

# Almost Sure Convergence to Consensus in Markovian Random Graphs

Ion Matei, Nuno Martins and John S. Baras

**Abstract**—In this paper we discuss the consensus problem for a network of dynamic agents with undirected information flow and random switching topologies. The switching is determined by a Markov chain, each topology corresponding to a state of the Markov chain. We show that in order to achieve consensus almost surely and from any initial state the sets of graphs corresponding to the closed positive recurrent sets of the Markov chain must be jointly connected. The analysis relies on tools from matrix theory, Markovian jump linear systems theory and random processes theory. The distinctive feature of this work is addressing the consensus problem with “Markovian switching” topologies.

## I. INTRODUCTION

A consensus problem, which lies at the foundation of distributed computing, consists of a group of dynamic agents who seek to agree upon certain quantities of interest by exchanging information among them according to a set of rules. This problem can model many phenomena involving information exchange between agents such as cooperative control of vehicles, formation control, flocking, synchronization, parallel computing, etc. Thus the consensus problem has been widely studied in the literature. Distributed computation over networks has a long history in control theory starting with the work of Borkar and Varaiya [1], Tsitsiklis, Bertsekas and Athans [15], [16] on asynchronous agreement problems and parallel computing. Olfati-Saber and Murray introduced in [11], [12] the theoretical framework for solving consensus problems. Jadbabaie et al. studied in [6] alignment problems involving reaching an agreement. Relevant extensions of the consensus problem were done by Ren and Beard [10] and by Moreau in [8]. The communication networks between agents may change in time due to link failures, packet drops, appearance or disappearance of nodes etc. Many of the variations in topology may happen randomly which lead to the consideration of consensus problems under a stochastic framework. Hatano and Mesbahi consider in [7] an agreement problem over random information networks where the existence of an information channel between a pair of elements at each time instance is probabilistic and independent of other channels. In [14] Salehi and Jadbabaie provide necessary and sufficient conditions for reaching consensus in the case of a discrete linear system, where the communication flow is given by a graph derived from a random graph process, independent of other time instances. Under a similar model for the network communication topology Porfiri and Stilwell give sufficient conditions for almost sure convergence to consensus in [9].

Ion Matei, Nuno Martins and John S. Baras are with the Institute for Systems Research and the Department of Electrical and Computer Engineering, University of Maryland, College Park, imatei, nmartins, baras@umd.edu

Modeling the variations in the communication topologies as independent events may prove to be not realistic enough. Consider the example of an agent which may adjust the power allocated to transmissions in order to overcome the failure of a link due to a large distance between agents. The actions of such an agent determines a change in the communication topology which is dependent on the state of the network at previous time instants.

In this paper we consider the consensus problem for a group of dynamic agents with undirected information flow and random switching topologies. The switching process is governed by a Markov chain whose states correspond to possible communication topologies. We formulate the necessary and sufficient conditions such that the agents reach almost surely consensus.

The outline of the paper is as follows. In Section II we present the setup and formulation of the problem. In Section III we state our main result and give an intuitive explanation. In Section IV we provide first a set of theoretical tools used in proving the main result and then we proceed with the proof of the main theorem.

## II. PROBLEM FORMULATION

In this section we introduce the problem setup for the almost sure convergence to consensus within a discrete-time context. We consider a group of  $n$  dynamic agents for which the information flow is modeled as an undirected graph  $G = (\mathcal{V}, \mathcal{E}, A)$  of order  $n$ . The set  $\mathcal{V} = \{1, \dots, n\}$  represents the set of vertices,  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is the set of edges and  $A = [a_{ij}]$  is the symmetric unweighted adjacency matrix with  $a_{ij}$  being 1 if a link exists between vertices  $(i, j)$  and zero otherwise. By *graph Laplacian* we understand the matrix  $L$  whose entries are:

$$l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^n a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}$$

Equivalently we can write the Laplacian of an undirected graph as  $L = D - A$ , where  $A$  is the adjacency matrix of the graph and  $D$  is the diagonal matrix of vertex degrees  $d_{ii} = \sum_{i \neq j} a_{ij}$ . Throughout this paper we will consider (except Section V) undirected, unweighted graphs.

*Definition 2.1: (Jointly Connected Graphs)* Let  $\{G_i\}_{i=1}^s$  be a set of undirected graphs. We say that this set is jointly connected if the union of the graphs in the set generates a connected graph where by union we understand the reunion of edges of all graphs in the set (we used the same definition as in [6]).

Consider a finite-state Markov chain  $M(k)$  which takes values in the discrete set  $\mathcal{S} = \{1, \dots, s\}$  and with an  $s \times s$

transition probability matrix  $P = (p_{ij})$  ( $\sum_{j=1}^s p_{ij} = 1, i \in S$ ).

*Definition 2.2: (Discrete Markovian Random Graph)* Let  $\{G_i\}_{i=1}^s$  be a set of undirected graphs of order  $n$ . By a *Discrete Markovian Random Graph* we understand a random graph  $G_{M(k)}$  which varies in time according to an underlying Markov chain  $M(k)$

$$G_{M(k)} = G_i \text{ if } M(k) = i$$

for all positive integer values of  $k$ .

We denote by  $X(k)$  the  $n$ -dimensional column vector representing the state of the agents. We consider a discrete stochastic dynamic system governing the evolution of the state vector:

$$X(k+1) = F_{M(k)}X(k), X(0) = X_0 \quad (1)$$

where the  $n \times n$  matrix  $F_{M(k)}$  describing the update rule is a random matrix taking values in the finite set of matrices  $\{F_i\}_{i=1}^s$ . Each matrix  $F_i$  is given by

$$F_i = I - \epsilon L_i, \quad 1 \leq i \leq s \quad (2)$$

with  $1/\epsilon$  greater than  $n$  and  $L_i$  the Laplacian of the undirected, unweighted graph  $G_i$ . The initial condition  $X_0$  is considered deterministic. To simplify the reference to the update law (2), we denote the protocol defined through  $F_{M(k)}$  by *Protocol  $A_1$* . Notice that (1) represents a discrete Markovian jump linear system for which we can use results from the theory of Markovian jump linear systems [2] to study its convergence properties.

The matrix  $F_i$  is also referred to as the *Perron* matrix of graph  $G_i$  with parameter  $\epsilon$ . There are several interesting properties regarding *Perron* matrices, such as that  $F_i$  is a stochastic matrix which is also an ergodic matrix if the graph  $G_i$  is (strongly) connected (see for instance [13]). In the particular case of an undirected graph (mainly considered here), the matrix  $F_i$  is symmetric, hence doubly stochastic. The same property is achieved in the case of the *balanced* directed graphs (see [12] for further details).

We define the *agreement space* as the subspace generated by the vector of all ones  $\mathcal{A} = \text{span}(\mathbb{1})$ , where we denote by  $\mathbb{1}$  the vector of all ones.

*Definition 2.3:* We say that the vector  $X(k)$  converges almost surely to consensus if it asymptotically reaches the agreement space in the almost sure sense

$$X(k) \xrightarrow{a.s.} \mathcal{A}$$

In this paper we analyze under what conditions the state vector  $X(k)$  reaches the agreement space almost surely. As we will show later under the updating rule expressed in (2) if the vector  $X(k)$  converges to consensus then it converges to a value given by the element wise average of the initial vector  $X_0$ :

$$X(k) \xrightarrow{a.s.} \text{av}(X_0)\mathbb{1}$$

where  $\text{av}(X_0) = \frac{\mathbb{1}^T X_0}{n}$  (the element wise average of the initial vector). Starting from this remark we formulate the following definition.

*Definition 2.4:* We say the state vector  $X(k)$  reaches *average consensus* almost surely if

$$X(k) \xrightarrow{a.s.} \text{av}(X_0)\mathbb{1}$$

By defining the error vector  $e(k)$  as the difference between the vectors  $X(k)$  and  $\text{av}(X_0)\mathbb{1}$

$$e(k) = X(k) - \text{av}(X_0)\mathbb{1} \quad (3)$$

we can write an equivalent form of *Definition 2.4*.

*Definition 2.5:* We say the state vector  $X(k)$  reaches *average consensus* almost surely if

$$\|e(k)\|^2 \xrightarrow{a.s.} 0$$

We are now ready to formulate the consensus problem.

*Problem 2.1:* Under the problem setup introduced above state the necessary and sufficient conditions such that the random vector  $X(k)$  converges almost surely to average consensus from any initial state  $X_0$ .

### III. MAIN RESULT

In this section we introduce the necessary and sufficient conditions for reaching *average consensus* in the almost sure sense together with some intuitive explanations of these conditions. We defer the rigorous mathematical proof for the next section.

Consider the problem setup presented in Section II. By the *Decomposition Theorem* of the states of a Markov chain (see [5]) the state space  $\mathcal{S}$  can be partitioned uniquely as

$$\mathcal{S} = \{T \cup C_1 \cup \dots \cup C_q\}$$

where  $T$  is the set of transient states and  $C_1, \dots, C_q$  are irreducible closed sets of (positive) recurrent states. Since  $M(k)$  is a finite state Markov chain there exists at least one (positive) recurrent closed set. We make the assumption that the distribution of  $M(0)$  is such that the probability of  $M(0)$  to belong to any of the sets  $T$  or  $C_i$  is non-zero and with non-zero probability  $M(k)$  can make a transition from  $T$  to any of the sets  $C_j$ . Let  $\mathcal{G}_i = \{G_{j_1}, G_{j_2}, \dots, G_{j_{|C_i|}}\}$  be the sets of graphs corresponding to the states in the sets  $C_i$  with  $i \in \{1, \dots, q\}$  and where by  $|C_i|$  we denote the cardinality of  $C_i$ .

*Theorem 3.1: (almost sure convergence to average consensus)* Consider the Markovian jump linear system as in (1). Then the random vector  $X(k)$  converges almost surely to *average consensus* for any initial state  $X_0$  if and only if each of the sets of graphs  $\mathcal{G}_i$  corresponding to the closed sets  $C_i$  are jointly connected.

We defer for the next section the proof of *Theorem 3.1* and of the supporting results. We rather provide here an intuitive

explanation. Regardless of the initial state of  $M(k)$ , there exist a time instant after which  $M(k)$  will be constrained to take values only in one of the closed sets  $C_i$ . Since  $C_i$  are irreducible and (positive) recurrent the probability of  $M(k)$  to visit each of the states belonging to  $C_i$  will never converge to zero. Thus  $M(k)$  will visit each of these states infinitely many times and consequently since the graphs corresponding to these states are jointly connected the agents will be connected infinitely many times. This is sufficient for the state vector  $X(k)$  to converge to consensus. On the other hand if we assume the existence of at least one set  $C_i$  such that the corresponding set of graphs  $\mathcal{G}_i$  is not jointly connected, then with non-zero probability  $M(k)$  may be isolated in such a set after a while. Since the graphs are not jointly connected, there will be at least one agent which will not exchange information with the others. This is enough to conclude (in the case of undirected graphs) that the dynamic system can not reach consensus regardless of the initial state.

#### IV. PROOF OF THE MAIN RESULT

In this section we detail the proof of *Theorem 3.1* and introduce a number of supporting results and their proofs. The proof of our results is based on the convergence properties of some matrices which arise from the analysis of the state vector's second moment. We start by stating a number of results from [6] which will be useful in our analysis. Then we continue with a series of corollaries and lemmas that will prove instrumental for showing *Theorem 3.1*.

*Theorem 4.1:* (Wolfowitz) Let  $A_1, \dots, A_s$  be a finite set of  $n \times n$  ergodic matrices with the property that for each sequence of matrices  $A_{i_1}, \dots, A_{i_j}$  of positive length the product matrix  $A_{i_j} \cdots A_{i_1}$  is an ergodic matrix. Then for each infinite sequence  $A_{i_1}, A_{i_2}, \dots$  there exist a vector  $c$  such that

$$\lim_{j \rightarrow \infty} A_{i_j} A_{i_{j-1}} \cdots A_{i_1} = \mathbb{1}c^T \quad (4)$$

Details and a proof of this theorem can be found in [17].

*Lemma 4.1:* Let  $m$  be a positive integer with  $m \leq 2$  and let  $\{A_i\}_{i=1}^m$  be a set of nonnegative  $n \times n$  matrices. Suppose the diagonal entries of each matrix  $A_i$  are all positive and let  $\mu$  and  $\rho$  denote the smallest and the largest of these respectively. Then the following property holds:

$$A_1 A_2 \cdots A_m \geq \left( \frac{\mu^2}{2\rho} \right)^{m-1} (A_1 + A_2 + \dots + A_m) \quad (5)$$

*Proof:* See [6] page 991. ■

In the special case of doubly stochastic matrices we can derive the following corollary from *Theorem 4.1*.

*Corollary 4.1:* Let  $A_1, \dots, A_s$  a finite set of  $n \times n$  doubly stochastic ergodic matrices with the property that for each sequence of matrices  $A_{i_1}, \dots, A_{i_j}$  of positive length the product matrix  $A_{i_j} \cdots A_{i_1}$  is an ergodic (doubly stochastic)

matrix. Then for each infinite sequence  $A_{i_1}, A_{i_2}, \dots$  we have

$$\lim_{j \rightarrow \infty} A_{i_j} A_{i_{j-1}} \cdots A_{i_1} = \frac{1}{n} \mathbb{1}\mathbb{1}^T \quad (6)$$

*Proof:* By *Theorem 4.1* we have that

$$\lim_{j \rightarrow \infty} A_{i_j} A_{i_{j-1}} \cdots A_{i_1} = \mathbb{1}c^T.$$

Since the matrices considered are doubly stochastic and ergodic their transposes are ergodic as well. Hence applying again *Theorem 4.1* on the transpose versions of the set of matrices  $A_1, \dots, A_s$  we obtain that there exist a vector  $d$  such that

$$\lim_{j \rightarrow \infty} (A_{i_j} A_{i_{j-1}} \cdots A_{i_1})^T = \mathbb{1}d^T$$

But since the stochastic matrix  $\mathbb{1}c^T$  must be equal to  $d\mathbb{1}^T$  we obtain that:

$$\mathbb{1}c^T = d\mathbb{1}^T = \frac{1}{n} \mathbb{1}\mathbb{1}^T \quad \blacksquare$$

*Lemma 4.2:* Consider a set of jointly connected undirected graphs  $\{G_i\}_{i=1}^s$  and a finite set of indices  $\{i_k\}_{k=1}^j$  which contains (at least once) each of the values  $\{1, \dots, s\}$ . Let  $\{F_i\}_{i=1}^s$  be a set of  $n \times n$  (symmetric) matrices given by  $F_i = I - \epsilon L_i$  (as in (2)), where  $L_i$  is the Laplacian of the graph  $G_i$  and  $1/\epsilon$  is a value larger than  $n$ . Then the matrix product  $F_{i_1} F_{i_2} \dots F_{i_j}$  is ergodic. The same property holds for the matrix product  $(F_{i_1} \otimes F_{i_1})(F_{i_2} \otimes F_{i_2}) \dots (F_{i_j} \otimes F_{i_j})$ .

*Proof:* The proof is based on *Lemma 4.1* and it is similar to the proof of *Lemma 1* in [6]. By construction all matrices  $\{F_i\}_{i=1}^s$  are non-negative matrices with positive diagonal entries (since the diagonal entries are obtained by subtracting from 1 some sub-unitary values). Then by *Lemma 4.1* we have

$$F_{i_1} F_{i_2} \dots F_{i_j} \leq \gamma(F_1 + F_2 + \dots + F_s)$$

where  $\gamma$  depends on the largest and smallest entry of matrices  $\{F_i\}_{i=1}^s$ . Since the diagonal entries of each  $F_i$  are positive  $\gamma$  can not be zero. We can further write

$$F_{i_1} F_{i_2} \dots F_{i_j} \geq \gamma(F_1 + F_2 + \dots + F_s) = \gamma s F$$

where  $F = I - \epsilon/s \sum_{i=1}^s L_i$ . From the hypothesis that the set  $\{G_i\}_{i=1}^s$  are jointly connected  $L = \sum_{i=1}^s L_i$  corresponds to the Laplacian of a connected undirected weighted graph. Therefore  $L$  has only one eigenvalue zero with the rest positive eigenvalues smaller than  $ns$  and therefore  $F$  is a stochastic ergodic matrix. As a consequence there exist a finite positive number  $k$  such that all entries of  $F^k$  are positive. Then

$$(F_{i_1} F_{i_2} \dots F_{i_j})^k \geq \gamma^k F^k$$

and therefore  $(F_{i_1} F_{i_2} \dots F_{i_j})^k$  has all entries strictly positive which is enough to conclude its ergodicity. To derive the

ergodicity of the matrix product involving the Kronecker product of matrices  $F_i$ 's notice that

$$\begin{aligned} & ((F_{i_1} \otimes F_{i_1})(F_{i_2} \otimes F_{i_2}) \dots (F_{i_j} \otimes F_{i_j}))^k = \\ & (F_{i_1} F_{i_2} \dots F_{i_j})^k \otimes (F_{i_1} F_{i_2} \dots F_{i_j})^k \end{aligned}$$

and since  $(F_{i_1} F_{i_2} \dots F_{i_j})^k$  has all entries strictly positive the same is true for  $(F_{i_1} F_{i_2} \dots F_{i_j})^k \otimes (F_{i_1} F_{i_2} \dots F_{i_j})^k$ . ■

*Lemma 4.3:* Let  $\{A_{ij}\}_{i,j=1}^s$  be a set of  $n \times n$  doubly stochastic ergodic matrices. Let  $P = [p_{ij}]$  be an  $s \times s$  stochastic matrix and consider the  $ns \times ns$  dimensional matrix  $Q$  whose  $(i, j)^{th}$  block is defined as  $Q_{ij} = p_{ij} A_{ij}$ . Then

$$\lim_{k \rightarrow \infty} Q^k = (\Phi \otimes I) \text{diag}(\frac{1}{n} \mathbb{1} \mathbb{1}^T) \quad (7)$$

where  $\text{diag}(\frac{1}{n} \mathbb{1} \mathbb{1}^T)$  is an  $ns \times ns$  block diagonal with matrices  $\frac{1}{n} \mathbb{1} \mathbb{1}^T$  on the main diagonal. The matrix  $\Phi$  is a stochastic matrix whose entries depend on the properties of matrix  $P$ :

$$(\Phi)_{ij} = \begin{cases} \lim_{k \rightarrow \infty} (P^k)_{ij} & \text{if the limit exists} \\ (P^l)_{ij} & \text{if } P^d = P, l \in \{0, \dots, d\} \end{cases} \quad (8)$$

*Proof:* The proof of this *Lemma* is based on *Corollary 4.1*. The matrix  $Q$  can be explicitly written as:

$$Q = \begin{pmatrix} p_{11} A_{11} & \dots & p_{1s} A_{1s} \\ \vdots & \ddots & \vdots \\ p_{s1} A_{s1} & \dots & p_{ss} A_{ss} \end{pmatrix} \quad (9)$$

We can express the  $(i, j)^{th}$  block entry of matrix  $Q^k$  as follows:

$$Q_{ij}^{(k)} = \sum_{1 \leq i_1, \dots, i_{k-1} \leq n} p_{ii_1} A_{ii_1} p_{i_1 i_2} A_{i_1 i_2} \dots p_{i_{k-1} j} A_{i_{k-1} j} \quad (10)$$

Notice from (10) that as  $k$  goes to infinity in each of the  $(i, j)$  block entries of the matrix  $Q^k$  there will be infinite sums containing infinite products of ergodic (doubly stochastic) matrices of the form  $A_{ii_1} A_{i_1 i_2} \dots$ . By *Corollary 4.1* every such infinite product of matrices converges to  $\frac{1}{n} \mathbb{1} \mathbb{1}^T$ . Therefore as  $k$  goes to infinity we can express (10) as:

$$\lim_{k \rightarrow \infty} Q_{ij}^{(k)} = \frac{1}{n} \mathbb{1} \mathbb{1}^T \lim_{k \rightarrow \infty} \sum_{1 \leq i_1, \dots, i_{k-1} \leq n} p_{ii_1} p_{i_1 i_2} \dots p_{i_{k-1} j} \quad (11)$$

It is not difficult to observe that the right-hand sum in (11) is the  $(i, j)^{th}$  entry of the matrix  $P^k$ . Whether or not this sum will converge depends entirely on the properties of the Markov chain behind the transition matrix  $P$ . Thus for large values of  $k$  the matrix  $\Phi = P^k$  will be a stochastic matrix whose entries are given by:

$$(\Phi)_{ij} = \begin{cases} \lim_{k \rightarrow \infty} (P^k)_{ij} & \text{if the limit exists} \\ (P^l)_{ij} & \text{if } P^d = P, l \in \{0, \dots, d\} \end{cases} \quad \blacksquare$$

*Lemma 4.4:* Consider a set of  $n \times n$  matrices  $\{F_i\}_{i=1}^s$  given by  $F_i = I - \epsilon L_i$  (as in (2)) where  $\{L_i\}_{i=1}^s$  are the Laplacians of  $\{G_i\}_{i=1}^s$ , a jointly connected set of undirected and un-weighted graphs of order  $n$ . Let  $P$  be a  $s \times s$  transition probability matrix corresponding to an irreducible (positive) recurrent finite-state Markov chain. Then

$$\lim_{k \rightarrow \infty} Q^k = \text{diag}(\frac{1}{n} \mathbb{1} \mathbb{1}^T) (\Phi \otimes I) \quad (12)$$

$$\lim_{k \rightarrow \infty} \tilde{Q}^k = \text{diag}(\frac{1}{n^2} \mathbb{1} \mathbb{1}^T) (\Phi \otimes I) \quad (13)$$

where

$$Q = \text{diag}(\{F_i\}_{i=1}^s) (P \otimes I) \quad (14)$$

and

$$\tilde{Q} = \text{diag}(\{F_i \otimes F_i\}_{i=1}^s) (P \otimes I) \quad (15)$$

The matrices  $\text{diag}(\{F_i\}_{i=1}^s)$  and  $\text{diag}(\{F_i \otimes F_i\}_{i=1}^s)$  are  $ns \times ns$  and  $n^2 s \times n^2 s$  respectively block diagonal matrices. The matrices  $\text{diag}(\frac{1}{n} \mathbb{1} \mathbb{1}^T)$  and  $\text{diag}(\frac{1}{n^2} \mathbb{1} \mathbb{1}^T)$  are block diagonal matrices of dimension  $ns \times ns$  and  $n^2 s \times n^2 s$  respectively and  $\Phi$  is a stochastic matrix given by

$$(\Phi)_{ij} = \begin{cases} \lim_{k \rightarrow \infty} (P^k)_{ij} & \text{if the limit exists} \\ (P^l)_{ij} & \text{if } P^d = P, l \in \{0, \dots, d\} \end{cases} \quad (16)$$

*Proof:* The proof follows the next steps. We will show there exist a positive integer  $l^*$  such that the matrix  $Q^{l^*} = (\text{diag}(\{F_i\}_{i=1}^s) (P \otimes I))^{l^*}$  has a structure as in (9), i.e. is a  $ns \times ns$  matrix where each  $s \times s$  block matrix is expressed by an ergodic (doubly) stochastic matrix weighted by some positive scalar. These weights values will sum up to one on rows. Then we can apply *Lemma 4.3* to conclude the proof.

Similar to (10) each block matrix  $Q_{ij}^{(k)}$  of the matrix  $Q^k$  can be expressed as

$$Q_{ij}^{(k)} = \sum_{1 \leq i_1, \dots, i_{k-1} \leq s} p_{ii_1} A_{ii_1} p_{i_1 i_2} A_{i_1 i_2} \dots p_{i_{k-1} j} A_{i_{k-1} j} \quad (17)$$

where this time  $A_{ij} = F_i$ . The reader should notice that if we sum each of the probabilities product multiplying the product of matrices in (17) we get  $p_{ij}^{(k)}$  which is the  $(i, j)$  entry of the matrix  $P^k$ . Each of these product of probabilities represent possible paths (sequence of transitions) of length  $k$  from state  $i$  to state  $j$ . By an inductive argument we can show that the matrix  $Q^k$  can be expressed as

$$Q^k = \begin{pmatrix} p_{11}^{(k)} A_{11}^{(k)} & \dots & p_{1s}^{(k)} A_{1s}^{(k)} \\ \vdots & \ddots & \vdots \\ p_{s1}^{(k)} A_{s1}^{(k)} & \dots & p_{ss}^{(k)} A_{ss}^{(k)} \end{pmatrix}$$

where  $p_{ij}^{(k)}$  is the  $(ij)^{th}$  entry of the matrix  $P^k$  and  $A_{ij}^{(k)}$  are some  $n \times n$  (doubly) stochastic matrices.

By the irreducibility assumption we can always find a path of state transitions which starts at a state  $i$  and ends at a state  $j$  which contains (at least once) every possible transition (hence transitions with nonzero probability) in the Markov chain. Let  $l$  denote the length of such a path. Clearly

the probability of this path is nonzero and has the general form  $p_{i i_1} p_{i_1 i_2} \dots p_{i_{l-1} j}$ , with  $1 \leq i_1, \dots, i_{l-1} \leq s$ . Then the previously mentioned product of probabilities will be part of the conditional probability  $p_{ij}^{(l)} = (P^l)_{ij}$ -the probability to start at state  $i$  and arrive at state  $j$  in  $l$  steps. Then by (17)  $p_{i i_1} p_{i_1 i_2} \dots p_{i_{l-1} j}$  is also encountered in the block matrix  $Q_{ij}^{(l)}$  multiplying a sequence of matrices  $F_i$ , with  $1 \leq i \leq s$ , since it is a possible path from state  $i$  to state  $j$  in  $l$  steps. Because this product of probabilities contains all possible paths (and a matrix  $F_i$  is associated to a transition from state  $i$  to any other state), then each of the matrices  $F_i$  will appear at least once. On the contrary we would have an isolated state which contradicts the hypothesis of irreducibility. We can rewrite the block matrix  $Q_{ij}^{(l)}$  as follows

$$Q_{ij}^{(l)} = p_{i i_1} p_{i_1 i_2} \dots p_{i_{l-1} j} F_i F_{i_1} \dots F_{i_{l-1}} + \dots$$

By *Lemma 4.2* the product of matrices  $F_i F_{i_1} \dots F_{i_{l-1}}$  is an ergodic (doubly) stochastic matrix. We mentioned above that  $Q_{ij}^{(l)}$  can be expressed as  $Q_{ij}^{(l)} = p_{ij}^{(l)} A_{ij}^{(l)}$ , where  $A_{ij}^{(l)}$  is a doubly stochastic matrix. Then  $A_{ij}^{(l)} = 1/p_{ij}^{(l)} (p_{i i_1} p_{i_1 i_2} \dots p_{i_{l-1} j} F_i F_{i_1} \dots F_{i_{l-1}} + \dots)$ . From the ergodicity of  $F_i F_{i_1} \dots F_{i_{l-1}}$  we can deduce the ergodicity of  $A_{ij}^{(l)}$  since we know there exist a positive integer  $k$  such that  $(F_i F_{i_1} \dots F_{i_{l-1}})^k$  has all entries positive and since  $(A_{ij}^{(l)})^k = (1/p_{ij}^{(l)})^k ((F_i F_{i_1} \dots F_{i_{l-1}})^k + \dots)$  which implies that all entries of  $A_{ij}^{(l)}$  will also be positive. Let  $l^*$  be the largest path length such that we can arrive from any state  $i$  to any state  $j$  by traveling through all possible state transitions. Then from the above argument it should be clear that the matrix  $Q^{l^*}$  is given by

$$Q^{l^*} = \begin{pmatrix} p_{11}^{(l^*)} A_{11}^{(l^*)} & \dots & p_{1s}^{(l^*)} A_{1s}^{(l^*)} \\ \vdots & \ddots & \vdots \\ p_{s1}^{(l^*)} A_{s1}^{(l^*)} & \dots & p_{ss}^{(l^*)} A_{ss}^{(l^*)} \end{pmatrix}$$

where for the non-zero matrix blocks we have that  $A_{ij}^{(l^*)}$  are ergodic (doubly) stochastic matrices. Then since  $\lim_{k \rightarrow \infty} Q^k = \lim_{k \rightarrow \infty} (Q^{l^*})^k$ , by *Lemma 4.3* we can determine (12). In order to show (13) we follow the same line as before and use the part of the *Lemma 4.2* involving the Kronecker products. ■

At this point we are ready for the proof of *Theorem 4.1*.

#### A. Sufficiency

Without loss of generality assume that the Markov chain  $M(k)$  is irreducible and (positive) recurrent. Thus we have only one closed set of states corresponding to a jointly connected set of graphs  $\{G_i\}_{i=1}^s$ . At the end of the sufficiency proof we will explain why we do not lose generality by making this assumption.

The proof has the following development. We will first analyze the convergence properties of the second moment

of the state vector  $X(k)$ . From this analysis we will assert that the second moment of the error vector  $e(k)$  converges exponentially to zero. Then by using the generalized Markov inequality together with the first Borel-Catelli lemma [5] we can conclude the almost sure convergence to zero of the error vector and implicitly the almost sure convergence of the state vector to the average consensus state.

Let the  $n \times n$  symmetric matrix  $Q(k)$  denote the second moment of the error vector  $e(k)$

$$Q(k) = E[X(k)X(k)^T]$$

where we used  $E$  to denote the expectation operator. Using an approach similar to [2] consider the matrices  $Q_i(k)$

$$Q_i(k) = E[X(k)X(k)^T \chi_{\{M(k)=i\}}] \quad (18)$$

where  $\chi_{\{M(k)=i\}}$  is the indicator function of the event  $\{M(k) = i\}$ . Then the second moment  $Q(k)$  can be expressed as the following sum:

$$Q(k) = \sum_{i=1}^s Q_i(k) \quad (19)$$

The set of discrete coupled Lyapunov equations governing the evolution of the matrices  $Q_i(k)$  is given by:

$$Q_i(k+1) = \sum_{j=1}^s p_{ji} F_j Q_j(k) F_j^T, \quad i \in \mathcal{S} \quad (20)$$

with initial conditions  $Q_i(0) = q_i X_0 X_0^T$  where  $q = (q_i)$  is the initial distribution of the Markov chain  $q_i = Pr(M(0) = i)$ .

We can further obtain a vectorized form of equations (20) with the advantage of getting a discrete linear system

$$\eta(k+1) = \Lambda \eta(k) \quad (21)$$

where  $\eta(k)$  is an  $n^2 s$  dimensional vector formed by the columns of all matrices  $Q_i(k)$  and  $\Lambda$  is an  $n^2 s \times n^2 s$  matrix given by

$$\Lambda = \begin{pmatrix} p_{11} F_1 \otimes F_1 & \dots & p_{s1} F_s \otimes F_s \\ \vdots & \ddots & \vdots \\ p_{1s} F_1 \otimes F_1 & \dots & p_{ss} F_s \otimes F_s \end{pmatrix} \quad (22)$$

The initial vector  $\eta(0)$  has the following structure

$$\eta(0)^T = [q_1 col_1(X_0 X_0^T)^T, \dots, q_1 col_n(X_0 X_0^T)^T, \dots, \dots, q_s col_1(X_0 X_0^T)^T, \dots, q_s col_n(X_0 X_0^T)^T]$$

where by  $col_i$  we understand the  $i^{th}$  column of the considered matrix. We notice that the current setup satisfies all the conditions of *Lemma 4.4* (matrix  $\Lambda$  is just a transposed version of the matrix  $\tilde{Q}$  in (15)) and hence we get

$$\lim_{k \rightarrow \infty} \Lambda^k = (\Phi^T \otimes I) diag\left(\frac{1}{n^2} \mathbb{1} \mathbb{1}^T\right)$$

where  $\Phi$  is given by (16). Using the observation that

$$\frac{1}{n^2} \mathbb{1} \mathbb{1}^T \begin{pmatrix} q_i \text{col}_1(X_0 X_0^T) \\ \vdots \\ q_i \text{col}_n(X_0 X_0^T) \end{pmatrix} = av(X_0)^2 q_i \mathbb{1},$$

the limiting value of vector  $\eta(k)$  is given by

$$\lim_{k \rightarrow \infty} \eta(k) = av(X_0)^2 \begin{pmatrix} \sum_{j=1}^s \Phi_{j1} q_j \mathbb{1} \\ \vdots \\ \sum_{j=1}^s \Phi_{js} q_j \mathbb{1} \end{pmatrix},$$

where  $\Phi_{ij}$  are entries of the stochastic matrix defined in (16). By collecting the entries of  $\lim_{k \rightarrow \infty} \eta(k)$  we obtain

$$\lim_{k \rightarrow \infty} Q_i(k) = av(X_0)^2 \left( \sum_{j=1}^s \Phi_{ji} q_j \right) \mathbb{1} \mathbb{1}^T$$

and from (19) we finally obtain

$$\lim_{k \rightarrow \infty} Q(k) = av(X_0)^2 \mathbb{1} \mathbb{1}^T \quad (23)$$

since  $\sum_{i,j=1}^s \Phi_{ji} q_j = 1$ .

Through an almost identical process as in the case of the second moment we find that

$$\lim_{k \rightarrow \infty} E[X(k)] = av(X_0) \mathbb{1}. \quad (24)$$

The steps for showing (24) are similar to the ones just introduced. We obtain a linear system as in (21) with the difference that there will be no Kronecker products inside the block matrices of  $\Lambda$ , but only the matrices  $F_i$  multiplied by the appropriate conditional probabilities. Therefore by applying the results of *Lemma 4.4* we get (24).

From (23) and (24) we deduce that the second moment of the error vector converges asymptotically (and thus exponentially since linear systems are involved) to zero

$$\lim_{k \rightarrow \infty} E[e(k)e(k)^T] = 0,$$

and therefore we can find some positive constants  $\alpha$  and  $\beta$  such that the following is satisfied

$$E[\|e(k)\|^2] = \text{trace}(E[e(k)e(k)^T]) \leq \alpha \beta^k \|e(0)\|^2 \quad (25)$$

for any positive  $k$ . Then by the generalized Markov inequality we can write

$$\sum_{k \geq 0} Pr(\|e(k)\|^2 > \epsilon) \leq \sum_{k \geq 0} \frac{E[\|e(k)\|^2]}{\epsilon} \quad (26)$$

for any positive  $\epsilon$  which by (25) is a bounded quantity. Therefore by the first Borel-Cantelli Lemma [5] we determine the almost sure convergence of the error vector to zero and implicitly the almost sure convergence of the state vector to average consensus.

We now turn to explain why it was enough to assume the Markov chain to be represented by a single closed irreducible and (positive) recurrent set of states. According to the initial distribution, with some probability the initial state will belong to either a closed positive recurrent set of

states or to the transient set. The first case corresponds to the problem setup for the sufficiency proof. In the second case since the state is transient then there exist a finite positive integer  $\tau$  such that  $M(\tau)$  belongs to a closed set  $C_i$ . We showed above that

$$X(k) \xrightarrow{a.s.} av(X(\tau)) \mathbb{1}.$$

The state vector at time  $\tau$  is given by

$$X(\tau) = F_{i_1} F_{i_2} \dots F_{i_{\tau-1}} X_0$$

where  $\{i_1, i_2, \dots, i_{\tau-1}\}$  is a set of indices representing states in the transient set. Since the matrices  $F_i$  are doubly stochastic we have that  $\mathbb{1}^T F_i = \mathbb{1}^T$  for  $1 \leq i \leq s$ . Then

$$av(X_\tau) = \frac{\mathbb{1}^T X(\tau)}{n} = \frac{\mathbb{1}^T F_{i_1} \dots F_{i_{\tau-1}} X_0}{n} = av(X_0).$$

Therefore we have that

$$X(k) \xrightarrow{a.s.} av(X_0) \mathbb{1}.$$

## B. Necessity

Before proving the necessity lets us introduce the probabilistic framework of the equation (1). The Markov chain  $M(k)$  takes values in the discrete set  $\mathcal{S} = \{1, 2, \dots, s\}$ . Let  $\Omega_k$  be the space of elementary outcomes

$$\Omega_k = \mathcal{S}_0 \times \mathcal{S}_1 \times \dots \times \mathcal{S}_{k-1},$$

where  $\mathcal{S}_i$  are copies of  $\mathcal{S}$  and  $\times$  denotes product space. More explicitly we can write

$$\Omega_k = \{\bar{\omega}_k : \bar{\omega}_k = (\omega_0 \omega_1 \dots \omega_{k-1}), \omega_i \in \mathcal{S}\}.$$

The random vector  $X_k$  is a map from the space of elementary outcomes  $\Omega_k$  to the vector space  $\mathbb{R}^n$

$$\bar{\omega}_k \longrightarrow X_k(\bar{\omega}_k)$$

A realization of  $X_k$  is given by

$$X_k(\bar{\omega}_k) = F_{M_{k-1}(\omega_{k-1})} F_{M_{k-2}(\omega_{k-2})} \dots F_{M_0(\omega_0)} X_0,$$

The probability measure for  $X_k$  is expressed as

$$\begin{aligned} Pr(X_k \in S) &= Pr(\{\bar{\omega}_k : X_k(\bar{\omega}_k) \in S\}) \\ &= \sum_{1 \leq \omega_0, \dots, \omega_{k-1} \leq s, X_k(\bar{\omega}_k) \in S} Pr(\omega_0, \omega_1, \dots, \omega_{k-1}) \end{aligned}$$

where  $S$  is a subset of  $\mathbb{R}^n$  and

$$Pr(\omega_0, \omega_1, \dots, \omega_{k-1}) = p_{\omega_{k-2} \omega_{k-1}, \dots, p_{\omega_0 \omega_1}} Pr(\omega_0)$$

with  $p_{\omega_{k-i-1} \omega_{k-i}}$  entries of the probability transition matrix  $P$ .

Let  $A_k(\epsilon)$  define the following event  $A_k(\epsilon) = \{\bar{\omega}_k : \|X_k(\bar{\omega}_k) - av(X_0) \mathbb{1}\|^2 > \epsilon\}$  for some positive  $\epsilon$ . Then by [5],  $X_k$  converge almost surely to  $av(X_0) \mathbb{1}$  if and only if  $\lim_{m \rightarrow \infty} Pr(\bigcup_{k \leq m} A_k(\epsilon)) = 0$  for any  $\epsilon > 0$ . Notice that for any  $m$  we have

$$Pr\left(\bigcup_{k \geq m} A_k(\epsilon)\right) \geq Pr(A_m(\epsilon)) \quad (27)$$

Suppose there exist an irreducible and positive recurrent set  $C_{i^*}$  such that the corresponding graph set  $\mathcal{G}_{i^*}$  is not jointly connected. Since we assumed that the initial distribution of  $M(k)$  is such that there is a non-zero probability for  $M_0$  to belong to any of the sets  $T$ ,  $C_i$ , and since with non-zero probability we can reach any of the positive recurrent closed set  $C_i$  from  $T$ , then there exist a  $\tau$  such that the set  $C_{i^*}$  can be reached with non-zero probability. Once inside the set  $C_{i^*}$ , the state vector  $X_k$  will not be able to converge to  $av(X_0)\mathbb{1}$  from any initial state since  $\mathcal{G}_{i^*}$  is not jointly connected and therefore there exist at least two agent which will never exchange information. Therefore we can find an  $\epsilon$  such that  $Pr(A_m(\epsilon)) > 0$  for any  $m \geq \tau$ . Then using (27) we can conclude that  $X_k$  can not converge almost surely to average consensus for any initial condition.

## V. DISCUSSION AND EXTENSIONS OF THE MAIN RESULT

To the authors' knowledge we address for the first time the stochastic consensus problem with communication topologies modeled as Markovian random graphs. Markovian random graphs are more realistic and more general since they include the independent and identically distributed random graphs used until now in the literature [7], [14], [9].

While we assumed a rather simple model for the communication topologies (unweighted, undirected graphs) in the following we will discuss some extensions of our result to more general models involving different protocols and directed graphs. We note that the proof of *Theorem 3.1* is mainly based on the results of *Lemma 4.2* and *Lemma 4.4*. The key idea of *Lemma 4.4* is the fact that the matrix product  $F_{i_1}F_{i_2}\dots F_{i_k}$  (where  $\{i_1, \dots, i_k\} \in \mathcal{S}$  is a finite set of indices which contains at least once all values between 1 and  $s$ ) produces an ergodic matrix. Thus for any protocol and communication model among agents which conserve this property we can potentially show almost sure convergence to the agreement. We can replace for example *Protocol A<sub>1</sub>*, by another update rule which we will refer as *Protocol A<sub>2</sub>* given by

$$F_i = (I + D_i)^{-1}(I + A_i)$$

(where  $D_i$  is the diagonal matrix of vertex degrees and  $A_i$  is the adjacency matrix) or with a more general protocol (*Protocol A<sub>3</sub>*) where each of the matrices  $F_i$  are constructed as

$$F = (f_{ij})$$

with  $f_{ij} = \alpha_{ij}G_{ij} / \sum_{j=1}^n \alpha_{ij}G_{ij}$ ,  $\alpha_{ij}$  being some positive scalars and  $G_{ij} = 1$  if there is an information flow between vertices  $i$  and  $j$  (and by convention  $G_{ii} = 1$ ). Both these protocols generate stochastic matrices  $F_i$  for which it can be shown that the property stated in *Lemma 4.2* is preserved (see [6], [10]). Notice however that unlike *Protocol A<sub>1</sub>* the matrices  $F_i$  are no longer symmetric, thus no longer doubly stochastic. As a consequence, with slight reformulation of the *Lemmas 4.3*, *4.4* and *Theorem 4.1* and some technical changes in their proofs (which will not be addressed here) it can be shown that the state vector reaches the agreement space almost surely, but not necessarily *average consensus*.

This is due mainly to the fact that since the matrices  $F_i$  are no longer doubly stochastic, according to *Theorem 4.1*, the infinite matrix product  $F_{i_1}F_{i_2}\dots$  will converge to some matrix  $\mathbb{1}c^T$  (for some vector  $c$  with positive entries) and not necessarily to  $\frac{1}{n}\mathbb{1}\mathbb{1}^T$  as in the case of doubly stochastic matrices.

We can also extend our result to the case when the communications topologies are modelled by directed weighted graphs. The protocols  $A_1$ ,  $A_2$  and  $A_3$  must be updated accordingly. For *Protocols A<sub>1</sub>* and  $A_2$  we use the notion *outer degree* of a vertex to construct the diagonal matrices  $D_i$ . The adjacency matrix  $A_i$  will not necessarily be symmetric and therefore the update matrices  $F_i$  will not satisfy in general the doubly stochastic property. The same is true in the case of *Protocol A<sub>3</sub>*, for which the change consists in taking  $G_{ij}^k = 1$  if there is an information flow going from vertex  $i$  to vertex  $j$ . As pointed out at the beginning of this section, in order to show the sufficiency part of *Theorem 3.1* we need the matrix product  $F_{i_1}F_{i_2}\dots F_{i_k}$  to be ergodic. Similar to the undirected graphs case, such property is achieved if all the sets  $\mathcal{G}_i$  are *jointly strongly connected*. As pointed out in [10], *Lemma 4.2* holds under a weaker condition of existence of a *directed spanning tree* for the union of graphs in each of the sets  $\mathcal{G}_i$ . It turns out that this condition is also necessary for almost sure convergence to consensus. Thus in the case of directed graphs and under any of the protocols mentioned above, *Theorem 4.1* can be reformulated as follows.

*Theorem 5.1:* Consider the Markovian jump linear system as in (2.2). Then under any of the Protocols  $A_1$ ,  $A_2$  or  $A_3$ , the state vector  $X(k)$  converges almost surely to consensus for any initial condition  $X_0$  if and only if each of the graphs resulting from the union of graphs in each set  $\mathcal{G}_i$  admits a spanning tree.

As shown in [12], [13], average consensus can be reached even in the case of directed graphs if the graphs are *balanced*. By using *Protocol A<sub>1</sub>*, balanced graphs generate doubly stochastic matrices  $F_i$ . Then if we assume that each of the sets  $\mathcal{G}_i$  is either *jointly strongly connected* or it "jointly" posses a spanning tree, then we can show almost sure convergence to *average consensus* identically as in the case of undirected graphs and *Protocol A<sub>1</sub>*. Although we will not enter in details here it turns out that having all graphs balanced is also a necessary condition to reach average consensus.

The implications of modeling the communication flow between agents as Markovian random graphs are more subtle than they may appear at first glance. In [14], Salehi and Jadbabaie pointed out that in the case of independent and identically distributed random graphs a necessary and sufficient condition for reaching the agreement space almost surely is that  $|\lambda_2(E[F(k)])| < 1$ , where  $\lambda_2$  is the second largest eigenvalue of  $E[F(k)]$  and  $F(k)$  is the update rule at time  $k$ . In the case of Markovian random graphs a similar condition such as  $|\lambda_2(E[F_{M(k)}])| < 1$ ,  $k \geq 0$  or  $|\lambda_2(\lim_{k \rightarrow \infty} E[F_{M(k)}])| < 1$  is not necessary. For example

consider a Markovian random graph  $G_{M(k)}$  which can take only two values  $\{G_1, G_2\}$ . The graphs  $G_1$  and  $G_2$  have both three vertices. In the first graph only node 1 and 2 is connected and in the second one only nodes 2 and 3 are connected. Clearly the set  $\{G_1, G_2\}$  is jointly connected. Assume that the underlying Markov chain  $M(k)$  of  $G_{M(k)}$  has the probability transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus the process  $G_{M(k)}$  oscillates between the two graphs at each time instant. Choosing for instance to start with graph  $G_1$  we get that

$$E[F_{M(k)}] = \begin{cases} F_1 & \text{if } k \text{ is odd} \\ F_2 & \text{otherwise} \end{cases}$$

Clearly since  $G_1$  or  $G_2$  are not connected the multiplicity of eigenvalue 1 of both  $F_1$  and  $F_2$  will not be one. Therefore  $\lambda_2(E[F_{M(k)}]) < 1$  will not hold. Moreover since  $E[F_{M(k)}]$  does not actually converge the previous condition will not hold in the limit either.

## VI. CONCLUSION

In this paper we analyzed a stochastic consensus problem for a group of agents with undirected information flow. The novelty of our approach consists in using Markovian random graphs as models for the communication flows among agents. This model is more general and more realistic. We showed that a necessary and sufficient condition for the state vector to converge to average consensus almost surely consists in having the sets of (undirected) graphs corresponding to the positive recurrent closed sets of the Markov chain jointly connected. Under the Markovian random graph modeling, the dynamic stochastic equation determining the evolution of the agents became a Markovian jump linear system, which proved to be instrumental in showing the almost sure convergence. We discussed extensions of our result to other communication protocols and to directed communication flows. We pointed out that in the case of directed graphs, if the set of graphs corresponding to possible communication topologies are jointly strongly connected or admit "jointly" a spanning tree, consensus (in the almost sure sense) can be reached as well. Our analysis relied on several tools from algebraic matrix theory, matrix theory and Markovian jump linear system theory.

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