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PREPRINTS

The asymptotic consensus problem on convex metric spaces ⋆

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Abstract: We consider the consensus problem of a group of dynamic agents whose communication network is modeled by a directed time-varying graph. In this paper we generalize the asymptotic consensus problem to convex metric spaces. A convex metric space is a metric space endowed with a convex structure. Using this convex structure we define convex sets and in particular the convex hull of a (finite) set. Under minimal connectivity assumptions, we show that if at each iteration an agent updates its state by choosing a point from a particular subset of the convex hull generated by the agent's current state and the states of its neighbors, then asymptotic agreement is achieved. In addition, we give bounds on the distance between the consensus point and the initial values of the agents.

Keywords: Consensus, metric spaces, convexity.

1. INTRODUCTION

A consensus problem consists of a group of dynamic agents who seek to agree upon certain quantities of interest by exchanging information among them according to a set of rules. This problem can model many phenomena involving information exchange between agents such as cooperative control of vehicles, formation control, flocking, synchronization, parallel computing, etc. Distributed computation over networks has a long history in control theory starting with the work of Borkar and Varaya (1982), Tsitsiklis (1984) and Tsitsiklis et al. (1986) on asynchronous agreement problems and parallel computing. A theoretical framework for solving consensus problems was introduced by Saber and Murray (2003, 2004), while Jadbabaie et al. (2004) studied alignment problems for reaching an agreement. Relevant extensions of the consensus problem were done by Ren and Beard (2005), by Moreau (2005) or, more recently, by Nedic and Ozdaglar (2008); Nedic et al. (to appear).

Typically agents are connected via a network that changes with time due to link failures, packet drops, node failure, etc. Such variations in topology can happen randomly which motivates the investigation of consensus problems under a stochastic framework. Hatano and Mesbahi (2005) consider an agreement problem over random information networks, where the existence of an information channel between a pair of elements at each time instance is probabilistic and independent of other channels. Porfiri and Stilwell (2007) provide sufficient conditions for reaching consensus almost surely in the case of a discrete linear system, where the communication flow is given by a directed graph derived from a random graph process, independent of other time instances. Under a similar model of the communication topology, Salehi and Jadbabaie (2008) give necessary and sufficient conditions for almost sure convergence to consensus , while Salehi and Jadbabaie (2010) extend the applicability of their necessary and sufficient conditions to strictly stationary ergodic random graphs. Extensions to the case where the random graph modeling the communication among agents is a Markovian random process are given by Matei et al. (2008) and Matei et al. (2009).

A convex metric space is a metric space endowed with a convex structure. The main goal of this paper is to generalize the asymptotic consensus problem to the more general case of convex metric spaces and emphasize the fundamental role of convexity and in particular of the convex hull of a finite set of points. Tsitsiklis (1984) showed that, under some minimal connectivity assumptions on the communication network, if an agent updates its value by choosing a point (in $\mathbb{R}^n$) from the (interior) of the convex hull of its current value and the current values of its neighbors, then asymptotic convergence to consensus is achieved. We will show that this idea extends naturally to the more general case of convex metric spaces.

Our main contributions are as follows. *First*, after citing relevant results concerning convex metric spaces, we study the properties of the distance between two points belonging to two, possible overlapping convex hulls of two finite sets of points. These properties will prove to be crucial in proving the convergence of the agreement algorithm. *Second*, we provide a dynamic equation for an upper bound of the vector of distances between the current values of the agents. We show that the agents asymptotically reach agreement, by showing that this upper bound asymptotically converges to zero. *Third*, we characterize the agreement point compared to the initial values of the agents, be giving upper bounds on the distance between the agreement point and the initial values in terms of the distances between the initial values of the agents.

The paper is organized as follows. Section 2 introduces the main concepts about convex metric spaces. Section 3 formulates the problem addressed in this note followed by the statement of our main results in Section 4. Section 5 presents a set

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of auxiliary results regarding properties of convex hulls in a convex metric space and regarding the time evolution of the distances between the states of the agents under a particular state update scheme. These auxiliary results are used to prove our main result in Section 6.

Some basic notations: Given $W \in \mathbb{R}^{n\times n}$, by $[W]_{ij}$ we refer to the (i, j) element of the matrix. The *underlying graph* of W is a graph of order n for which every edge corresponds to a non-zero, non-diagonal entry of W. We will denote by $\mathbb{1}_{\{A\}}$ the indicator function of event A. Given some space $\mathcal{X}$ we denote by $\mathcal{P}(\mathcal{X})$ the set of all subsets of $\mathcal{X}$.

We would like to point out that this note does not contain all the proofs of the results. For the missing proofs, the reader is invited to consult the extended version of this note given by the reference Matei and Baras (2010).

2. DEFINITIONS AND RESULTS ON CONVEX METRIC SPACES

In this subsection we first present a set of definitions and basic results about convex metric spaces. For more details about the following definitions and results the reader is invited to consult Sharma and Dewangan (1995),Takahashi (1970). Next, we introduced the convex hull notion of a finite set of points belonging to a convex metric space.

2.1 Main definitions

Definition 1. Let $(\mathcal{X}, d)$ be a metric space. A mapping $\psi : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ is said to be a *convex structure* on $\mathcal{X}$ if

$$d(u, \psi(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \tag{1}$$

for all $x, y, u \in \mathcal{X}$ and for all $\lambda \in [0, 1]$.

Definition 2. The metric space $(\mathcal{X}, d)$ together with the convex structure ψ is called *convex metric space*.

A Banach space and each of its subsets are convex metric space. There are examples of convex metric spaces not embedded in any Banach space. The following two examples are taken from Takahashi (1970).

Example 1. Let I be the unit interval $[0, 1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and $\lambda \in I$, we define a mapping ψ by $\psi(I_i, I_j, \lambda) = [\lambda a_i + (1 - \lambda)a_j, \lambda b_i + (1 - \lambda)b_j]$ and define a metric d in X by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a\in I}\{|\inf_{b\in I_i}\{|a - b|\} - \inf_{c\in I_j}\{|a - c|\}|\}.$$

Example 2. We consider a linear space L which is also a metric space with the following properties:

(a) For $x, y \in L$, $d(x, y) = d(x - y, 0)$;
(b) For $x, y \in L$, and $\lambda \in [0, 1]$,

$$d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0).$$

Hence L, together with the convex structure $\psi(x, y, \lambda) = \lambda x + (1 - \lambda)y$, is a convex metric space.

Definition 3. Let $\mathcal{X}$ be a convex metric space. A nonempty subset $K \subset \mathcal{X}$ is said to be *convex* if $\psi(x, y, \lambda) \in K$, $\forall x, y \in K$ and $\forall \lambda \in [0, 1]$.

We define the set valued mapping $\tilde{\psi} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ as

$$\tilde{\psi}(A) \triangleq \{\psi(x, y, \lambda) \mid \forall x, y \in A, \forall \lambda \in [0, 1]\}, \tag{2}$$

where A is an arbitrary set in $\mathcal{X}$.

In Takahashi (1970) it is shown that, in a convex metric space, an arbitrary intersection of convex sets is also convex and therefore the next definition makes sense.

Definition 4. The *convex hull* of the set $A \subset \mathcal{X}$ is the intersection of all convex sets in $\mathcal{X}$ containing A and is denoted by *conv*(A).

Another characterization of the convex hull of a set in $\mathcal{X}$ is given in what follows. By defining $A_m \triangleq \tilde{\psi}(A_{m-1})$ with $A_0 = A$ for some $A \subset \mathcal{X}$, it is discussed in Sharma and Dewangan (1995) that the set sequence $\{A_m\}_{m\geq 0}$ is increasing and $\limsup A_m$ exits, and $\limsup A_m = \liminf A_m = \lim A_m = \bigcup_{m=0}^{\infty} A_m$.

Proposition 1. (Sharma and Dewangan (1995)). Let $\mathcal{X}$ be a convex metric space. The convex hull of a set $A \subset \mathcal{X}$ is given by

$$conv(A) = \lim A_m = \bigcup_{m=0}^{\infty} A_m. \tag{3}$$

It follows immediately from above that if $A_{m+1} = A_m$ for some m, then $conv(A) = A_m$.

2.2 On the convex hull of a finite set

For a positive integer n, let $A = \{x_1, \ldots, x_n\}$ be a finite set in $\mathcal{X}$ with convex hull *conv*(A) and let z belong to *conv*(A). By Proposition 1 it follows that there exits a positive integer m such that $z \in A_m$. But since $A_m = \tilde{\psi}(A_{m-1})$ it follows that there exits $z_1, z_2 \in A_{m-1}$ and $\lambda_{(1,2)} \in [0, 1]$ such that $z = \psi(z_1, z_2, \lambda_{(1,2)})$. Similarly, there exits $z_3, z_4, z_5, z_6 \in A_{m-2}$ and $\lambda_{(3,4)}, \lambda_{(5,6)} \in [0, 1]$ such that $z_1 = \psi(z_3, z_4, \lambda_{(3,4)})$ and $z_2 = \psi(z_5, z_6, \lambda_{(5,6)})$. By further decomposing z_3,z_4,z_5 and z_6 and their followers until they are expressed as functions of elements of A and using a graph theory terminology, we note the z can be viewed as the root of a weighted binary tree with leaves belonging to the set A. Each node α (except the leaves) has two children α_1 and α_2, and are related through the operator ψ in the sense $\alpha = \psi(\alpha_1, \alpha_2, \lambda)$ for some $\lambda \in [0, 1]$. The weights of the edges connecting α with α_1 and α_2 are given by λ and $1 - \lambda$ respectively.

From the above discussion we note that for any point $z \in conv(A)$ there exits a non-negative integer m such that z is the root of a binary tree of height m, and has as leaves elements of A. The binary tree rooted at z may or may not be a *perfect binary tree*, i.e. a full binary tree in which all leaves are at the same depth. That is because on some branches of the tree the points in A are reached faster than on others. Let n_i denote the number of times x_i appears as a leaf node, with $\sum_{i=1}^{n} n_i \leq 2^m$ and let m_{i_l} be the length of the i_l^{th} path from the root z to the node x_i, for $l = 1 \ldots n_i$. We formally describe the paths from the root z to x_i as the set

$$P_{z,x_i} \triangleq \left\{\left(\{y_{i_l,j}\}_{j=0}^{m_{i_l}}, \{\lambda_{i_l,j}\}_{j=1}^{m_{i_l}}\right) \mid l = 1 \ldots n_i\right\}, \tag{4}$$

where $\{y_{i_l j}\}_{j=0}^{m_{i_l}}$ is the set of points forming the $i_l{}^{th}$ path, with $y_{i_l,0} = z$ and $y_{i_l,m_{i_l}} = x_i$ and where $\{\lambda_{i_l,j}\}_{j=1}^{m_{i_l}}$ is the set of weights corresponding to the edges along the paths, in particular $\lambda_{i_l,j}$ being the weight of the edge $(y_{i_l,j-1}, y_{i_l,j})$. We define the aggregate weight of the paths from root z to node x_i as

$$\mathcal{W}(P_{z,x_i}) \triangleq \sum_{l=1}^{n_i} \prod_{j=1}^{m_{i_l}} \lambda_{i_l,j}. \tag{5}$$

It is not difficult to note that all the aggregate weights of the paths from the root z to the leaves $\{x_1, \ldots, x_n\}$ sum up to one, i.e.

$$\sum_{i=1}^{n} \mathcal{W}(P_{z,x_i}) = 1.$$

Definition 5. We say that a point z belongs to the *interior* of *conv*(A) and we denote this by $z \in int(conv(A))$, if all elements of A belong to the set of leaves of the binary tree rooted at z.

Definition 6. Given a small enough positive scalar $\underline{\lambda} < 1$ we define the following subset of *int*(*conv*(A)) consisting of all points in *int*(*conv*(A)) whose aggregate weights are lower bounded by $\underline{\lambda}$, i.e.

$$C_{\underline{\lambda}}(A) \triangleq \{z \mid z \in int(conv(A)), \mathcal{W}(P_{z,x_i}) \geq \underline{\lambda},\ \forall x_i \in A\}. \tag{6}$$

Remark 1. By a *small enough* value of $\underline{\lambda}$ we understand a value such that the inequality $\mathcal{W}(P_{z,x_i}) \geq \underline{\lambda}$ is satisfied for all i. Obviously, for n agents $\underline{\lambda}$ needs to satisfy

$$\underline{\lambda} \leq \frac{1}{n},$$

but usually we would want to choose a value much smaller then $1/n$ since this implies a reacher set $C_{\underline{\lambda}}(A)$.

3. PROBLEM FORMULATION

We consider a convex metric space $(\mathcal{X}, d)$ and a set of n agents indexed by i which take values on $\mathcal{X}$. Denoting by k the time index, the agents exchange information based on a communication network modeled by a time varying, directed graph $G(k) = (V, E(k))$, where V is the finite set of vertices (the agents) and $E(k)$ is the set of edges. An edge (communication link) $e_{ij}(k) \in E(k)$ exists if node i receives information from node j. Each agent has an initial value in $\mathcal{X}$. At each subsequent time-slot is adjusting its value based on the observations about the values of its neighbors. The goal of the agents is *to asymptotically agree on the same value*. In what follows we denote by $x_i(k) \in \mathcal{X}$ the value or *state* of agent i at time k.

Definition 7. We say that the agents asymptotically reach *consensus* (or agreement) if

$$\lim_{k\to\infty} d(x_i(k), x_j(k)) = 0,\ \forall i, j,\ i \neq j. \tag{7}$$

Similar to the communication models used by Tsitsiklis et al. (1986), Blondel et al. (2005) and Nedic and Ozdalgar (2009), we impose minimal assumptions on the connectivity of the communication graph $G(k)$. Basically these assumption consists in having the communication graph connected *infinitely often* and having *bounded intercommunication interval* between neighboring nodes.

Assumption 1. (Connectivity). The graph (V, E_∞) is connected, where E_∞ is the set of edges (i, j) representing agent pairs communicating directly infinitely many times, i.e.,

$$E_\infty = \{(i, j) \mid (i, j) \in E(k) \text{ for infinitely many indices } k\}$$

Assumption 2. (Bounded intercommunication interval). There exists an integer $B \geq 1$ such that for every $(i, j) \in E_\infty$ agent j sends his/her information to the neighboring agent i at least once every B consecutive time slots, i.e. at time k or at time $k + 1$ or ... or (at latest) at time $k + B - 1$ for any $k \geq 0$.

Assumption 2 is equivalent to the existence of an integer $B \geq 1$ such that

$$(i, j) \in E(k) \cup E(k + 1) \cup \ldots \cup E(k + B - 1),\ \forall k, \forall (i, j) \in E_\infty.$$

Let $\mathcal{N}_i(k)$ denote the communication neighborhood of agent i, which contains all nodes sending information to i at time k, i.e. $\mathcal{N}_i = \{j \mid e_{ji}(k) \in E(k)\} \cup \{i\}$, which by convention contains the node i itself. We denote by $A_i(k) \triangleq \{x_j(k), \forall j \in \mathcal{N}_i(k)\}$ the set of the states of agent i's neighbors (its own included).

4. STATEMENT OF THE MAIN RESULT

The following theorem states our main result regarding the asymptotic agreement problem on metric convex space.

Theorem 1. Let Assumptions 1 and 2 hold for $G(k)$ and let $\underline{\lambda} < 1$ be a positive scalar sufficiently small. If agents update their state according to the scheme

$$x_i(k + 1) \in C_{\underline{\lambda}}(A_i(k)),\ \forall i, \tag{8}$$

then they asymptotically reach consensus, i.e.

$$\lim_{k\to\infty} d(x_i(k), x_j(k)) = 0,\ \forall i, j,\ i \neq j. \tag{9}$$

Remark 2. We can iteratively generate points for which we can make sure that they belong to the interior of the convex hull of a finite set $A = \{x_1, \ldots, x_n\}$. Given a set of positive scalars $\{\lambda_1, \ldots, \lambda_{n-1}\} \in (0, 1)$, consider the iteration

$$y_{i+1} = \psi(y_i, x_{i+1}, \lambda_i) \text{ for } i = 1 \ldots n - 1 \text{ with } y_1 = x_1. \tag{10}$$

It is not difficult to note that y_n is guaranteed to belong to the interior of *conv*(A). In addition, if we impose the condition

$$\underline{\lambda}^{\frac{1}{n-1}} \leq \lambda_i \leq \frac{1 - (n - 1)\underline{\lambda}}{1 - (n - 2)\underline{\lambda}},\ i = 1 \ldots n - 1, \tag{11}$$

and $\underline{\lambda}$ respects the inequality

$$\underline{\lambda}^{\frac{1}{n-1}} \leq \frac{1 - (n - 1)\underline{\lambda}}{1 - (n - 2)\underline{\lambda}}, \tag{12}$$

then $y_n \in C_{\underline{\lambda}}(A)$. We should note that for any $n \geq 2$ we can find a small enough value of $\underline{\lambda}$ such that inequality (12) is satisfied.

Remark 3. We would like to point out that the result refers strictly to the convergence of the distances between states and not to the convergence of the states themselves. It may be the case that the sequences $\{x_i(k)\}_{k\geq 0}$ $i = 1 \ldots n$ do not have a limit and still the distances $d(x_i(k), x_j(k))$ decrease to zero as k goes to infinity. In other words the agents asymptotically agree on the same value which may be very well variable. However, as stated in the next corollary this is not the case and in fact the states of the agents do converge to the same value.

Corollary 1. Let Assumptions 1 and 2 hold for $G(k)$ and let $\underline{\lambda} < 1$ be a positive scalar sufficiently small. If agents update their state according to the scheme

$$x_i(k + 1) \in C_{\underline{\lambda}}(A_i(k)),\ \forall i, \tag{13}$$

then there exists $x^* \in \mathcal{X}$ such that

$$\lim_{k\to\infty} d(x_i(k), x^*) = 0,\ \forall i. \tag{14}$$

Remark 4. A procedure for generating points for which is guaranteed to belong to $C_{\underline{\lambda}}(A_i(k))$ is described in Remark 2. The idea of picking $x_i(k+1)$ from $C_{\underline{\lambda}}(A_i(k))$ rather than $int(conv(A_i(k)))$ is in the same spirit as the assumption imposed on the non-zero consensus weights in Tsitsiklis (1984), Nedic and Ozdalgar (2009), Blondel et al. (2005), i.e. they are assumed lower bounded by a positive, sub-unitary scalar. Setting $x_i(k+1) \in int(conv(A_i(k)))$ may not necessarily guarantee asymptotic convergence to consensus. Indeed, consider the case where $\mathcal{X} = \mathbb{R}$ with the standard Euclidean distance. A convex structure on $\mathbb{R}$ is given by $\psi(x, y, \lambda) = \lambda x + (1-\lambda)y$, for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. Assume that we have two agents which exchange information at all time slots and therefore $A_1(k) = \{x_1(k), x_2(k)\}$, $A_2(k) = \{x_1(k), x_2(k)\}$, $\forall k \geq 0$. Let $x_1(k+1) = \lambda(k)x_1(k) + (1-\lambda(k))x_2(k)$, where $\lambda(k) = 1 - 0.1e^{-k}$ and let $x_2(k+1) = \mu(k)x_1(k) + (1-\mu(k))x_2(k)$, where $\mu(k) = 0.1e^{-k}$. Obviously, $x_i(k+1) \in int(conv(A_i(k)))$, $i = 1, 2$ for all $k \geq 0$. It can be easily argued that

$$d(x_1(k+1), x_2(k+1)) \leq [\lambda(k)(1-\mu(k)) + \mu(k)(1-\lambda(k))]\, d(x_1(k), x_2(k)). \tag{15}$$

We note that $\lim_{K\to\infty} \prod_{k=0}^{K} (\lambda(k)(1-\mu(k)) + (1-\lambda(k))\mu(k)) = \lim_{K\to\infty} \prod_{k=0}^{K}(1 - 0.2e^{-k} + 0.02e^{-2k}) = 0.73$ and therefore under inequality (15) asymptotic convergence to consensus is not guaranteed. In fact it can be explicitly shown that the agents do not reach consensus. From the dynamic equation governing the evolution of $x_i(k)$, $i = 1, 2$, we can write

$$\mathbf{x}(k+1) = \begin{pmatrix} \lambda(k) & 1-\lambda(k) \\ \mu(k) & 1-\mu(k) \end{pmatrix} \mathbf{x}(k),\ \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x}(k)^T = [x_1(k), x_2(k)]$, and we obtain that

$$\lim_{k\to\infty} \mathbf{x}(k) = \begin{pmatrix} 0.8540 & 0.1451 \\ 0.1451 & 0.8540 \end{pmatrix} \mathbf{x}_0$$

and therefore it can be easily seen that consensus is not reached from any initial state.

5. AUXILIARY RESULTS

This section is divided in two parts. In the first part we study the properties of the distance between two points belonging to two convex hulls of two (possibly overlapping) finite sets of points (Proposition 2). In the second part, under a particular state update scheme, we show that the entries of the vector of distances between the states of the agents at time $k+1$ are upper bounded by linear combinations of the entries of the same vector but at time k. The coefficients of the linear combinations are the entries of a time varying matrix for which we prove a number of properties (Lemma 1). In addition, we analyze the properties of the transition matrix of the aforementioned time varying matrix (Lemma 2).

The next result characterizes the distance between two points $x, y \in \mathcal{X}$ belonging to the convex hulls of two (possibly overlapping) finite sets X and Y.

Proposition 2. Let $X = \{x_1, \ldots, x_{n_x}\}$ and $Y = \{y_1, \ldots, y_{n_y}\}$ be two finite sets on $\mathcal{X}$ and let $\underline{\lambda} < 1$ be a positive scalar small enough.

(a) If $x \in int(conv(X))$ and $y \in \mathcal{X}$ then

$$d(x, y) \leq \sum_{i=1}^{n_x} \lambda_i d(x_i, y), \tag{16}$$

for some $\lambda_i > 0$ with $\sum_{i=1}^{n_x} \lambda_i = 1$.

(b) If $x \in int(conv(X))$ and $y \in int(conv(Y))$ then

$$d(x, y) \leq \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \lambda_{ij} d(x_i, y_j), \tag{17}$$

for some $\lambda_{ij} > 0$ with $\sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \lambda_{ij} = 1$.

(c) If $x \in C_{\underline{\lambda}}(X)$, $y \in C_{\underline{\lambda}}(Y)$, then

$$\lambda_i \geq \underline{\lambda} \text{ and } \lambda_{ij} \geq \underline{\lambda}^2,\ \forall\, i, j, \tag{18}$$

where λ_i and λ_{ij} where introduced in part (a) and part (b), respectively.

(d) If $x \in C_{\underline{\lambda}}(X)$, $y \in C_{\underline{\lambda}}(Y)$ and $X \cap Y \neq \emptyset$, then

$$\sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \lambda_{ij} \mathbb{1}_{\{d(x_i, y_j) \neq 0\}} \leq 1 - \underline{\lambda}^2, \tag{19}$$

where λ_{ij} were introduced in part (b).

Proof. See the proof of Proposition 2.2 of Matei and Baras (2010).

5.1 On the time evolution of the distances between the states of the agents

The next result characterizes the dynamics of an upper bound for the vector of distances between the agents, under a particular state update scheme.

Lemma 1. Given a small enough positive scalar $\underline{\lambda} < 1$, assume that agents update their states according to the scheme $x_i(k+1) \in C_{\underline{\lambda}}(A_i(k))$, for all i. Let $\mathbf{d}(k) \triangleq (d(x_i(k), x_j(k)))$ for $i \neq j$ be the N dimensional vector of all distances between the states of the agents, where $N = \frac{n(n-1)}{2}$. Then we obtain that

$$\mathbf{d}(k+1) \leq \mathbf{W}(k)\mathbf{d}(k),\ \mathbf{d}(0) = \mathbf{d}_0, \tag{20}$$

where the inequality is meant componentwise and the $N \times N$ dimensional matrix $\mathbf{W}(k)$ has the following properties:

(a) $\mathbf{W}(k)$ is non-negative and there exits a positive scalar $\eta \in (0, 1)$ such that

$$[\mathbf{W}(k)]_{\bar{i}\bar{i}} \geq \eta,\ \forall\, \bar{i}, k \tag{21}$$

$$[\mathbf{W}(k)]_{\bar{i}\bar{j}} \geq \eta,\ \forall\, [\mathbf{W}(k)]_{\bar{i}\bar{j}} \neq 0,\ \bar{i} \neq \bar{j},\ \forall\, k. \tag{22}$$

(b) If $\mathcal{N}_i(k) \cap \mathcal{N}_j(k) \neq \emptyset$, then the row $\bar{i}$ of matrix $\mathbf{W}(k)$, corresponding to the pair of agents (i, j), has the property

$$\sum_{\bar{j}=1}^{N} [\mathbf{W}(k)]_{\bar{i}\bar{j}} \leq 1 - \eta, \tag{23}$$

where η is the same as in part (a).

(c) If $\mathcal{N}_i(k) \cap \mathcal{N}_j(k) = \emptyset$ then the row $\bar{i}$ corresponding to the pair of agents (i, j) sums up to one, i.e.

$$\sum_{\bar{j}=1}^{N} [\mathbf{W}(k)]_{\bar{i}\bar{j}} = 1. \tag{24}$$

In particular if $G(k)$ is completely disconnected (i.e. agents do not send any information), then $\mathbf{W}(k) = I$.

(d) the rows of $\mathbf{W}(k)$ sum up to a value smaller or equal than one, i.e.

$$\sum_{\bar{j}=1}^{N} [\mathbf{W}(k)]_{\bar{i}\bar{j}} \leq 1,\ \forall\, \bar{i}, k. \tag{25}$$

Proof. Given two agents i and j, by part (b) of Proposition 2 the distance between their states can be upper bounded by

$$d(x_i(k+1), x_j(k+1)) \leq \sum_{p\in\mathcal{N}_i(k), q\in\mathcal{N}_j(k)} w^{ij}_{pq}(k) d(x_p(k), x_q(k)), \quad (26)$$

for all $i \neq j$ and where $w^{ij}_{pq}(k) > 0$ and $\sum_{p\in\mathcal{N}_i(k),q\in\mathcal{N}_j(k)} w^{ij}_{pq}(k) = 1$. By defining $\mathbf{W}(k) \triangleq (w^{ij}_{pq}(k))$ for $i \neq j$ and $p \neq q$ (where the pairs (i, j) and (p, q) refer to the rows and columns of $\mathbf{W}(k)$, respectively), inequality (20) follows. We continue with proving the properties of matrix $\mathbf{W}(k)$.

(a) Since all $w^{ij}_{pq}(k) > 0$ for all $i \neq j$, $p \in \mathcal{N}_i(k)$ and $q \in \mathcal{N}_j(k)$, we obtain that $\mathbf{W}(k)$ is non-negative. By part (c) of Proposition 2, there exists $\eta \triangleq \underline{\lambda}^2$ such that $w^{ij}_{pq}(k) \geq \eta$ for all non-zero entries of $\mathbf{W}(k)$. Also, since $i \in \mathcal{N}_i(k)$ and $j \in \mathcal{N}_j(k)$ for all $k \geq 0$ it follows that the term $w^{ij}_{ij}(k)d(x_i(k), x_j(k))$, with $w^{ij}_{ij}(k) \geq \eta$ will always be present in the right-hand side of the inequality (26), and therefore $\mathbf{W}(k)$ has positive diagonal entries.

(b) Follows from part (d) of Proposition 2, with $\eta = \underline{\lambda}^2$.

(c) If $\mathcal{N}_i(k) \cap \mathcal{N}_j(k) = \emptyset$ then no terms of the form $w^{ij}_{pp}(k)d(x_p(k), x_p(k))$ will appear in the sum of the right hand side of inequality (26). Hence $\sum_{p\in\mathcal{N}_i(k),q\in\mathcal{N}_j(k)} w^{ij}_{pq}(k) = 1$ and therefore

$$\sum_{\bar{j}=1}^{N} [\mathbf{W}(k)]_{\bar{i}\bar{j}} = 1.$$

If $G(k)$ is completely disconnected, then the sum of the right hand side of inequality (26) will have only the term $w^{ij}_{ij}(k)d(x_i(k), x_j(k))$ with $w^{ij}_{ij}(k) = 1$, for all $i, j = 1 \ldots n$. Therefore $\mathbf{W}(k)$ is the identity matrix.

(d) The result follows from parts (b) and (c).

Let $\bar{G}(k) = (\bar{V}, \bar{E}(k))$ be the underlying graph of $\mathbf{W}(k)$ and let $\bar{i}$ and $\bar{j}$ refer to the rows and columns of $\mathbf{W}(k)$, respectively. Note that under this notation, index $\bar{i}$ corresponds to a pair (i, j) of distinct agents. It is not difficult to see that the set of edges of $\bar{G}(k)$ is given by

$$\bar{E}(k) = \{((i, j), (p, q)) \mid (i, p) \in E(k), (j, q) \in E(k), i \neq j,\ p \neq q\}. \quad (27)$$

Proposition 3. Let Assumptions 1 and 2 hold for $G(k)$. Then, similar properties hold for $\bar{G}(k)$ as well, i.e.

(a) the graph $(\bar{V}, \bar{E}_\infty)$ is connected, where
$$\bar{E}_\infty = \{(\bar{i}, \bar{j}) \mid (\bar{i}, \bar{j}) \in \bar{E}(k) \text{ infinetly many indices } k\};$$

(b) there exists an integer $\bar{B} \geq 1$ such that every $(\bar{i}, \bar{j}) \in \bar{E}_\infty$ appears at least once every $\bar{B}$ consecutive time slots, i.e. at time k or at time $k + 1$ or … or (at latest) at time $k + \bar{B} - 1$ for any $k \geq 0$.

Proof. See the proof of Proposition 4.1 of Matei and Baras (2010).

Let $\Phi(k, s) \triangleq \mathbf{W}(k-1)\mathbf{W}(k-2)\cdots\mathbf{W}(s)$, with $\Phi(k, k) = \mathbf{W}(k)$ denote the transition matrix of $\mathbf{W}(k)$ for any $k \geq s$. It should be obvious from the properties of $\mathbf{W}(k)$ that $\Phi(k, s)$ is a non-negative matrix with positive diagonal entries and $\|\Phi(k, s)\|_\infty \leq 1$ for any $k \geq s$.

Lemma 2. Let $\mathbf{W}(k)$ be the matrix introduced in Lemma 1. Let Assumptions 1 and 2 hold for $G(k)$. Then, there exits a row index $\bar{i}^*$ such that

$$\sum_{\bar{j}=1}^{N} [\Phi(s+m, s)]_{\bar{i}^*\bar{j}} \leq 1 - \eta^m,\ \forall\ s, m \geq \bar{B} - 1, \quad (28)$$

where η is the lower bound on the non-zero entries of $\mathbf{W}(k)$ and $\bar{B}$ is the positive integer from the part (b) of the Proposition 3.

Proof. See the proof of Lemma 4.2 of Matei and Baras (2010).

Corollary 2. Let $\mathbf{W}(k)$ be the matrix introduced in Lemma 1 and let Assumptions 1 and 2 hold for $G(k)$. We then have

$$[\Phi(s + (N-1)\bar{B} - 1, s)]_{ij} \geq \eta^{(N-1)\bar{B}}\ \forall s, i, j, \quad (29)$$

where η is the lower bound on the non-zero entries of $\mathbf{W}(k)$ and $\bar{B}$ is the positive integer from the part (b) of the Proposition 3.

Proof. By Proposition 3 and Lemma 1 all the assumptions of Lemma 2 of Nedic and Ozdalgar (2009) are satisfied, from which the result follows.

6. PROOF OF THE MAIN RESULT

6.1 Proof of Theorem 1

Proof. We have that the vector of distances between the states of the agents respects the inequality

$$\mathbf{d}(k+1) \leq \mathbf{W}(k)\mathbf{d}(k),$$

where the properties of $\mathbf{W}(k)$ are described by Lemma 1. It immediately follows that

$$\|\mathbf{d}(k+1)\|_\infty \leq \|\mathbf{d}(k)\|_\infty, \text{ for } k \geq 0. \quad (30)$$

Let $\bar{B}_0 \triangleq (N-1)\bar{B} - 1$, where $\bar{B}$ is the positive integer from the part (b) of the Proposition 3. In the following we show that all row sums of $\Phi(s + 2B_0, s)$ are upper-bounded by a positive scalar strictly less than one. Indeed since $\Phi(s + 2\bar{B}_0, s) = \Phi(s + 2\bar{B}_0, s + \bar{B}_0)\Phi(s + \bar{B}_0, s)$ we obtain that

$$\sum_{\bar{j}=1}^{N} [\Phi(s+2\bar{B}_0, s)]_{\bar{i}\bar{j}} = \sum_{\bar{j}=1}^{N} [\Phi(s+2\bar{B}_0, s+\bar{B}_0)]_{\bar{i}\bar{j}} \sum_{\bar{h}=1}^{N} [\Phi(s+\bar{B}_0, s)]_{\bar{j}\bar{h}},$$

for all $\bar{i}$. By Lemma 2 we have that there exists a row $\bar{j}^*$ such that

$$\sum_{\bar{h}=1}^{N} [\Phi(s + \bar{B}_0, s)]_{\bar{j}^*\bar{h}} \leq 1 - \eta^{\bar{B}_0}, \forall s,$$

and since $\sum_{\bar{h}=1}^{N} [\Phi(s + \bar{B}_0, s)]_{\bar{j}\bar{h}} \leq 1$ for any $\bar{j}$, we get

$$\sum_{\bar{j}=1}^{N} [\Phi(s + 2\bar{B}_0, s)]_{\bar{i}\bar{j}} \leq \sum_{\bar{j}=1, \bar{j}\neq\bar{j}^*}^{N} [\Phi(s + 2\bar{B}_0, s + \bar{B}_0)]_{\bar{i}\bar{j}} + $$
$$+ [\Phi(s + 2\bar{B}_0, s + \bar{B}_0)]_{\bar{i}\bar{j}^*}(1 - \eta^{\bar{B}_0}) =$$
$$= \sum_{\bar{j}=1}^{N} [\Phi(s + 2\bar{B}_0, s + \bar{B}_0)]_{\bar{i}\bar{j}} - [\Phi(s + 2\bar{B}_0, s + \bar{B}_0)]_{\bar{i}\bar{j}^*}\eta^{\bar{B}_0}.$$

By Corollary 2 it follows that

$$[\Phi(s + 2\bar{B}_0, s + \bar{B}_0)]_{\bar{i}\bar{j}} \geq \eta^{\bar{B}_0+1},\ \forall \bar{i}, \bar{j}, s,$$

and since $\sum_{\bar{j}=1}^{N} [\Phi(s + 2\bar{B}_0, \bar{B}_0)]_{\bar{i}\bar{j}} \leq 1$ we get that

$$\sum_{\bar{j}=1}^{N} [\Phi(s + 2\bar{B}_0, s)]_{\bar{i}\bar{j}} \leq 1 - \eta^{2\bar{B}_0+1},\ \forall \bar{i}, s.$$

Therefore

$$\|\Phi(s+2\bar{B}_0,s)\|_\infty \leq 1-\eta^{2\bar{B}_0+1},\ \forall s. \quad (31)$$

It follows that

$$\|\mathbf{d}(t_k)\|_\infty \leq \left(1-\eta^{2\bar{B}_0+1}\right)^k \|\mathbf{d}(0)\|_\infty,\ \forall k\geq 0, \quad (32)$$

where $t_k = 2k\bar{B}_0$ which shows that the subsequence $\{\|\mathbf{d}(t_k)\|_\infty\}_{k\geq 0}$ asymptotically converges to zero. Combined with inequality (30) we farther obtain that the sequence $\{\|\mathbf{d}(k)\|_\infty\}_{k\geq 0}$ asymptotically converges to zero. Therefore the agents asymptotically reach consensus.

6.2 *Proof of Corollary 1*

Proof. The main idea of the proof consist of showing that the set $conv(A(k))$, where $A(k) = \{x_i(k), i = 1\ldots n\}$, converges to a set containing one point.

We first note that since $A_i(k) \subseteq A(k)$ it can be easily argued that $conv(A_i(k)) \subseteq conv(A(k))$, for all i and k. Also, since $C_{\underline{\lambda}}(A_i(k)) \subseteq conv(A_i(k))$ it follows that $C_{\underline{\lambda}}(A_i(k)) \subseteq conv(A(k))$ and consequently $x_i(k+1) \in conv(A(k))$. Therefore, we have that $conv(A(k+1)) \subseteq conv(A(k))$ for all k and from the theory of limit of sequence of sets, it follows that

$$\liminf conv(A(k)) = \limsup conv(A(k)) = \lim conv(A(k)) = A_\infty,$$

where $A_\infty = \bigcap_{k\geq 0} conv(A(k))$. We denote the diameter of the set $A(k)$ by

$$\delta(A(k)) = \sup\{d(x,y) \mid x,y \in A(k)\},$$

and by Proposition 2 of Sharma and Dewangan (1995) we have that

$$\delta(conv(A(k))) = \delta(A(k)).$$

From Theorem 1 we have that

$$\lim_{k\to\infty} d(x_i(k),x_j(k)) = 0,\ \forall i\neq j,$$

and consequently

$$\lim_{k\to\infty}\delta(A(k)) = \lim_{k\to\infty}\delta(conv(A(k))) = 0,$$

which also means that

$$\delta(A_\infty) = 0,$$

i.e. the set A_∞ contains only one point, say $x^* \in \mathcal{X}$, or $A_\infty = conv(x^*)$, or

$$\lim_{k\to\infty} conv(A(k)) = conv(x^*).$$

But since $x_i(k+1) \in C_{\underline{\lambda}}(A_i(k)) \subseteq conv(A(k))$ for all i,k it follows that

$$\lim_{k\to\infty} d(x_i(k),x^*) = 0,\ \forall\ i,$$

i.e. the states of the agents converge to the same point $x^* \in \mathcal{X}$.

7. CONCLUSION

In this note we emphasized the importance of the convexity concept and in particular the importance of the convex hull notion for reaching consensus. We did this by generalizing the asymptotic consensus problem to the case of convex metric space. For a group of agents taking values in a convex metric space, we introduced an iterative algorithm which ensures asymptotic convergence to agreement under some minimal assumptions on the communication graph.

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