first and then quantize Y_N based on quantized Y_{N-1}, or (b) hold on until N and quantize Y_{N-1}^N in one shot.

1. In choice (a), no delay is introduced so $J^d(N-1) = 0$, $J^d(N) = 0$. Let $\omega_{N-1} = (\mathcal{Q}_{N-1}^1, \mathcal{Q}_N^1)$ be an admissible quantization decision, where $\mathcal{Q}_{N-1}^1, \mathcal{Q}_N^1 \in \Theta^1$ are the quantization schemes for Y_{N-1} and Y_N, respectively. Then

$$\begin{aligned} J_{N-1}(\Pi_{N-2}, \omega_{N-1}) &= E[1 + H[\mathcal{Q}_{N-1}^1(Y_{N-1})] + \lambda_e \rho(\hat{\Pi}_{N-1}, \tilde{\Pi}_{N-1}) \\ &\quad + \underbrace{1 + H[\mathcal{Q}_N^1(Y_N)|\mathcal{Q}_{N-1}^1(Y_{N-1})] + \lambda_e \rho(\hat{\Pi}_N, \tilde{\Pi}_N)}_{T_1}\ ] \end{aligned} \tag{13}$$

$$\begin{aligned} &= 1 + H[\mathcal{Q}_{N-1}^1(Y_{N-1})] + E[\lambda_e \rho(\hat{\Pi}_{N-1}, \tilde{\Pi}_{N-1}) \\ &+ \underbrace{E[1 + H[\mathcal{Q}_N^1(Y_N)|\mathcal{Q}_{N-1}^1(Y_{N-1})] + \lambda_e \rho(\hat{\Pi}_N, \tilde{\Pi}_N)|\mathcal{Q}_{N-1}^1(Y_{N-1})]}_{T_2 = J_N(\hat{\Pi}_{N-1}, \mathcal{Q}_N^1)}\ ]. \end{aligned} \tag{14}$$

Note that $\hat{\Pi}_{N-1}$ depends on $\mathcal{Q}_{N-1}^1(Y_{N-1})$, and $\hat{\Pi}_N$ depends on both $\mathcal{Q}_{N-1}^1(Y_{N-1})$ and $\mathcal{Q}_N^1(Y_N)$. Rewriting the term T_1 in (13) as T_2 in (14) translates the requirement that $\mathcal{Q}_N^1$ depend only on $\mathcal{Q}_{N-1}^1(Y_{N-1})$ into an amenable form. In particular, the optimal $\mathcal{Q}_N^1$ to minimize T_2 will depend on $\mathcal{Q}_{N-1}^1(Y_{N-1})$ through $\hat{\Pi}_{N-1}$. Minimizing (14) with respect to ω_{N-1} and plugging (6) for $\hat{\Pi}_{N-1}$, we get

$$\begin{aligned} &V_{N-1}^a(\Pi_{N-2}) \\ &= 1 + \min_{\mathcal{Q}^1 \in \Theta^1} \{H[\mathcal{Q}^1(Y_{N-1})] + \lambda_e E[\rho(f_1^1(\Pi_{N-2}, \mathcal{Q}^1(Y_{N-1})), \tilde{\Pi}_{N-1}) + V_N(f_1^1(\Pi_{N-2}, \mathcal{Q}^1(Y_{N-1})))]\}. \end{aligned}$$

2. In choice (b), we have $J^d(N-1) = 1$ and $J^d(N) = 0$. Minimizing over $\mathcal{Q}^2 \in \Theta^2$, we have

$$\begin{aligned} & V^b_{N-1}(\Pi_{N-2}) \\ & = 1 + \lambda_d + \min_{\mathcal{Q}^2 \in \Theta^2} \{ H[\mathcal{Q}^2(Y^N_{N-1})] + \lambda_e E[\sum_{n=N-1}^{N} \rho(f_2^{n-N+2}(\Pi_{N-2}, \mathcal{Q}^2(Y^N_{N-1})), \tilde{\Pi}_n)]\}. \end{aligned}$$

Then $V_{N-1}(\Pi_{N-2}) = \min\{V^a_{N-1}(\Pi_{N-2}), V^b_{N-1}(\Pi_{N-2})\}$, which satisfies (12).

Similar arguments can be used to prove the cases for $i < N-2$. $\square$

Remark 2.1 *In solving (11) and (12) one also obtains the optimal quantization policy for each stage. Concatenating the optimal quantization schemes (with variable block length) yields the optimal quantization decision for our original problem (3).*

3 Numerical Results

In this section we discuss issues in numerically solving the DP equations (11) and (12), and present simulation results.

3.1 Partition enumeration

In solving the DP equations, we need to enumerate and compare all partition (encoding) schemes for the finite, discrete sets $\mathcal{Y}^j$, $1 \leq j \leq B$, where $\mathcal{Y}^j$ is the product space of j copies of $\mathcal{Y}$. Each partition for $\mathcal{Y}^j$ corresponds to an element of Θ^j. For a discrete set D, the number of partitions grow rapidly with the cardinality n_D of D. How to enumerate all partitions without repetition is an important issue since repetitions might substantially add to the computational complexity.

We have developed an effective method to eliminate all redundant partitions. The procedure consists of two steps. In the first step a tree-structured algorithm is used to find all the partition patterns. Then in the second step we remove remaining redundant partitions by making use of the "characteristic numbers". We now describe these two steps in detail.

By the *partition pattern* for a (disjoint) partition of the set D , we mean a nonincreasing sequence of positive integers where each integer corresponds to the cardinality of one cluster in the partition. For example, if we partition a 7-element set into 3 clusters, one with 3 elements and the other two with 2 elements each. Then the partition pattern is (3 2 2). Different partitions may share the same partition pattern. Our enumeration approach is to first list all partition patterns, then list all corresponding partitions for each pattern.

To list all patterns we construct a tree as follows. Each node of the tree has two properties, "current value" and "remainder". The root node has "current value"= 0 and "remainder" $= n_D$. The root has n_D children whose "current value"s are $n_D, n_D - 1, \cdots, 1$, respectively. The "remainder" of each node equals its parent node's "remainder" minus its own "current value". For a non-root node with "current value"

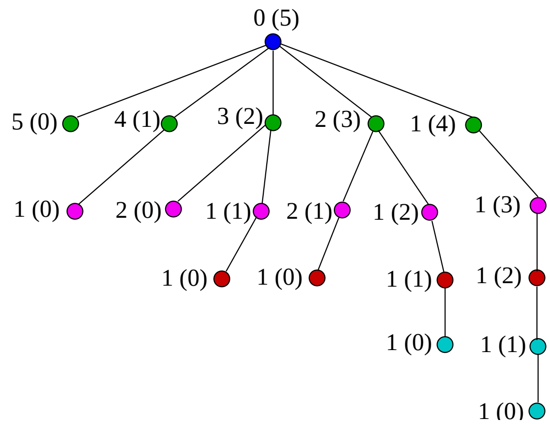


Figure 2: Tree-structured pattern generation for partitions.

i_1 and "remainder" i_2, it will have $i_0 = \min(i_1, i_2)$ children whose "current value"s are $i_0, i_0 - 1, \cdots, 1$, respectively. A node is a *leaf* if its "remainder" is 0. Every path from the root to a leaf is identified with a partition pattern if we read off the "current values" of the nodes (except the root) along the path.

Figure 2 illustrates the pattern generation for $n_D = 5$. For each node the number inside the parenthesis is its "remainder" while the number outside is its "current value". We immediately have all the patterns: (5), (4 1), (3 2), (3 1 1), (2 2 1), (2 1 1 1), (1 1 1 1 1). We note that if the "current value" of a node is 1, then either it's a leaf or it has only one child, and hence there is only one path passing this node. This observation helps reducing complexity in practice.

Given a pattern, we generate all the corresponding partitions by choosing appropriate numbers of elements from D and putting them into groups. Take the example for $n_D = 5$ and let (3 2) be the pattern. By selecting 3 elements for the first group and leaving the rest 2 elements for the second, we can list all 10 partitions for this pattern, and none of them are redundant.

However, if an integer number greater than 1 occurs more than once in a pattern, repetitive enumeration of certain partitions will occur. For the pattern (2 2 1), naive enumeration gives 20 partitions; however, a more careful investigation reveals that only 15 of them are distinct. Such redundancies can be virtually removed using the *characteristic numbers* of partitions. To be specific, we first order all elements in D and map them one-to-one to elements of $\{1, 2, \cdots, n_D\}$ (denoting the map as $\mathcal{I}$), the latter being called the *indices* of the former.

Assume that the current pattern of interest contains two i_p for some $i_p > 1$ (the method can be extended easily to the case of more than two i_p existing in the pattern). Given a partition $\mathcal{P}_1$ consistent with the pattern, we locate the two clusters C_1, C_2 that have i_p elements. We then let

$$c_1 = \sum_{d \in C_1} (\mathcal{I}(d))^3, \quad c_2 = \sum_{d \in C_2} (\mathcal{I}(d))^3. \tag{15}$$

We call the pair (c_1, c_2) (after being ordered) the characteristic numbers of the partition $\mathcal{P}_1$. By modifying the construction method of characteristic numbers if necessary (although the one in (15) works quite well for the many cases we have tested), the mapping from (C_1, C_2) to (c_1, c_2) becomes one-to-one, and we can determine easily whether the partition being enumerated has been listed before through comparison of the characteristic numbers.

In addition to redundancy removal in partitioning, we also observe that in (12) many evaluations repeat for each i. This also helps to speed up the calculation.

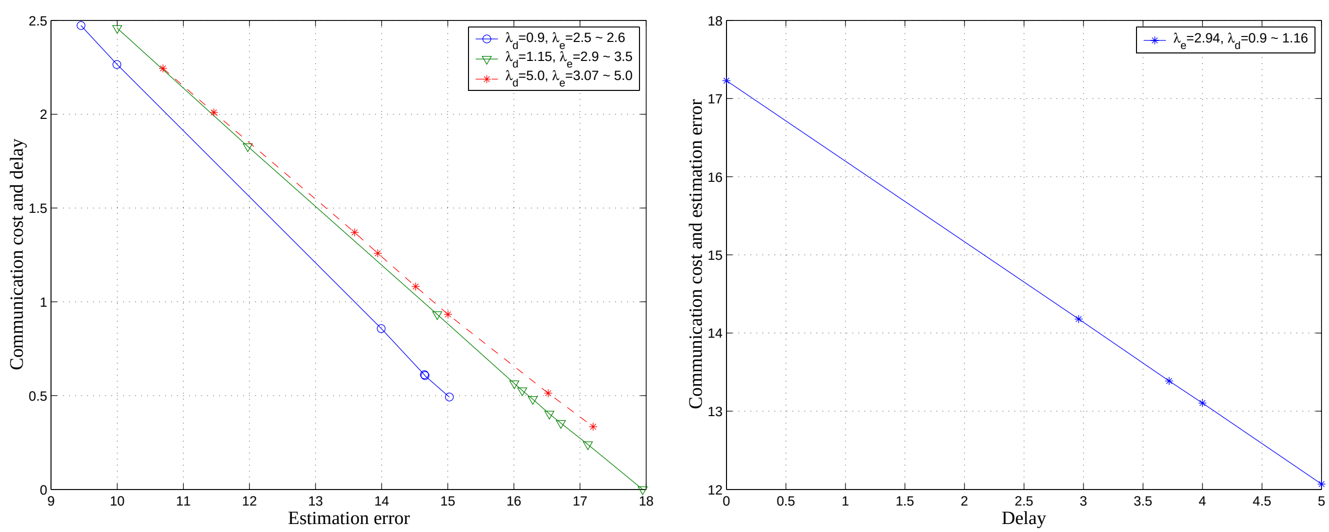
3.2 Simulation results

We have conducted calculation and simulation for a two-state, two-output hidden Markov chain. The maximum length B for block coding is 2, and $N = 10$. The matrices (a_{ij}) and (c_{ij}) we use are

$$(a_{ij}) = \begin{bmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{bmatrix}, (c_{ij}) = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix},$$

By varying the weighting constants λ_d and λ_e, we compute and store a family of optimal quantization policies. Then for the initial condition $\Pi_0 = (0.9, 0.1)$, we obtain 50 sample output trajectories by simulation. Each quantization policy is applied to these output trajectories, and the average accumulative communication cost $\bar{J}^h$, delay $\bar{J}^d$, and estimation error $\bar{J}^e$ are calculated.

In Figure 3(a), each curve shows the variation of combined communication cost and delay *vs.* the estimation error as λ_e is changed (λ_d is fixed for each curve). The vertical axis is $\bar{J}^h + \lambda_d \bar{J}^d$ and the horizontal axis is $\bar{J}^e$. We have also found that (not seen in the figure), for $\lambda_d = 5.0$, the accumulative delay cost $= 0$ (exclusively sequential quantization); for $\lambda_d = 0.9$, the accumulative delay cost $= 5.0$ (exclusively block-coding of length 2); while for $\lambda_d = 1.15$, variable-length block coding is observed. Figure 3(b) shows the variation of combined communication cost and estimation error *vs.* the delay as λ_d is changed, where the vertical axis is $\bar{J}^h + \lambda_e \bar{J}^e$ and the horizontal axis is $\bar{J}^d$. From the figures we see that jointly optimal quantization decisions vary with the weighting coefficients, and by appropriately choosing these coefficients we can achieve the desired tradeoff among different objectives.



(a) (b)

Figure 3: (a) Weighted combination of communication cost and delay *vs.* estimation error (points with lower estimation error corresponding to higher λ_e); (b) Weighted combination of communication cost and estimation error *vs.* delay (points with smaller delay corresponding to higher λ_d).

4 Joint Quantization and Control

In this section we formulate and solve the joint quantization and control problem.

4.1 Problem formulation

Consider Figure 1. We now restrict ourselves to sequential vector quantization of Y_n, i.e., at any time n, we quantize Y_n and transmit it. Denote the quantization scheme at n as $\mathcal{Q}_n \in \Theta^1$, and denote $q_n = \mathcal{Q}_n(Y_n)$. Let $\delta_n = (\mathcal{Q}_n, U_n)$. We call $\{\delta_n\}$ *jointly admissible* if U_n depends only on $\{\Pi_0, q_0^n, U_0^{n-1}\}$, and $\mathcal{Q}_n$ depends only on $\{\Pi_0, q_0^{n-1}, U_0^{n-1}\}$. Fix $N \geq 1$. We denote by Δ_0 the space of jointly admissible quantization schemes and controls from time 0 to time $N-1$.

Given the initial condition Π_0 for X_0 and $\{\delta_n\}_0^{N-1} \in \Delta_0$, we define the cost function

$$J(\Pi_0, \{\delta_n\}_0^{N-1}) = E[\sum_{n=0}^{N-1} \lambda_c J^c(n) + J^p(n)], \tag{16}$$

and the value function

$$V(\Pi_0) = \min_{\{\delta_n\}_0^{N-1} \in \Delta_0} J(\Pi_0, \{\delta_n\}_0^{N-1}), \tag{17}$$

where $\lambda_c \geq 0$ is a weighting constant, and $J^c(n)$ and $J^p(n)$ are the costs relating to communication and performance at time n, respectively. $J^c(n)$ takes the form of $J^h(n)$ in Section 2 if entropy coding for q_n is used, and $J^c(n) = log_2|q_n|$ if a plain coding for q_n is used, where $|q_n|$ denotes the number of possible outcomes of q_n. In the following we let $J^c(n) = h(q_n)$ for some suitable function $h(\cdot)$. We assume that $J^p(n)$ depends on the state and the control, $J^p(n) = g_n(X_n, U_n)$, for some function $g_n(\cdot, \cdot)$.

4.2 The dynamic programming equation

As in solving the joint quantization and estimation problem, we first define a sequence of joint quantization and control problems. For $0 \leq i \leq N-1$, we let

$$J_i(\Pi_i, \{\delta_n\}_i^{N-1}) = E[\sum_{n=i}^{N-1} \lambda_c h(\mathcal{Q}_n(Y_n)) + g_n(X_n, U_n)], \tag{18}$$

and

$$V_i(\Pi_i) = \min_{\{\delta_n\}_i^{N-1} \in \Delta_i} J_i(\Pi_i, \{\delta_n\}_i^{N-1}), \tag{19}$$

where Δ_i is the space of jointly admissible quantization schemes and controls from time i to time $N-1$, and Π_i is the initial condition for the i-th problem.

We denote by $\bar{\Pi}_i = \{\bar{\pi}_i(x_1), \cdots, \bar{\pi}_i(x_S)\}$ the conditional PMF of X_i given Π_0, q_0^{i-1} (and the corresponding quantization schemes), and U_0^{i-1}. We can derive a recursive formula for $\bar{\Pi}_i$,

$$\bar{\Pi}_{i+1} = \bar{f}(\bar{\Pi}_i, q_i, U_i), \tag{20}$$

for some function $\bar{f}(\cdot,\cdot,\cdot)$. To be specific, for $1 \le l \le S$,

$$\bar{\pi}_{i+1}(x_l) = \frac{\sum_{s=1}^{S} a_{ls}(U_i)\bar{\pi}_i(x_s)(\sum_{m=1}^{M} \mathbf{1}(\mathcal{Q}_i(y_m) = q_i)c_{sm})}{\sum_{t=1}^{S} \bar{\pi}_i(x_t)(\sum_{m=1}^{M} \mathbf{1}(\mathcal{Q}_i(y_m) = q_i)c_{tm})}. \tag{21}$$

Proposition 4.1 *For $\mathcal{Q} \in \Theta^1$, we denote by $\mathcal{A}^{\mathcal{Q}}$ the space of functions mapping the range of $\mathcal{Q}$ to $\mathcal{U}$. The value functions $\{V_i\}_{i=0}^{N-1}$ satisfy:*

$$V_{N-1}(\Pi_{N-1}) = \min_{\mathcal{Q}_{N-1}\in\Theta^1} \min_{\alpha_{N-1}\in\mathcal{A}^{\mathcal{Q}_{N-1}}} E[\lambda_c h(q_{N-1}) + E[g_{N-1}(X_{N-1}, \alpha_{N-1}(q_{N-1}))|q_{N-1}]\ ], \tag{22}$$

where $q_{N-1} = \mathcal{Q}_{N-1}(Y_{N-1})$, and for $0 \le i \le N-2$,

$$V_i(\Pi_i) = \min_{\mathcal{Q}_i\in\Theta^1} \min_{\alpha_i\in\mathcal{A}^{\mathcal{Q}_i}} E[\lambda_c h(q_i) + E[g_i(X_i, \alpha_i(q_i))|q_i] + V_{i+1}(\bar{f}(\Pi_i, q_i, \alpha_i(q_i)))], \tag{23}$$

where $q_i = \mathcal{Q}_i(Y_i)$. From the solutions $\{(\mathcal{Q}_i^*, \alpha_i^*)\}_{i=0}^{N-1}$ to (22) and (23) one can construct the jointly optimal quantization and control schemes.

Sketch of proof. We sketch the proof for the cases $i = N-1$ and $i = N-2$, and the rest can be proved analogously.

For $i = N-1$, we have

$$J(\Pi_{N-1}, (\mathcal{Q}_{N-1}, U_{N-1})) = E[\lambda_c h(q_{N-1}) + g_{N-1}(X_{N-1}, U_{N-1})] \tag{24}$$

$$= E[\lambda_c h(q_{N-1}) + E[g_{N-1}(X_{N-1}, U_{N-1})|q_{N-1}]\ ]. \tag{25}$$

Since U_{N-1} is measurable with respect to q_{N-1}, it can be expressed as $\alpha_{N-1}(q_{N-1})$ for some $\alpha_{N-1} \in \mathcal{A}^{\mathcal{Q}_{N-1}}$. Minimizing over $\mathcal{Q}_{N-1}$ and α_{N-1} thus gives (22).

For $i = N-2$, we first rewrite

$$\begin{aligned} J(\Pi_{N-2}, \{\delta_n\}_{N-2}^{N-1}) = \quad & E[\lambda_c h(q_{N-2}) + E[g_{N-2}(X_{N-2}, U_{N-2})|q_{N-2}] + \\ & \underbrace{E[\lambda_c h(q_{N-1}) + g_{N-1}(x_{N-1}, U_{N-1})|q_{N-2}, U_{N-2}]}_{T_3 = J_{N-1}(\bar{f}(\Pi_{N-2}, q_{N-2}, U_{N-2}), (\mathcal{Q}_{N-1}, U_{N-1}))}\ ]. \end{aligned} \tag{26}$$

Since U_{N-2} depends only on q_{N-2}, we have $U_{N-2} = \alpha_{N-2}(q_{N-2})$ for some $\alpha_{N-2} \in \mathcal{A}^{\mathcal{Q}_{N-2}}$. Then it's not hard to see that minimization of (26) leads to (23). □

Remark 4.1 *By rewriting the cost functions as in (25) and (26), we express the cost in terms of the "observables" to the receiver. As in the joint quantization/estimation problem, the conditional PMF $\bar{\Pi}_n$ of X_n turns out to be the information state (or the sufficient statistic), and it determines the optimal quantization/control scheme.*

4.3 An example problem

We take the machine repair problem from [12] (pp. 190) as an example. A machine can be in one of two states denoted by P (*Proper* state) and $\bar{P}$ (*Improper* state). If the machine starts in P and runs for one period, its new state will remain P with probability $\frac{2}{3}$, and if it starts in $\bar{P}$, it will remain in $\bar{P}$ with probability 1. At the beginning of each time period, one take an inspection to help determine the machine's state. There are two possible inspection outcomes denoted G (*Good*) and B (*Bad*). If the machine is in P, the inspection outcome is G with probability $\frac{3}{4}$; if the machine is in $\bar{P}$, the inspection outcome is B with probability $\frac{3}{4}$.

After each inspection, one of two possible actions can be taken, C (operate the machine for one period) or S (stop the machine and perform maintenance, then operate the machine for one period). The running cost for one period is 2 units if the machine is in state $\bar{P}$, and is 0 if it is in P. The action S makes sure that the machine is in P but it costs 1 unit.

To relate this problem to the joint quantization/control problem discussed earlier, we assume that the inspection outcome needs to be sent to a remote site for action decision. A plain coding scheme requires one bit to send the information G or B. The only other quantization scheme for $\{G, B\}$ is to cluster these two outcomes, in which case no information is sent and 0 bit is required for communication. The constant λ_c now carries an interpretation of communication cost per bit. A different interpretation of λ_c would be the cost for each inspection if we assume that the inspection is optional.

Given an initial condition of the machine, the problem is to decide at the beginning of each time period whether to communicate the inspection outcome and what action to take based on the received information, so that the total cost is minimized. One can show that the value function of this problem is concave and piecewise linear. We have obtained the explicit solution for $N = 2$. Now let $Prob[X_0 = P] = \frac{2}{3}$. Then one of the following four joint quantization/control strategies becomes optimal depending on the value λ_c:

(a) At time 0, send the inspection outcome Y_0, and let $U_0 = C(S,\ \text{resp.})$ if $Y_0 = G(B,\ \text{resp.})$; At time 1, send Y_1, and let $U_1 = C(S,\ \text{resp.})$ if $Y_1 = G(B,\ \text{resp.})$;

(b) At time 0, send Y_0, and let $U_0 = C(S,\ \text{resp.})$ if $Y_0 = G(B,\ \text{resp.})$; At time 1, if $Y_0 = G$, send Y_1 and let $U_1 = C(S,\ \text{resp.})$ if $Y_1 = G(B,\ \text{resp.})$, and if $Y_0 = B$, let $U_1 = C$ without transmitting Y_1;

(c) At time 0, send Y_0, and let $U_0 = C(S,\ \text{resp.})$ if $Y_0 = G(B,\ \text{resp.})$; At time 1, let $U_1 = C$ without transmitting Y_1;

(d) At time 0, let $U_0 = C$ without transmitting Y_0; at time 1, let $U_1 = S$ without transmitting Y_1.

The optimal strategy is

$$\begin{cases} (a) \text{ if } \lambda_c \leq \frac{1}{12} \\ (b) \text{ if } \frac{1}{12} < \lambda_c \leq \frac{5}{28} \\ (c) \text{ if } \frac{5}{28} \leq \lambda_c < \frac{11}{36} \\ (d) \text{ if } \lambda_c \geq \frac{11}{36} \end{cases}.$$

In Figure 4, we show the expected accumulative running and maintenance cost *vs.* the expected bits of communication for these four strategies. The thresholds of λ_c for switching of the optimal strategy correspond to the negative slopes of the line segments connecting the neighboring points in Figure 4. Hence when the communication cost per bit increases, the optimal strategy tends not to transmit the inspection outcome.

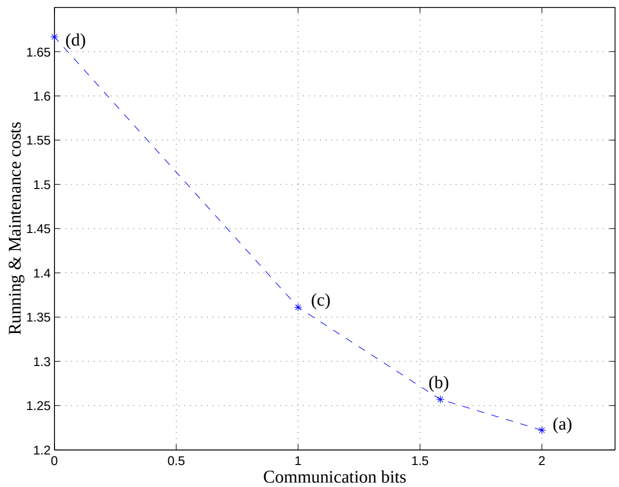


Figure 4: Running and maintenance costs *vs.* communication bits for jointly optimal strategies.

5 Conclusions and Future Work

In this paper we have studied the problem of joint quantization, estimation, and control of a hidden Markov chain. We first investigated the joint quantization and estimation problem, where vector quantization with variable-block length was considered. Then we formulated and solved the joint quantization and control problem. The common theme for these two problems is that a weighted combination of different costs is minimized. By varying the weighting coefficients, one can obtain a family of optimal quantization/control schemes that reflect different tradeoff strategies. Simulation and a simple example have been used to illustrate the results.

The framework presented in this paper can be extended to continuous-range systems. It is also possible to incorporate the case of noisy communication. Ongoing work involves joint consideration of quantization of the control input.

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