ON A CLASS OF HYSTERETIC, DYNAMICAL SYSTEMS WITH APPLICATION TO SMART MATERIAL ACTUATORS

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Abstract. A class of dynamical systems with hysteresis is considered, where the hysteresis non-linearity is coupled to an ordinary differential equation (ODE) in an unusual manner. Such systems arise in various contexts, e.g., smart material modeling, open-loop control of smart material actuators, and feedback control of systems involving hysteretic components. In this paper the Preisach operator is used to represent the hysteresis. The well-posedness of the hysteretic systems is established under general assumptions. With more specific assumptions, a number of system-theoretic properties are analyzed, including reachability, observability, input-output stability, and stability of equilibria. Existence of periodic solutions under a periodic input is shown. Finally numerical integration of such systems is discussed.

Key words. dynamical system with hysteresis, Preisach operator, smart actuators, well-posedness, stability, reachability, observability

AMS subject classifications. 93B07, 93C23, 93D05

1. Introduction. Dynamical systems with hysteresis are of interest in physics, mathematics, and engineering [25, 27, 40, 6]. This paper is focused on the analysis of the following novel class of hysteretic, dynamical systems:

\[
\begin{align*}
\dot{x}(t) + c \dot{z}(t) &= f(x(t), z(t), u(t)) \\
z(t) &= \mathcal{H}[x(\cdot)](t)
\end{align*}
\]

(1.1)

where \(x(\cdot)\) and \(z(\cdot)\) are scalar functions of time \(t\), \(u(\cdot)\) is a scalar input, \(\mathcal{H}\) is a hysteresis operator, and \(c > 0\) is some constant. The interest in (1.1) is motivated by a number of modeling and control problems involving hysteresis, especially in connection to smart material actuators.

The term *smart materials* refers to materials that exhibit inherent coupling of their mechanical properties with applied electric, magnetic, or thermal fields [33]. Examples of smart materials include piezoelectrics [32], magnetostrictives [11], shape memory alloys (SMAs) [30], magnetorheological (MR) [3] and electrorheological (ER) fluids [43], and electroactive polymers [2]. Direct energy transduction between different domains offers built-in sensing and actuation capabilities of smart materials. Comparing to traditional sensors and actuators (e.g., motors and hydraulic actuators), smart material sensors and actuators are potentially advantageous since they require fewer moving parts and are better suited for miniaturized systems. However, hysteresis presents itself as a ubiquitous nonlinearity in smart materials and hinders their wider application. Tremendous research has been done in modeling and control with the aim of understanding physical origins of hysteresis and mitigating its adverse effects.
Systems of the form (1.1) arise in different contexts, as illustrated in Section 2. These include, e.g., a dynamic hysteresis model for thin magnetostrictive actuators [36], open-loop control of piezoelectric actuators, and feedback control of systems with hysteretic components (e.g., SMA actuators). In (1.1) a key element is the hysteresis operator $H[\cdot]$. In this paper the Preisach operator [27] is used to represent $H$. Unlike the physics-based hysteresis models (see, e.g., [23, 39, 34]), the Preisach operator is of phenomenological type and it models hysteresis as a (weighted) aggregate effect of all possible delayed relay elements. Originally proposed for modeling ferromagnetic hysteresis, the Preisach operator has been used for modeling hysteresis in diverse disciplines such as super-conductivity [27], economics [10], and geosciences [18]. In particular it has been used for representing hysteresis in various smart materials [20], e.g., piezoelectrics [13, 9], shape memory alloys [16], magnetostrictives [28, 36, 21], and electroactive polymers [8]. Note that a similar operator using delayed relays of finite slopes, called Krasnosel’skii-Pokrovskii (KP) operator, has also been used [1, 12].

When an (infinite-dimensional) Preisach operator is coupled to an ODE as in (1.1), many interesting questions arise from basic issues like well-posedness, to various system-theoretic issues for this nonlinear dynamical system. The goal of this paper is two-fold: 1) to establish the well-posedness of (1.1) under appropriate, general assumptions on the Preisach operator and the function $f(\cdot, \cdot, \cdot)$; 2) to investigate whether and to what extent important concepts for dynamical systems can be applied. A brief discussion of the contributions follows.

The well-posedness of hysteretic systems of the form

\[
\begin{aligned}
\dot{x}(t) &= f(x(t), z(t), u(t)) \\
z(t) &= H[x(t)](t)
\end{aligned}
\]  

(1.2)

has been studied [6]. The key difference from (1.1) is that (1.2) has no coupling term $\dot{z}(t)$, the derivative of the hysteresis output. The well-posedness of (1.1) is established in this paper by integrating analysis techniques for ODEs with essential properties of the Preisach operator.

System-theoretic properties of (1.1) are analyzed by taking the (infinite dimensional) memory curve (see Section 3) of the Preisach operator as the state, $u(\cdot)$ as the input, and $x(\cdot)$ and $z(\cdot)$ as the output. Specific structures for $f(\cdot, \cdot, \cdot)$ with varying degrees of generality are assumed in the investigation of (approximate) reachability, observability, input-output stability, and stability of equilibria.

Oscillations in dynamical systems with hysteresis have been a subject of active research [26, 4, 5, 31, 29, 7]. Existence of periodic solutions to (1.1) under periodic excitation is a question of practical interest as physical systems involving smart material actuators often demonstrate looping behaviors. By defining a mapping on the space of memory curves, we establish the existence of periodic solutions using Schauder fixed point theorem [42].

Numerical integration of the hysteretic system (1.1) is necessary since in general there is no analytical solutions. Both forward Euler and backward Euler schemes turn out to be implicit, and are closely related to inversion algorithms for the Preisach operator. The schemes are shown to have the first order of consistency, as Euler schemes for integration of traditional ODEs do.

The remainder of the paper is organized as follows. Motivating examples for the class of systems considered are provided in Section 2. The Preisach operator is briefly reviewed in Section 3. Well-posedness is established in Section 4. In Section 5 system-theoretic properties are discussed. Section 6 is devoted to the existence of pe-
2. Motivating examples.

2.1. A dynamic hysteresis model for magnetostrictive actuators. A cascaded model structure for thin magnetostrictive actuators was proposed by Venkataraman [38], as shown in Fig. 2.1(a). Here \( I \) is the current input to the actuator, and \( y \) is its displacement output, and \( M \) is the bulk magnetization along the rod direction. The block \( W \) (illustrated in Fig. 2.1(b)) models the ferromagnetic hysteresis and the eddy current losses, and \( G(s) \), a second-order linear system, models the magnetoelastic dynamics of the magnetostrictive rod.

In \( W \) a hysteretic inductor and a resistor (representing eddy current losses) are connected in parallel. Let \( I_1 \) be the current through the inductor. As the voltages across the two circuit elements are equal, one has

\[
N_m A_m \dot{B}(t) = R_{\text{eddy}}(I(t) - I_1(t)),
\]

where \( B \) is the magnetic flux density, and \( N_m \) and \( A_m \) are the number of coil turns and cross-sectional area of the rod, respectively. Letting \( H \) be the magnetic field along the rod, one has

\[
B(t) = \mu_0 (H(t) + M(t)),
\]

where \( \mu_0 \) is the permeability of free space. Since \( H(t) = c_0 I_1(t) \), where \( c_0 \) is the coil factor, one obtains the following model for \( W \) considering the ferromagnetic hysteresis \( M(t) = \mathcal{H}[H(\cdot)](t) \):

\[
\begin{cases}
\dot{H}(t) + M(t) = -\frac{c_m}{c_0} H(t) + c_m I(t) \\
M(t) = \mathcal{H}[H(\cdot)](t)
\end{cases},
\]

where \( c_m \triangleq \frac{R_{\text{eddy}}}{\mu_0 N_m A_m} \). It is not hard to see (2.3) is of the form (1.1) if one lets \( x = H \), \( z = M \), \( u = I \), \( c = 1 \), and \( f(x, z, u) = -\frac{c_m}{c_0} x + c_m u \). Fig. 2.2 shows that, with the Preisach operator to model \( \mathcal{H}[\cdot] \), the model (2.3) together with other blocks in Fig. 2.1 predicts well the dynamic hysteresis behavior in magnetostrictive actuators in a wide frequency range.

2.2. Open-loop control of a piezoelectric actuator. Fig. 2.3 shows a circuit schematic where a voltage \( V \) is applied to a piezoelectric actuator. Assume that the piezoelectric material has thickness \( d \) and electrode area \( A \). It is modeled as a nonlinear capacitor (ignoring the displacement-induced voltage under actuation) with hysteretic relationship between the polarization \( P \) and the electric field \( E \). The resistor \( R \) represents electrical resistance in the circuit. Let \( V_p \) denote the voltage across the piezo actuator, i.e., \( V_p(t) = V(t) - RI(t) \). The electric displacement \( D \) satisfies:

\[
D(t) = \varepsilon_0 E(t) + P(t)
= \varepsilon_0 E(t) + \mathcal{H}[E(\cdot)](t),
\]

where \( \varepsilon_0 \) is the permittivity of free space. Utilizing the charge \( Q(t) = AD(t) \), \( I(t) = \dot{Q}(t) \), and \( V_p(t) = dE(t) \), one derives

\[
\begin{cases}
\dot{E}(t) + \frac{1}{\varepsilon_0} \dot{P}(t) = -dc_p E(t) + c_p V(t) \\
P(t) = \mathcal{H}[E(\cdot)](t)
\end{cases},
\]
2.3. Feedback control of hysteretic systems. The integration of inverse compensation and feedback proves to be effective in the control of hysteresis (see, e.g., [36]). Using certain properties of the hysteresis model, one could also control a hysteretic system directly with some linear feedback controller. For instance, passivity and dissipativity properties of the Preisach operator were explored for velocity control [16] and position control [14] of SMA actuators, respectively. Fig. 2.4 shows the diagram of position control for an SMA wire. The controller $C(s)$ dictates the power

$$
\Delta x = \frac{1}{c_0} c_p x. \quad \text{Again, Eq. (2.4) is of the form (1.1) by taking } x = E, z = P, u = V, c = \frac{1}{c_0}, \text{and } f(x, z, u) = -d c_p x + c_p u.
$$
to be imparted into the SMA wire through Joule heating, which is proportional to the square of current $I^2$. The block $\frac{b}{s + a}$ models the heating dynamics and its output is the temperature $T$. $T$ is then linked to the position $y$ through a hysteretic map $H[\cdot]$. Let $C(s)$ be a Proportional-Derivative (PD) controller: $C(s) = k_p + k_ds$. The closed-loop system is then described by:

$$\begin{align*}
\dot{T}(t) + bk_d\dot{y}(t) &= -aT(t) - bk_p y(t) + (bk_p y_{ref}(t) + bk_d \dot{y}_{ref}(t)) \\
y(t) &= H[T(\cdot)](t)
\end{align*}$$

which is of the form (1.1) if one lets $x = T$, $z = y$, $u = bk_p y_{ref} + bk_d \dot{y}_{ref}$, $c = bk_d$, and $f(x, z, u) = -ax - bk_p z + u$.

One can show that if a Proportional-Integral (PI) controller is used to control the velocity $\dot{y}$, the resulting closed-loop system is also of the form (1.1).

Note that, with $C(s) = k_p + k_ds$, the system in Fig. 2.4 can also be regarded as using a lead (or lag) compensator to control a generic hysteresis operator $H$, hence the characterization (2.5) is general and not limited to the example of SMA control.

3. The Preisach operator. The Preisach operator is briefly reviewed in this section to fix the notation and provide the background for later developments. A detailed treatment can be found in [27, 40, 6].

For $r > 0$ and $s$, define a delayed relay $\hat{\gamma}_{r,s}$ with thresholds $s - r$ and $s + r$, as illustrated in Fig. 3.1. Note that $s$ is the mid-point of the two thresholds while $r$ is the half-width of the delayed relay. For $x(\cdot) \in C([0, T])$ (the space of continuous functions on $[0, T]$) and an initial configuration $\zeta \in \{-1, 1\}$, $\omega = \hat{\gamma}_{r,s}[x(\cdot), \zeta]$ is defined as, for $t \in [0, T]$,

$$\omega(t) \triangleq \begin{cases} 
-1 & \text{if } x(t) < s - r \\
1 & \text{if } x(t) > s + r \\
\omega(t^-) & \text{if } s - r \leq u(t) \leq s + r
\end{cases},$$

where $\omega(0^-) = \zeta$ and $t^- \triangleq \lim_{\epsilon \to 0, \epsilon > 0} t - \epsilon$. One calls $\hat{\gamma}_{r,s}$ an elementary Preisach hysteron, or just hysteron for brief.
The Preisach operator is a weighted superposition of hysterons. Define the Preisach plane

\[ P_0 \triangleq \{(r, s) \in \mathbb{R}^2 : r > 0\}, \]

where \((r, s) \in P_0\) is identified with \(\hat{\gamma}_{r,s}\). For \(x \in C([0,T])\) and a Borel measurable configuration \(\zeta_0\) of all delayed relays, \(\zeta_0 : P_0 \to \{-1, 1\}\), the output of the Preisach operator \(\Gamma\) is defined as

\[ \Gamma[x, \zeta_0](t) = \int_{P_0} \hat{\gamma}_{r,s}[x(\cdot), \zeta_0(r, s)](t) d\nu(r, s), \quad (3.1) \]

where \(\nu\) is a finite Borel measure on \(P_0\), called the Preisach measure. In this paper \(\nu\) is called nonsingular if \(|\nu|\) is absolutely continuous with respect to the two-dimensional Lebesgue measure. For a nonsingular \(\nu\), (3.1) can be rewritten as

\[ \Gamma[x, \zeta_0](t) = \int_{P_0} \mu(r, s) \hat{\gamma}_{r,s}[x(\cdot), \zeta_0(r, s)](t) drds, \quad (3.2) \]

for some Borel measurable function \(\mu\), called the Preisach density function. It is assumed in this paper that \(\mu\) has a compact support

\[ P \triangleq \{(r, s) \in P_0 : s - r \geq -r_0, \ s + r \leq r_0\}, \text{ for } r_0 > 0. \]

\(P\), the finite triangle in Fig. 3.2(a), is also called the Preisach plane when no confusion arises.

At any time \(t\), \(P\) is a disjoint union of two sets, \(P_+(t)\) and \(P_-(t)\), where \(P_+(t)\) (\(P_-(t)\), resp.) consists of points \((r, s)\) such that the output of \(\hat{\gamma}_{r,s}\) at \(t\) is +1 (-1, resp.). Under mild conditions, each of \(P_+(t)\) and \(P_-(t)\) is a connected set, which is illustrated in Fig. 3.2:

- Assume \(x(t_0) \leq -r_0\) at some initial time \(t_0\). Then every hysteron in \(P\) outputs -1, and thus \(P_- = P, \ P_+ = \emptyset\);
- Next suppose that \(x(t)\) increases monotonically to some maximum value at \(t_1\). As \(x(t)\) increases, the output of \(\hat{\gamma}_{r,s}\) switches to +1 if \(s + r = x(t)\), and thus a line segment \(s + r = x(t)\) separates \(P_-(t)\) and \(P_+(t)\) (Fig. 3.2(b));
- Now suppose that \(x(t)\) decreases monotonically to some minimum value at \(t_2\). As \(x(t)\) decreases, the output of \(\hat{\gamma}_{r,s}\) switches to -1 if \(s - r = x(t)\), and thus a line segment \(s - r = x(t)\) with decreasing \(x(t)\) becomes part of the boundary between \(P_-(t)\) and \(P_+(t)\) (Fig. 3.2(c)).
Further input reversals could lead to a boundary with more segments (3.2(d)), each of which has a slope of +1 or −1. The boundary always intersects the $s$–axis at $s = x(t)$.

For a Preisach operator with nonsingular measure $\nu$, its output is solely dependent on the boundary separating $P_-$ and $P_+$ since the contribution from hysterons on the (one-dimensional) boundary is zero. The boundary $\psi$ is thus called the memory curve of the Preisach operator. Although practically a memory curve is only composed of segments of slope ±1, the set of memory curves is defined to include all curves with Lipschitz constant ±1:

**Definition 3.1.** The set $\Psi$ of memory curves is defined to be the set of continuous functions

$$\psi : [0, r_0] \to \mathbb{R}$$

such that

1. $|\psi(r_1) - \psi(r_2)| \leq |r_1 - r_2|, \forall r_1, r_2 \in [0, r_0]$;
2. $\psi(r_0) = 0$.

$\Psi$ defined as above is a complete metric space [15], which will facilitate analysis in the sequel. In addition this will allow one to include certain initial hysteron config-
Four steps:

1. Assumption that $\psi(r) = 0, \forall r \in [0, r_0]$, can represent the demagnetized virgin state in ferromagnetics [27, 40].

The memory curve $\psi_0$ at $t = 0$ is called the initial memory curve and it represents the initial condition of the Preisach operator. One can identify a configuration of hysterons $\zeta$ with a memory curve $\psi$ in the following way: $\zeta(r, s) = 1 (-1, \text{resp.})$ if $(r, s)$ is below (above, resp.) the graph of $\psi$. Note that it does not matter whether $\zeta$ takes 1 or $-1$ on the graph of $\psi$. In the sequel we will put the initial memory curve $\psi_0$ as the second argument of $\Gamma$, where $\Gamma[\cdot, \psi_0] \equiv \Gamma[\cdot, c_{\psi_0}]$.

The following theorem summarizes some basic properties of the Preisach operator:

**Theorem 3.2.** Let $\nu$ be a Preisach measure. Let $x, x_1, x_2 \in C([0, T])$ and $\psi_0 \in \Psi$.

1. **(Rate independence)** If $\phi : [0, T] \rightarrow [0, T]$ is an increasing continuous function satisfying

$$\phi(0) = 0 \text{ and } \phi(T) = T,$$

then $\Gamma[x \circ \phi, \psi_0](t) = \Gamma[x, \psi_0](\phi(t)), \forall t \in [0, T]$, where “$\circ$” denotes composition of functions;

2. **(Continuity)** If $\nu$ is nonsingular, then $\Gamma[\cdot, \psi_0] : C([0, T]) \rightarrow C([0, T])$ is strongly continuous (in the sup norm);

3. **(Lipschitz continuity)** Let $\nu$ be nonsingular with a density function $\mu$. Then for any $\psi_0 \in \Phi$, $\Gamma[\cdot, \psi_0]$ is Lipschitz continuous on $C([0, T])$ with Lipschitz constant $C_L$, where

$$C_L = 2 r_0 \sup_{(r, s) \in P} \mu(r, s),$$

and “$\sup$” denotes the essential supremum;

4. **(Piecewise monotonicity)** Let $\nu \geq 0$. If $x$ is either nondecreasing or nonincreasing on some interval in $[0, T]$, then so is $\Gamma[x, \psi_0]$;

5. **(Order preservation)** Let $\nu \geq 0$. If $x_1 \leq x_2$ on $[0, T]$, then $\Gamma[x_1, \psi_0] \leq \Gamma[x_2, \psi_0]$ on $[0, T]$.

4. Well-posedness. The well-posedness of the following system is studied, where the generic hysteresis operator $\mathcal{H}$ in (1.1) is replaced by the Preisach operator $\Gamma$:

$$\begin{cases}
\dot{x}(t) + c z(t) = f(x, z, u) \\
\dot{z}(t) = \Gamma[x(\cdot), \psi_0](t)
\end{cases}$$

Eq. (4.1) involves time derivatives of both $x$ and $z$. It is well known that, in general, a Preisach operator does not map $C^1$ (the space of continuously differential functions) into $C^1$. Indeed, when corners in the memory curve are eliminated, discontinuities occur in the output derivative if the Preisach measure does not vanish in a neighborhood of the corner [40]. Hence we will interpret (4.1) in the sense of Carathéodory [41].

**Theorem 4.1.** Assume that $f(x, z, u)$ is Lipschitz continuous in $x$ and $z$, and continuous in $u$. Let the Preisach measure $\nu$ be nonnegative and nonsingular. Let $u(\cdot)$ be piecewise continuous. Then for any $\psi_0 \in \Psi$, for any $T > 0$, there exists a unique pair $\{x(\cdot), z(\cdot)\} \in C([0, T]) \times C([0, T])$ satisfying (4.1) almost everywhere.

**Proof.** 1. The existence of solutions will be proved first. The proof consists of four steps:

- **Step 1.** For an integer $N$, solve for nodal points $\{x_N^{(m)}, z_N^{(m)}\}_{m=0}^N$ through an Euler scheme:
• Step 2. Obtain piecewise linear functions \( \{x_N^m(\cdot), z_N^m(\cdot)\} \) by interpolating \( \{x_N^m, z_N^m\}_{m=0}^{N} \).

• Step 3. Show \( x_N(\cdot) \to \tilde{x}(\cdot) \) and \( z_N(\cdot) \to \tilde{z}(\cdot) \) uniformly as \( N \to \infty \), with \( \{\tilde{x}(\cdot), \tilde{z}(\cdot)\} \in C([0, T]) \times C([0, T]) \).

• Step 4. Show \( \{\tilde{x}(\cdot), \tilde{z}(\cdot)\} \) satisfies (4.1).

The proof is laid out in detail next.

From \( \psi_0 \), one can evaluate initial values \( x(0) \) and \( z(0) \). Eq. (4.1) is equivalent to the following: \( \forall t \in [0, T] \),

\[
\begin{cases}
x(t) + cz(t) = x(0) + cz(0) + \int_0^t f(x(\tau), z(\tau), u(\tau))d\tau \\
z(t) = \Gamma[x(\cdot), \psi_0](t)
\end{cases}
\]

Eq. (4.1) can be approximated with an Euler scheme: for \( N > 0 \) and \( h_N = \frac{T}{N} \), solve consecutively

\[
\begin{align*}
x_N^{(m+1)} - x_N^{(m)} &= f(x_N^{(m)}, z_N^{(m)}, u_N^{(m)}) \\
z_N^{(m+1)} &= \Gamma[x_N^{(m+1)}, \psi_m]
\end{align*}
\]

for \( 0 \leq m \leq N - 1 \), with \( x_N^{(0)} = x(0), z_N^{(0)} = z(0), u_N^{(m)} = \frac{1}{h_N} \int_{h_N}^{(m+1)h_N} u(\tau)d\tau, \) and \( \psi_m \) the memory curve resulting from application of the sequence \( \{x_N(i)\}_{i=1}^m \). To make sense out of (4.3), it is tacitly understood that the input of \( \Gamma \) is changed monotonically from \( x_N^{(m)} \) to \( x_N^{(m+1)} \). From the continuity and the piecewise monotonicity properties of \( \Gamma[\cdot, \psi_m] \), the left-hand side of the first equation in (4.3) is a continuous and strictly increasing function of \( x_N^{(m+1)} \). Thus (4.3) admits a unique solution for \( x_N^{(m+1)} \) and thus for \( z_N^{(m+1)} \).

One can show the boundedness of \( x_N^{(m)}, z_N^{(m)} \). The boundedness of \( z_N^{(m)} \) follows from the fact that the Preissach measure is finite: \( |z_N^{(m)}| \leq z_{sat} \), where \( z_{sat} \) denotes the saturation output of \( \Gamma \). By the piecewise monotonicity of \( \Gamma \), \( x_N^{(m+1)} - x_N^{(m)} \) and \( z_N^{(m+1)} - z_N^{(m)} \) have the same sign. From (4.3),

\[
\left|\frac{x_N^{(m+1)} - x_N^{(m)}}{h_N}\right| \leq |f(x_N^{(m)}, z_N^{(m)}, u_N^{(m)})| \\
\leq |f(x_N^{(m)}, z_N^{(m)}, u_N^{(m)}) - f(x_N^{(0)}, z_N^{(m)}, u_N^{(m)})| + |f(x_N^{(0)}, z_N^{(m)}, u_N^{(m)})|.
\]

(4.4)

From \( \psi_0 \in \Psi, |x_N^{(0)}| \leq r_0 \). Since \( f(x, z, u) \) is Lipschitz continuous in \( x \),

\[
|f(x_N^{(m)}, z_N^{(m)}, u_N^{(m)}) - f(x_N^{(0)}, z_N^{(m)}, u_N^{(m)})| \leq c_1|x_N^{(m)} - x_N^{(0)}| \leq c_1|x_N^{(m)}| + c_2,
\]

for constants \( c_1 > 0, c_2 > 0 \). Due to the piecewise continuity of \( u(\cdot) \), \( |u_N^{(m)}| \leq c_3 \), where \( c_3 > 0 \) is a constant independent of \( N \). As \( f(x, z, u) \) is continuous in its arguments,

\[
|f(x_N^{(0)}, z_N^{(m)}, u_N^{(m)})| \leq c_4.
\]

Thus (4.4) implies

\[
\left|\frac{x_N^{(m+1)} - x_N^{(m)}}{h_N}\right| \leq c_1|x_N^{(m)}| + c_5,
\]

(4.5)
where \( c_5 = c_2 + c_1 \). From (4.5),

\[
|x_N^{(m+1)}| \leq (1 + c_1 h_N) |x_N^{(m+1)}| + c_5 h_N
\]

\[
\leq (1 + c_1 h_N)^{m+1} |x_N^{(0)}| + c_5 h_N \sum_{k=0}^{m} (1 + c_1 h_N)^k
\]

\[
\leq r_0 (1 + c_1 h_N)^N + \frac{c_5}{c_1} ((1 + c_1 h_N)^N - 1)
\]

\[
\leq (r_0 + \frac{c_5}{c_1}) \left( 1 + \frac{c_1 T}{N} \right)^N
\]

\[
\leq (r_0 + \frac{c_5}{c_1}) e^{c_1 T} = \bar{C}, \tag{4.6}
\]

and \( \bar{C} > 0 \) is independent of \( N \). The boundedness of \( x_N^{(m)} \) is thus established.

One then obtains \( x_N(\cdot), z_N(\cdot) \in C([0,T]) \) by linearly interpolating \( \{x_N^{(m)}\} \) and \( \{z_N^{(m)}\} \), i.e.,

\[
x_N(t) = \alpha x_N^{(m)} + (1 - \alpha) x_N^{(m+1)}, \quad \text{for} \quad t = (m + \alpha) h_N, 0 \leq \alpha \leq 1,
\]

and analogously for \( z_N(\cdot) \). By (4.5) and (4.6), \( x_N(\cdot) \) is Lipschitz continuous with Lipschitz constant \( L = c_1 \bar{C} + c_5 \). Therefore, \( \{x_N(\cdot)\}_{N \geq 1} \) is an equicontinuous and equibounded family of functions, and from the Ascoli-Arzelà Theorem, by extracting a subsequence if necessary, \( x_N(\cdot) \to \hat{x}(\cdot) \in C([0,T]) \) uniformly as \( N \to \infty \). By replacing the left-hand side of (4.5) with \( c e^{(c^{(m+1)} - x_N^{(m)})} h_N^{N+1} \), one obtains the Lipschitz continuity of \( z_N(\cdot) \) with Lipschitz constant \( \frac{C}{\bar{C}} \). Thus \( z_N(\cdot) \to \tilde{z}(\cdot) \in C([0,T]) \) uniformly. It’s easy to see that \( \hat{x}(\cdot) \) and \( \tilde{z}(\cdot) \) are also Lipschitz continuous and thus differentiable almost everywhere.

Now define \( e_N(t) = \hat{x}_N(t) + c \hat{z}_N(t) - f(x_N(t), z_N(t), u(t)) \) for \( t \) where \( \hat{x}_N(t) \) and \( \hat{z}_N(t) \) exist. By the definitions of \( x_N(\cdot) \) and \( z_N(\cdot) \), \( e_N(t) \) is well defined a.e. and

\[
e_N(t) = f(x_N^{(m)}(\cdot), z_N^{(m)}(\cdot), u_N^{(m)}(\cdot)) - f(x_N(t), z_N(t), u(t)), \quad \forall t \in (mh_N, (m + 1)h_N).
\]

Integrating

\[
\hat{x}_N(t) + c \hat{z}_N(t) = f(x_N(t), z_N(t), u(t)) + e_N(t)
\]

from 0 to \( t \) leads to

\[
x_N(t) + cz_N(t) = x_N(0) + cz_N(0) + \int_0^t f(x_N(\tau), z_N(\tau), u(\tau)) d\tau + \int_0^t e_N(\tau) d\tau. \tag{4.7}
\]

As \( N \to \infty \), one can show \( \int_0^T |e_N(\tau)| d\tau \to 0 \), and \( \hat{x}(\cdot) \) and \( \tilde{z}(\cdot) \) satisfy the first part of (4.2). We are left to show \( \tilde{z}(\cdot) = \Gamma[\tilde{z}(\cdot), \psi_0(\cdot)] \), \( \forall t \in [0,T] \).

Let \( \tilde{z}_N \equiv \Gamma[x_N(\cdot), \psi_0(\cdot)] \). By the continuity of \( \Gamma \), \( \tilde{z}_N \to \Gamma[\tilde{z}(\cdot), \psi_0(\cdot)] \) since \( x_N(\cdot) \to \hat{x}(\cdot) \). Furthermore, \( \tilde{z}_N(mh_N) = z_N(mh_N) \), \( 0 \leq m \leq N \). This, together with the piecewise monotonicity of \( \Gamma \) and Lipschitz continuity of \( z_N \), leads to

\[
\sup_{t \in [0,T]} |z_N(t) - \tilde{z}_N(t)| \leq \frac{L h_N}{c}.
\]
Therefore as $N \to \infty$, $\{x_N\}$ and $\{\tilde{x}_N\}$ have the same limit, i.e.,
\[ \tilde{z}(t) = \Gamma[\tilde{x}(\cdot), \psi_0](t), \forall t \in [0, T]. \]

2. Next the uniqueness of the solution will be proved by contradiction. Assume that there exist two solutions $\{x_1(\cdot), z_1(\cdot)\}$ and $\{x_2(\cdot), z_2(\cdot)\}$ and $x_1(t') \neq x_2(t')$ for some $t' > 0$ (note $x_1(0) = x_2(0)$). Define $e_x = x_2 - x_1$ and $e_z = z_2 - z_1$. From (4.2),
\[ e_x(t) + ce_z(t) = \int_0^t (f(x_2(\tau), z_2(\tau), u(\tau)) - f(x_1(\tau), z_1(\tau), u(\tau)))d\tau. \] (4.8)

Define $\bar{t}$ to be
\[ \bar{t} = \sup_{t' \leq t'} \{ t : e_x(\tau) \equiv 0, \forall \tau \in [0, t] \}. \]

By the continuity of $e_x$, there exists $\delta_1 > 0$ such that $e_x(t)$ has a constant sign, say (without loss of generality), $> 0$, on $(\bar{t}, \bar{t} + \delta_1]$. By the order preservation property of $\Gamma$, $e_z(t) \geq 0$, $\forall t \in [\bar{t}, \bar{t} + \delta_1]$. Therefore, $\forall t \in [0, \bar{t} + \delta_1],
\begin{align*}
|e_x(t)| & \leq \int_0^t |f(x_2(\tau), z_2(\tau), u(\tau)) - f(x_1(\tau), z_1(\tau), u(\tau))|d\tau \\
& \leq \int_0^t |f(x_2(\tau), z_2(\tau), u(\tau)) - f(x_1(\tau), z_2(\tau), u(\tau))| \\
& \quad + |f(x_1(\tau), z_2(\tau), u(\tau)) - f(x_1(\tau), z_1(\tau), u(\tau))|d\tau \\
& \leq \int_0^t L_x|e_x(\tau)| + L_z|e_z(\tau)|d\tau,
\end{align*}
(4.9)

where $L_x$ and $L_z$ are Lipschitz constants of $f$ with respect to $x$ and $z$, respectively. One can also find $\delta_2$, $0 < \delta_2 \leq \delta_1$, such that $x_1$ and $x_2$ are both monotonic on $[\bar{t}, \bar{t} + \delta_2]$ (they could be both increasing, both decreasing, or $x_1$ increasing and $x_2$ decreasing). Since $x_1$ and $x_2$ agree on $[0, \bar{t}]$, for $t \in [0, \bar{t} + \delta_2]$, the following holds:
\[ |e_z(t)| \leq C_L|e_x(t)|, \] (4.10)

where $C_L$ is the Lipschitz constant of $\Gamma$ as defined in (3.3). Combining (4.9) and (4.10), one gets
\[ |e_x(t)| \leq (L_x + C_L L_z) \int_0^t |e_x(\tau)|d\tau, \forall t \in [0, \bar{t} + \delta_2], \]

which implies $|e_x(t)| \leq 0$ by the Gronwall inequality, $\forall t \in [0, \bar{t} + \delta_2]$. However, this contradicts $|e_x(t)| > 0$, $\forall t \in (\bar{t}, \bar{t} + \delta_2]$. $\square$

5. **System-theoretic properties.** The state of (4.1) is the infinite-dimensional memory curve $\psi[t]$ since both $x(t)$ and $z(t)$ can be evaluated from $\psi[t]$. In this section system-theoretic properties of the dynamical system (4.1) are studied by taking $u$ as the input, and $\{x, z\}$ as the output. In particular, the notions of (approximate) reachability, observability, input-output stability and stability of equilibria are explored for (4.1) for more specific forms of $f(x, z, u)$. 

5.1. Approximate reachability. An appropriate metric is needed for the discussion of reachability and approximate reachability of (4.1). For a nonsingular Preisach measure $\nu$, define a metric $\| \cdot \|_\Psi$ on the set $\Psi$ of memory curves: for $\psi_1, \psi_2 \in \Psi$,

$$\| \psi_1 - \psi_2 \|_\Psi \triangleq \int \int |\zeta_{\psi_1}(r, s) - \zeta_{\psi_2}(r, s)| \mu(r, s) dr ds,$$

(5.1)

where $\zeta_{\psi_i}$ denotes the hysteron configuration corresponding to $\psi_i$ (refer to Section 3). For a well-posed system as stated in Theorem 4.1, for $u(\cdot) \in PC([0, T])$ (the space of piecewise continuous functions on $[0, T]$), $x(\cdot)$ (input to the Preisach operator) is continuous, and one can verify that the corresponding memory curve $\psi[\cdot]$ is continuous in time with the metric (5.1). The latter is denoted as $\psi[\cdot] \in C([0, T], \Psi)$. The state evolution map $\Xi : C([0, T]) \times \Psi \to C([0, T], \Psi)$ can be defined as: $\forall u(\cdot) \in PC([0, T])$, $\forall \psi_0 \in \Psi$,

$$\psi[t] = \Xi[u(\cdot), \psi_0](t),$$

(5.2)

i.e., $\Xi$ describes the state transition under a certain input.

The following definitions are motivated by those for a Preisach operator [15]:

**Definition 5.1.**

- The state space $\Psi$ of (4.1) is reachable if, for any $\psi_1, \psi_2 \in \Psi$, there exist a finite $T > 0$ and $u(\cdot) \in PC([0, T])$, such that $\psi_2 = \Xi[u(\cdot), \psi_1](T)$.
- The state space $\Psi$ of (4.1) is approximately reachable if, for any $\psi_1, \psi_2 \in \Psi$ and any $\epsilon > 0$, there exist a finite $T > 0$, $u(\cdot) \in PC([0, T])$, and $\psi_\epsilon \in \Psi$ such that $\| \psi_2 - \psi_\epsilon \|_\Psi \leq \epsilon$ and $\psi_\epsilon = \Xi[u(\cdot), \psi_1](T)$.

**Theorem 5.2.** Let the assumptions on $\nu$ and $f(\cdot, \cdot, \cdot)$ in Theorem 4.1 be satisfied. The state space of (4.1) is not reachable. If for any $T > 0$, for any Lipschitz continuous $\bar{x}(\cdot), \tilde{x}(\cdot) \in C([0, T])$, the equation

$$f(\bar{x}(t), \tilde{x}(t), u(t)) = \dot{x}(t) + c\tilde{x}(t)$$

(5.3)

admits a solution for $u(\cdot) \in PC([0, T])$, then the state space of (4.1) is approximately reachable.

**Proof.** Define

$$\Psi_R \triangleq \{ \psi \in \Psi : \psi \text{ consists of segments of slopes } \pm 1 \}$$

(5.4)

The state transition is dictated by $u(\cdot)$, but through its effect on $x(\cdot)$, the input to $\Gamma$. For any $\psi_2 \in \Psi_R$, one can find $\psi_\epsilon \in \Psi_R$ such that no $x(\cdot) \in C([0, T])$ can yield $\psi_2$ for a finite $T$. Thus no $u(\cdot) \in PC([0, T])$ satisfies $\Xi[u(\cdot), \psi_1](T) = \psi_2$. Therefore $\Psi$ is not reachable.

On the other hand, it can be easily verified that $\Psi_R$ is a dense subset of $\Psi$. For any $\psi_2 \in \Psi$ and any $\epsilon > 0$, one can find $\psi_\epsilon \in \Psi_R$ with $\| \psi_2 - \psi_\epsilon \|_\Psi \leq \epsilon$. For any $\psi_1 \in \Psi$ and $T > 0$, there is an input $\bar{x}(\cdot) \in C([0, T])$ to the Preisach operator $\Gamma$ that yields $\psi_\epsilon$ at $T$. In particular, this can be achieved by reversing $\bar{x}$ in accordance with the corners of $\psi_\epsilon$. By the rate-independence property of $\Gamma$, such an $\bar{x}(\cdot)$ can be made continuously differentiable. Define $\tilde{x}$ by $\tilde{x}(t) = \Gamma[\bar{x}(\cdot), \psi_1](t)$. Note that $\tilde{x}$ is differentiable a.e. from the Lipschitz continuity of $\Gamma$. From the assumptions, one can obtain $u \in PC([0, T])$ by solving (5.3). Feeding this $u$ to (4.1), one has $x(\cdot) \equiv \bar{x}(\cdot)$ by the uniqueness of the solution to (4.1). Therefore, $\Xi[u(\cdot), \psi_1] = \psi_\epsilon$ and $\Psi$ is approximately reachable. $\square$

Note that the condition (5.3) is not stringent. In particular, it is satisfied if $f(\cdot, \cdot, \cdot)$ has the form $f(x, z, u) = f_0(x, z) + au$ with $a \neq 0$, and all the examples in Section 2 satisfy this condition.
5.2. Observability. The definition of observability is first given for (4.1).

Definition 5.3. For $\psi_1, \psi_2 \in \Psi$, $\psi_1$ is distinguishable from $\psi_2$, if there exist a finite $T > 0$ and $u(\cdot) \in PC([0,T])$, such that $x_1(t') \neq x_2(t')$ or $z_1(t') \neq z_2(t')$ for some $t' \in [0,T]$, where $\{x_i(\cdot), z_i(\cdot)\}$ is the solution to (4.1) with input $u(\cdot)$ and initial memory curve $\psi_i$, for $i = 1, 2$. The system (4.1) is observable if any state $\psi \in \Psi$ is distinguishable from any other state.

Before proceeding, we define two quantities, $\frac{dz}{dx}(\psi, +)$ and $\frac{dz}{dx}(\psi, -)$, which carry the interpretation of derivatives of $z$ with respect to $x$ as $x$ is increased and decreased, respectively. They will prove useful in the discussion of both observability and other properties.

Let the Preisach measure $\nu$ be nonnegative and nonsingular. For an increasing $x$, the memory curve evolves as illustrated in Fig. 5.1(a). Let $\Delta^+$ be the region of hysterons switched from $-1$ to $+1$, and $\Delta z^+$ be the change in output:

$$\Delta z^+ = 2 \int \int_{\Delta^+} \mu(r, s) dr ds.$$

Let

$$\frac{dz}{dx}(\psi, +) = \lim_{\Delta x \to 0^+} \frac{\Delta z^+}{\Delta x},$$

if the limit in (5.5) exists. Analogously one can define $\frac{dz}{dx}(\psi, -)$. For a continuous Preisach density function $\mu$, $\frac{dz}{dx}(\psi, \pm)$ are well-defined and given by:

$$\frac{dz}{dx}(\psi, +) = 2 \int_0^{r^+} \mu(r, x - r) dr,$$

$$\frac{dz}{dx}(\psi, -) = 2 \int_0^{r^-} \mu(r, x + r) dr,$$

where $r^\pm$ are defined as in Fig. 5.1. Due to the piecewise monotonicity of $\Gamma$ and the finiteness of $\nu$, the following holds: $\forall \psi \in \Psi$,

$$0 \leq \frac{dz}{dx}(\psi, \pm) \leq \bar{D},$$

(5.6)

for some constant $\bar{D} > 0$.

The observability conditions of (4.1) are stated in the following theorem:

Theorem 5.4. Let the assumptions on $\nu$ and $f(\cdot, \cdot, \cdot)$ in Theorem 4.1 hold.
• (Necessity) If the system (4.1) is observable, the following is true:
\[ \| \psi_1 - \psi_2 \|_\Psi > 0, \forall \psi_1, \psi_2 \in \Psi \text{ such that } \psi_1 \neq \psi_2. \] (5.7)

• (Sufficiency) Assume that the Preisach density \( \mu \) is piecewise continuous. Further assume that for any \( \psi \in \Psi \), there exist a finite \( T > 0 \) and \( u(\cdot) \in PC([0, T]) \), such that \( x(\cdot) \) traverses entire interval \([-r_0, r_0]\). Then the system (4.1) is observable if (5.7) is satisfied.

Proof. 1. First the necessity part will be proved by contradiction. Suppose that (5.7) is violated for some \( \psi_1, \psi_2 \in \Psi, \psi_1 \neq \psi_2 \). This implies that the Preisach density \( \mu \) vanishes throughout the region between the two curves (shaded area in Fig. 5.2(a)). Note that the Lebesgue measure of this area is positive. Based on \( \psi_2 \), one can obtain another memory curve \( \psi_3 \) such that (see Fig. 5.2(b) for illustration): \( \| \psi_3 - \psi_1 \|_\Psi = 0, \psi_3 \neq \psi_1 \), and \( \psi_3(0) = \psi_1(0) \) (memory curves \( \psi_1 \) and \( \psi_3 \) intersect the \( s \)-axis at the same point \( x \)).

It can be shown that \( \psi_1 \) and \( \psi_3 \) are not distinguishable. Indeed the following two observations can be made:

- The initial values \( \{ x(0), z(0) \} \) corresponding to \( \psi_1 \) and \( \psi_3 \), respectively, are identical;
- For any \( x(\cdot) \in C([0, T]) \), \( \Gamma[x(\cdot), \psi_1] = \Gamma[x(\cdot), \psi_3] \).

Now for any \( u(\cdot) \in PC([0, T]) \), one can establish that the two trajectories \( \{ x(\cdot), z(\cdot) \} \) starting from \( \psi_1 \) and \( \psi_3 \), respectively, are identical (using, e.g., the Euler scheme as in the proof of Theorem 4.1). Thus the system (4.1) is not observable.

![Illustration of the observability theorem (necessity). (a) Two memory curves \( \psi_1 \neq \psi_2 \), \( \| \psi_1 - \psi_2 \|_\Psi = 0 \) (solid line: \( \psi_1 \), dashed line: \( \psi_2 \)); (b) \( \psi_3 \) obtained from \( \psi_2 \), with \( \psi_3(0) = \psi_1(0) = x_1 \).](attachment:image.png)

2. Next the sufficiency part will be proved. Pick any two memory curves \( \psi_1, \psi_2 \in \Psi, \psi_1 \neq \psi_2 \). Assume \( \{ x_1(0), z_1(0) \} = \{ x_2(0), z_2(0) \} \), where \( \{ x_i(0), z_i(0) \} \) denotes the initial value corresponding to \( \psi_i \), for \( i = 1, 2 \); otherwise \( \psi_1 \) and \( \psi_2 \) are already distinguishable by taking \( t' = 0 \). Fig. 5.3(a) illustrates such a pair of memory curves. From the figure,
\[ z_1(0) - z_2(0) = 2 \int_\Omega \int_{D_1} \mu(r, s) dr ds - 2 \int_\Omega \int_{D_2} \mu(r, s) dr ds + \nu_\infty = 0, \] (5.8)
where \( \nu_\infty \) represents the contribution from the region \( r > r^* \). By hypothesis, there exist a finite \( T, t_1 \leq t_2 < T \), and \( u(\cdot) \in PC([0, T]) \), such that
\[ x_1(t) \begin{cases} \leq x^*, & t < t_1 \\ = x^*, & t = t_1 \\ > x^*, & t_1 < t \leq t_2 \end{cases} \]
where \( x^* \) is the value as indicated in Fig. 5.3(b).

**Fig. 5.3. Illustration of the observability theorem (sufficiency).** (a) Two memory curves \( \psi_1 \neq \psi_2 \) with \( \{x_1(0), z_1(0)\} = \{x_2(0), z_2(0)\} \) (solid line: \( \psi_1 \), dashed line: \( \psi_2 \), \( \psi_1 \) and \( \psi_2 \) overlapping on \([0, r_1] \)); (b) critical value of \( x \) to break the balance \( z_1 = z_2 \).

Starting from \( \psi_2 \), apply the same \( u(\cdot) \) as above. Note that \( \{x_2(t), z_2(t)\} = \{x_1(t), z_1(t)\}, \forall t \in [0, t_1] \). Starting from \( t = t_1 \), rewrite the first equation in (4.1) as

\[
\dot{x}(t) + \frac{dz}{dx}(\psi(t), \text{sgn}(\dot{x}(t))) \cdot \dot{x}(t) = f(x(t), z(t), u(t)),
\]

which implies

\[
\dot{x}(t) = \frac{f(x(t), z(t), u(t))}{1 + cg(t)}, \quad (5.9)
\]

\[
\frac{dz}{dx}(\psi_1(t), \text{sgn}(\dot{x}_1(t))) = \frac{dz}{dx}(\psi_2(t), \text{sgn}(\dot{x}_2(t))) \geq \epsilon. \quad (5.10)
\]

where \( g(t) \triangleq \frac{dz}{dx}(\psi(t), \text{sgn}(\dot{x}(t))) \). By the assumption on \( \mu \), \( g(t) \) is piecewise continuous. The piecewise continuity of \( \mu \), together with the condition (5.7), implies the existence of \( \delta > 0 \) and \( \epsilon > 0 \) such that, \( \forall t \in [t_1, t_1 + \delta] \),

\[\left| \frac{dz}{dx}(\psi_1(t), \text{sgn}(\dot{x}_1(t))) - \frac{dz}{dx}(\psi_2(t), \text{sgn}(\dot{x}_2(t))) \right| \geq \epsilon. \quad (5.11)\]

If for some \( t' \in [t_1, t_1 + \delta] \), \( z_1(t') \neq z_2(t') \), the claim is proved; otherwise, from (5.9) and (5.11), \( x_1(t) \) and \( x_2(t) \) differ for a finite time starting at \( t_1 \), which will lead to \( x_1(t') \neq x_2(t') \) for some \( t' \in [t_1, t_1 + \delta] \). The proof is now complete. \( \square \)

It is not a strong assumption that, from any initial \( \psi \in \Psi \), \( x(\cdot) \) can traverse \([-r_0, r_0] \) through application of a suitable \( u(\cdot) \). Indeed, all motivating examples in Section 2 meet this requirement.

### 5.3. Input-output stability

Recall the definitions of \( \mathcal{L} \) stability and finite-gain \( \mathcal{L} \) stability for a system [24] (page 197), where \( \mathcal{L} \) denotes signal spaces. Let \( \mathcal{L}_e \) denote the extended space of \( \mathcal{L} \):

\[
\mathcal{L}_e \triangleq \{ u : u_\tau \in \mathcal{L}, \forall \tau \in [0, \infty) \},
\]

where \( u_\tau \) denotes the truncation of \( u \): \( u_\tau(t) = u(t), 0 \leq t \leq \tau \), and \( u_\tau(t) = 0, t > \tau \). A mapping \( \mathcal{M} \) is \( \mathcal{L} \) stable if there exist a class \( \mathcal{K} \) function \( \alpha(\cdot) \) and \( b_0 \geq 0 \), such that,
∀u ∈ \mathcal{L}, \forall \tau \in [0, \infty),
\|(\mathcal{M}u)\|_{\mathcal{L}} \leq \alpha(\|u\|_{\mathcal{L}}) + b_0, \quad (5.12)
where \(\|\cdot\|_{\mathcal{L}}\) denotes the norm on \(\mathcal{L}\). It is finite-gain \(\mathcal{L}\) stable if (5.12) is replaced by
\[\|(\mathcal{M}u)\|_{\mathcal{L}} \leq \gamma \|u\|_{\mathcal{L}} + b_0, \quad (5.13)\]
for some \(\gamma \geq 0\). Two common function spaces are \(L_\infty\) and \(L_2\), and thus \(L_\infty\) stability and \(L_2\) stability are investigated for the system (4.1). Since the input of (4.1) was considered to be \(PC([0, T])\) (Theorem 4.1), \(L_\infty\) and \(L_2\) stability will be checked with input from \(L_\infty \cap PC([0, \infty))\) and \(L_2 \cap PC([0, \infty))\), respectively.

5.3.1. \(L_\infty\) stability. The \(L_\infty\) stability is stated in the following proposition:

**Theorem 5.5.** Let the assumptions on \(\nu\) and \(f(\cdot, \cdot, \cdot)\) in Theorem 4.1 hold. Assume
\[f(x, z, u) = f_1(x) + f_2(z, u), \quad (5.14)\]
where \(f_1(x)\) is a strictly decreasing function. Let \(|x(0)| \leq c_0\), for \(c_0 > 0\). Then (4.1) is \(L_\infty\) stable. Assume further
\[|f_1(x)| \geq c_1|x| + c_2, \quad (5.15)\]
\[|f_2(z, u)| \leq c_3|u| + c_4, \quad \forall z \in [-z_{sat}, z_{sat}], \forall u, \quad (5.16)\]
where \(c_1 > 0\), \(c_3 \geq 0\), \(c_4 \geq 0\) and \(c_2\) are constants, and \(z_{sat}\) denotes the saturation output of the Preisach operator \(\Gamma\) corresponding to \(\nu\). Then (4.1) is also finite-gain \(L_\infty\) stable.

**Proof.** As the Preisach measure \(\nu\) is finite, \(z_{sat} < \infty\) and \(|z(t)| \leq z_{sat}, \forall t\). This implies \(||z(\cdot)||_{L_\infty} \leq z_{sat}, \forall u(\cdot) \in PC([0, \infty))\). Therefore, the key is to bound \(||x(\cdot)||_{L_\infty}\).

Since \(f_2(z, u)\) is continuous in \(z\) and \(u\), for any \(\bar{u} > 0\),
\[C(\bar{u}) \triangleq \max_{|z| \leq z_{sat}, |u| \leq \bar{u}} |f_2(z, u)|\]
extists and is finite. It is clear that \(C(\bar{u})\) is nondecreasing in \(\bar{u}\).

Since \(f_1(x)\) is strictly decreasing, one can find \(x_L \leq x_U\), such that \(f_1(x_L) = C(\bar{u})\) and \(f_1(x_U) = -C(\bar{u})\). Therefore,
\[\begin{array}{ll}
f_1(x) + f_2(z, u) \geq 0, & \forall z \in [-z_{sat}, z_{sat}], \forall u \in [-\bar{u}, \bar{u}], \text{ if } x \leq x_L \\
f_1(x) + f_2(z, u) \leq 0, & \forall z \in [-z_{sat}, z_{sat}], \forall u \in [-\bar{u}, \bar{u}], \text{ if } x \geq x_U,
\end{array}\]
which, by the piecewise monotonicity of \(\Gamma\), implies
\[\begin{array}{ll}
\dot{x}(t) \geq 0, & \text{if } x(t) \leq x_L \\
\dot{x}(t) \leq 0, & \text{if } x(t) \geq x_U
\end{array}\]
Hence \(|x(t)| \leq \max\{c_0, |x_L|, |x_U|\} \leq c_0 + \max\{|x_L|, |x_U|\}\). Note that both \(x_L\) and \(x_U\) are nondecreasing functions of \(\bar{u}\), and that one can find \(c_1 > 0\) and a class \(K\) function \(\alpha(\cdot)\) satisfying
\[\max\{|x_L|, |x_U|\} \leq \alpha(\bar{u}) + d_0.\]
Hence \(||x(\cdot)||_{L_\infty} \leq \alpha(||u(\cdot)||_{L_\infty}) + c_0 + d_0, \forall u(\cdot) \in PC([0, \infty])\). This together with \(||z(\cdot)||_{L_\infty} \leq z_{sat}\) establishes the \(L_\infty\) stability of (4.1).
If (5.15) and (5.16) are satisfied, for both \( x = x_L \) and \( x = x_U \),
\[
c_1|x| + c_2 \leq |f(x)| = C(\bar{u}) \leq c_3 \bar{u} + c_4,
\]
implying
\[
\max\{|x_L|, |x_U|\} \leq \frac{c_1}{c_1} \bar{u} + c_4 - c_2.
\]
This leads to the finite-gain \( L_\infty \) stability of (4.1) with \( \gamma = \frac{c_2}{c_1} \) and \( b_0 = c_0 + |c_2 - c_4| + z_{sat} \), where \( \gamma \) and \( b_0 \) are defined as in (5.13). \( \square \)

Note that the conditions (5.15) and (5.16) are both satisfied in the three examples in Section 2.

### 5.3.2. \( L_2 \) stability

The system (4.1) is not \( L_2 \) stable in general. Consider, for instance, \( \psi^* \in \Psi \) with corresponding output \( \{x^*, z^*\} \). If \( f(x^*, z^*, 0) = 0 \) (i.e., \( \psi^* \) is an equilibrium of (4.1)) and \( z^* \neq 0 \), then for \( \psi_0 = \psi^* \) and \( u(\cdot) \equiv 0 \), \( z(\cdot) = z^* \neq 0 \). This implies \( \|z(\cdot)\|_{L_2} = \infty \) (and thus \( \|\{x(\cdot), z(\cdot)\}\|_{L_2} = \infty \) although \( \|u(\cdot)\|_{L_2} = 0 \).

For a special subclass of systems (4.1), one can establish the finite-gain \( L_2 \) stability when taking \( x(\cdot) \) alone instead of \( \{x(\cdot), z(\cdot)\} \) as the output. The first two examples in Section 2 fall into this subclass. The claim is made precise in the following proposition.

**Proposition 5.6.** Let the Preisach measure be nonnegative and nonsingular with piecewise continuous density \( \mu \). Let \( f(\cdot, \cdot, \cdot) \) in (4.1) have the form:
\[
f(x, z, u) = -c_1 x + c_2 u,
\]
where \( c_1 > 0 \) and \( c_2 \) are constants. Let \( |x(0)| \leq c_0 \), for \( c_0 > 0 \). Then \( \forall \psi_0 \in \Psi \),
\[
\|x(\cdot)\|_{L_2} \leq \tilde{\gamma}\|u(\cdot)\|_{L_2} + \tilde{b}_0,
\]
where
\[
\tilde{\gamma} = \sup_{\omega} \frac{c_2}{j\omega + \frac{\bar{D}}{1 + cD}}, \quad \tilde{b}_0 = \frac{c_0}{2c_1} \sqrt{(1 + cD)}
\]
and \( \tilde{D} \) is the constant in (5.6).

**Proof.** From (5.17), one can rewrite the first equation in (4.1) as
\[
\dot{x}(t) = -\frac{c_1}{1 + cg(t)} x(t) + \frac{c_2}{1 + cg(t)} u(t),
\]
where \( g(t) \triangleq \frac{dz}{dx}(\psi[t], sgn(\dot{x}(t))) \) as in (5.10).

Eq. (5.19) is a linear time-varying ODE and its solution is given by: \( x(t) = x_0(t) + x_1(t) \), where
\[
x_0(t) = e^{-\int_0^t \frac{c_1}{1 + cg(\tau)} d\tau} x(0),
\]
\[
x_1(t) = \int_0^t e^{-\int_\tau^t \frac{c_1}{1 + cg(\tau')} d\tau'} \frac{c_2}{1 + cg(\tau')} u(\tau) d\tau.
\]
From (5.6), one obtains
\[
\|x_0(\cdot)\|_{L_2} = \sqrt{\int_0^\infty e^{-2\int_0^\tau \frac{c_1}{1 + cg(\tau')} d\tau'} dt |x(0)|}
\]
\[
\leq c_0 \sqrt{\int_0^\infty e^{-2\int_0^\tau \frac{c_1}{1 + cg(\tau')} d\tau'} dt} = \tilde{b}_0,
\]
(5.22)
Again from (5.6),

\[ |x_1(t)| \leq x_2(t) \triangleq e^{-\frac{c_1}{1+c_D} t} \otimes c_2|u(t)|, \tag{5.23} \]

where "\( \otimes \)" denotes the convolution. Note that \( x_2 \) is just the output of a linear time invariant system

\[ G_0(s) = \frac{c_2}{s + \frac{c_1}{1+c_D}} \]

with the input \( |u(\cdot)| \). Denote Fourier transforms of \( x_2(\cdot), u(\cdot) \) and \( |u(\cdot)| \) by \( \hat{x}_2(j\omega), \hat{u}(j\omega) \) and \( \hat{|u}(j\omega)| \), respectively. Then

\[ \|x_1(\cdot)\|_{L_2}^2 \leq \|x_2(\cdot)\|_{L_2}^2 \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G_0(j\omega)\hat{u}'(j\omega)|^2 d\omega \]

\[ \leq \gamma^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}'(j\omega)|^2 d\omega \]

\[ = \gamma^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega \]

\[ = \gamma^2 \|u(\cdot)\|_{L_2}^2, \tag{5.28} \]

where the Parseval’s identity is used in (5.25), (5.27) and (5.28). Eq. (5.18) now follows from (5.22) and (5.28).

Note that the bound \( \gamma \) is just the \( H_{\infty} \) norm [17] of \( G_0(s) \).

5.4. Stability of equilibria. Set the input \( u(\cdot) \equiv 0 \). A memory curve \( \psi^* \in \Psi \) is an equilibrium of (4.1) if \( f(x^*, z^*, 0) = 0 \), where \( \{x^*, z^*\} \) corresponds to \( \psi^* \). This is true because \( \dot{x}(t) \cdot \dot{z}(t) \geq 0, \forall t, \) and \( \dot{x}(t) = \dot{z}(t) = 0 \) if \( \dot{x}(t) + c\dot{z}(t) = 0 \). This observation indicates that stability properties of equilibria are linked closely to \( f(\cdot, \cdot, \cdot) \). To check the stability and asymptotic stability (in the sense of Lyapunov [24]) of an equilibrium, one typically perturbs the state from the equilibrium and analyzes the system’s capability of returning to the neighborhood or the equilibrium itself. For the case of (4.1), an equilibrium \( \psi^* \) can be perturbed in different ways (recall the metric (5.1)) with a) increased \( x \) and \( z \), b) decreased \( x \) and \( z \), c) increased \( x \) but decreased \( z \), d) decreased \( x \) but increased \( z \), or e) either \( x \) or \( z \) or both kept unchanged. The diverse scenarios make it difficult to discuss the stability for a general \( f(\cdot, \cdot, \cdot) \). Instead, the investigation will be focused on the case where \( f(\cdot, \cdot, \cdot) \) is independent of \( z \):

\[ f(x, z, 0) = f_0(x). \tag{5.29} \]

Note that (5.29) covers the first two examples in Section 2.

Let \( f_0(0) = 0 \). For the system

\[ \begin{cases} \dot{x}(t) + c\dot{z}(t) = f_0(x(t)) \\ z(t) = \Gamma[x(\cdot), \psi_0](t) \end{cases}, \tag{5.30} \]

the set of equilibria \( \Psi_0 \) is characterized by

\[ \Psi_0 = \{ \psi \in \Psi : \psi(0) = 0 \}, \]
as illustrated in Fig. 5.4(a). Clearly, $\Psi_0$ is an invariant set of (5.30). It turns out the notion of asymptotic stability of an invariant set [24] (page 331) is relevant here.

**Proposition 5.7.** Assume that the Preisach measure is nonnegative, and nonsingular with a piecewise continuous density $\mu$. Let $x = 0$ be an asymptotically stable equilibrium for the system

$$\dot{x}(t) = f_0(x(t)).$$

Then $\Psi_0$ is an asymptotically stable invariant set of (5.30).

**Proof.** Rewrite the first equation in (5.30) as

$$\dot{x}(t) = \frac{f_0(x(t))}{1 + c \frac{dz}{dx}(\psi[t], \text{sgn}(\dot{x}(t)))}.$$

(5.32)

Since $x = 0$ is an asymptotically stable equilibrium of (5.31), one has $f(x) > 0$ if $x < 0$ and $f(x) < 0$ if $x > 0$. This together with (5.6) implies that, for (5.32),

$$\dot{x}(t) \begin{cases} > 0 & \text{if } x(t) < 0 \\ < 0 & \text{if } x(t) > 0 \end{cases}.$$

Therefore, starting from any $\psi_0 \in \Psi$, $\psi[t]$ approaches $\Psi_0$, as illustrated in Fig. 5.4(b) and (c). □

Note that each $\psi^* \in \Psi_0$ is stable and not asymptotically stable.

**6. Existence of periodic solutions.** The existence of periodic solutions to (4.1) under periodic input $u(\cdot)$ can be established if the system (4.1) is $L_\infty$ stable.

**Theorem 6.1.** Assume that $f(\cdot, \cdot, \cdot)$ and the Preisach measure $\nu$ satisfy the conditions of Theorem 4.1. Let (4.1) be $L_\infty$ stable. Let the input $u(\cdot) \in \text{PC}([0, \infty))$ be periodic with period $T$, i.e., $u(t + T) = u(t), \forall t \geq 0$. Pick $r_0$ sufficiently large such that $\psi[t]$ never leaves $\Psi$ for any $\psi_0 \in \Psi$. Let $\Xi : \text{PC}([0, \infty)) \times \Psi \rightarrow C([0, \infty), \Psi)$ be the state evolution map (recall (5.2)) for (4.1). Then there exists $\psi_0 \in \Psi$, such that $\Xi[u(\cdot), \psi_0](t + T) = \Xi[u(\cdot), \psi_0](t), \forall t \geq 0$.

**Proof.** Denote $L_1([0, r_0])$ the Banach space of integrable functions on $[0, r_0]$. One can show that $\Psi$ is a closed subset of $L_1([0, r_0])$. First any $\psi \in \Psi$ is a continuous function of $r$ on $[0, r_0]$, and thus $\psi \in L_1([0, r_0])$. Now let a sequence $\{\psi_n \in \Psi\}$ converge
to \( \tilde{\psi} \in L_1([0, r_0]) \) in the \( L_1 \) norm. By definition of \( \Psi \), \( \{ \psi_n \} \) is equicontinuous and equibounded. Therefore, by the Ascoli-Arzela Theorem, a subsequence \( \psi_{n_k} \rightarrow \tilde{\psi} \in \Psi \) uniformly on \([0, r_0]\), which implies \( \{ \psi_{n_k} \} \) converges to \( \tilde{\psi} \) in \( L_1 \). Hence \( \psi = \tilde{\psi} \) and \( \Psi \) is closed.

Given \( \psi_0 \in \Psi \) and a \( T \)-periodic \( u(\cdot) \), one has \( \Xi[u(\cdot), \psi_0](t) \in \Psi, \forall t \geq 0 \), by hypothesis. Define the map \( \Xi_T : \Psi \rightarrow \Psi \) by

\[
\Xi_T(\psi_0) \triangleq \Xi[u(\cdot), \psi_0](T), \quad \forall \psi_0 \in \Psi. \tag{6.1}
\]

It’s easy to verify that \( \Xi_T \) is continuous. Also \( \Xi_T \) is a compact mapping since \( \Psi \) itself is compact. Finally \( \Psi \) is a convex set. Therefore, \( \Xi_T \) has a fixed point by the Schauder fixed point theorem, which completes the proof. \( \Box \)

Theorem 6.1 implies that the corresponding solution \( \{ x(\cdot), z(\cdot) \} \) is also periodic.

7. Numerical integration. Numerical integration of (4.1) is of importance in verification of models against experimental data and in simulation of closed-loop hysteretic systems. It involves implicit equations due to the nature of the coupling in (4.1) and recursive methods are typically needed in solving these equations. Two schemes, forward Euler and backward Euler, are presented next. The accuracy of these methods is analyzed. A numerical example is given to provide some insight into the performance of the two schemes.

7.1. Forward Euler scheme. Given the memory curve \( \psi[t_0] \) at time \( t_0 \) and the input \( u(\cdot) \), approximate values of \( x \) and \( z \) at \( t_0 + h \) can be computed by the following scheme:

\[
\begin{cases}
\frac{x^* - x(t_0)}{h} + c \frac{z^* - z(t_0)}{h} = f(x(t_0), z(t_0), u(t_0)) \\
z^* = \Gamma[x^*, \psi[t_0]]
\end{cases}, \tag{7.1}
\]

where \( h \) is the time step. For proper interpretation, monotonic change from \( x(t_0) \) to \( x^* \) is assumed. Eq. (7.1) is called the forward Euler scheme since the unknowns, \( x^* \) and \( z^* \), do not appear in \( f \). However, it is an implicit equation of \( x^* \) as \( x^* \) and \( z^* \) are coupled through \( \Gamma \).

The existence and uniqueness of solution to (7.1) follow from the piecewise monotonicity and continuity of \( \Gamma \), as mentioned in the proof of Theorem 4.1. The equation can be solved by adapting methods for computing the inverse of the Preisach operator [22]. In particular, a recursive scheme can be used based on the piecewise monotonicity and the Lipschitz continuity of \( \Gamma \):

\[
\begin{cases}
\frac{x^{[n+1]} - x^{[n]}}{h} = \frac{x^{[n]} - x(t_0)}{h} + c \frac{z^{[n]} - z(t_0)}{h} - f(x(t_0), z(t_0), u(t_0)) \\
z^{[n+1]} = \Gamma[x^{[n+1]}, \psi[t_0]] \\
x^{[0]} = x(t_0), z^{[0]} = z(t_0)
\end{cases}, \tag{7.2}
\]

where \( C_L \) is the Lipschitz constant of \( \Gamma \). It can be verified that \( x^{[n]} \rightarrow x^* \) and \( z^{[n]} \rightarrow z^* \) (both monotonically) as \( n \rightarrow \infty \).

7.2. Backward Euler scheme. A backward Euler scheme is obtained by replacing \( x(t_0) \) and \( z(t_0) \) with their counterparts at \( t_0 + h \), \( x^* \) and \( z^* \), and replacing \( u(t_0) \) with \( u(t_0 + h) \):

\[
\begin{cases}
\frac{x^* - x(t_0)}{h} + c \frac{z^* - z(t_0)}{h} = f(x^*, z^*, u(t_0 + h)) \\
z^* = \Gamma[x^*, \psi[t_0]]
\end{cases}. \tag{7.3}
\]
Again, the equation is implicit in $x^*$. The existence and uniqueness of solution to (7.3) is no longer straightforward due to the term $f(x^*, z^*, u(t_0 + h))$. However, if one assumes that, for any fixed $u$, $f(x^*, \Gamma[x^*, \psi(t_0)], u)$ is a nonincreasing function of $x^*$ (which is the case for all three motivating examples in Section 2), then the following function

$$Q(x^*) \triangleq \frac{x^* - x(t_0)}{h} + c \frac{z^* - z(t_0)}{h} - f(x^*, z^*, u(t_0 + h))$$

is strictly increasing in $x^*$ (taking $z^*$ as a function of $x^*$), and it admits a unique solution for $Q(x^*) = 0$. Before presenting a scheme to compute $x^*$, we analyze the property of $Q(\cdot)$. For an input value $x_1$, let $z_1 = \Gamma[x_1, \psi(t_0)]$. It then follows that

$$|Q(x_1) - Q(x(t_0))|$$

$$= \left| \frac{x_1 - x(t_0)}{h} + c \frac{z_1 - z(t_0)}{h} - f(x_1, z_1, u(t_0 + h)) + f(x(t_0), z(t_0), u(t_0 + h)) \right|$$

$$\leq \frac{1}{h} |x_1 - x(t_0)| + c \frac{C_L}{h} |x_1 - x(t_0)| + |f(x_1, z_1, u(t_0 + h)) - f(x_1, z(t_0), u(t_0 + h))|$$

$$+ |f(x_1, z(t_0), u(t_0 + h)) - f(x(t_0), z(t_0), u(t_0 + h))|$$

$$\leq \frac{1 + cC_L}{h} |x_1 - x(t_0)| + 1 + (L_x + C_L L_z)|x_1 - x(t_0)|$$

$$= C_0|x_1 - x(t_0)|, \tag{7.4}$$

where (7.4) is from the Lipschitz continuity of $f$ and that of $\Gamma$, and $C_0 \triangleq \frac{1 + cC_L}{h} + L_x + C_L L_z$.

Based on (7.5) and the strict (increasing) monotonicity of $Q(\cdot)$, a recursive scheme is proposed to solve (7.3):

$$\begin{aligned}
&\left\{ 
\begin{array}{l}
x^{[n+1]} = x^{[n]} + \frac{1}{C_0} \left( \frac{x^{[n]} - x(t_0)}{h} + c \frac{z^{[n]} - z(t_0)}{h} - f(x^{[n]}, z^{[n]}, u(t_0 + h)) \right) \\
z^{[n+1]} = \Gamma[x^{[n+1]}, \psi(t_0)] \\
x^{[0]} = x(t_0), z^{[0]} = z(t_0)
\end{array}
\right. \tag{7.6}
\end{aligned}$$

As $n \to \infty$, $x^{[n]} \to x^*$ and $z^{[n]} \to z^*$ monotonically.

The following proposition establishes the order of accuracy for the forward Euler scheme (7.1). A similar result can be obtained for the backward Euler scheme, but it is omitted for brevity.

**Proposition 7.1.** Assume that $f(\cdot, \cdot, \cdot)$ and the Preisach measure $\nu$ satisfy the conditions of Theorem 4.1. Let the true solution to (4.1) be $\{x(\cdot), z(\cdot)\}$. Assume that $\frac{d}{d\tau}(\psi[t_0], \pm) \Delta \hat{x}(t_0)$, and $\hat{z}(t_0)$ exist. Let $\{x^*, z^*\}$ be the solution to (7.1). Then

$$|x^* - x(t_0 + h)| = O(h^2), \tag{7.7}$$

$$|z^* - z(t_0 + h)| = O(h^2). \tag{7.8}$$

**Proof.** Denote $g_0 = f(x(t_0), z(t_0), u(t_0))$. Taylor series expansion of $x(\cdot)$ and $z(\cdot)$ at $t_0$ yields:

$$\begin{aligned}
x(t_0 + h) = x(t_0) + \hat{x}(t_0)h + O(h^2), \tag{7.9} \\
z(t_0 + h) = z(t_0) + \hat{z}(t_0)h + O(h^2). \tag{7.10}
\end{aligned}$$
where
\[
\dot{x}(t_0) = \frac{g_0}{1 + \frac{dz}{dx}(\psi(t_0), sgn(g_0))},
\]
\[
\dot{z}(t_0) = \frac{dz}{dx}(\psi(t_0), sgn(g_0)) \dot{x}(t_0).
\]

From (7.1) and the piecewise monotonicity property of \(\Gamma\),
\[
|x^* - x(t_0)| \leq h|g_0|.
\]
This implies
\[
z^* - z(t_0) = \frac{dz}{dx}(\psi(t_0), sgn(g_0)) \cdot (x^* - x(t_0)) + O(|x^* - x(t_0)|^2)
\]
\[
= \frac{dz}{dx}(\psi(t_0), sgn(g_0)) \cdot (x^* - x(t_0)) + O(h^2).
\]
Combining (7.1) and (7.12) leads to
\[
x^* = x(t_0) + \frac{hg_0}{1 + \frac{dz}{dx}(\psi(t_0), sgn(g_0))} + O(h^2).
\]
Comparing (7.9) and (7.13), one gets the estimate (7.7). Eq. (7.8) is a consequence of (7.10), (7.12) and (7.13).

From Proposition 7.1, the forward Euler scheme is accurate to the first order. This is consistent with the accuracy order of the Euler method in numerical integration of standard ODEs.

Backward Euler methods perform better than forward ones for many problems, especially for stiff problems [19]. Fig. 7.1 compares the performance of the two schemes in integrating the dynamic hysteresis model (2.3). For step size \(h = 8 \times 10^{-5}\) second, the forward scheme yields spurious, high-frequency oscillations (which goes away for much smaller \(h\)) while the backward scheme yields clean numerical solutions.

8. Conclusions. A novel class of hysteretic systems was investigated from a system-theoretic viewpoint. Such systems were motivated by a number of physical problems, in particular, modeling and control problems related to smart material actuators. These systems extended conventional ODE models by incorporating a nonstandard hysteretic coupling. The Preisach operator was used as the hysteresis model throughout the paper.

The well-posedness of the dynamical, hysteretic system was proved with mild assumptions on the Preisach measure and the function \(f(\cdot, \cdot, \cdot)\). Note that the Lipschitz continuity of \(f\) can be relaxed to local Lipschitz continuity without affecting the well-posedness proof. In addition, although the Preisach operator was considered, the proof extends directly if other hysteresis operators (e.g., the Prandtl-Ishlinskii model [40]) are used provided they are continuous and piecewise monotonic. Furthermore, the results on \(L_\infty\) stability, existence of periodic solutions, and numerical schemes do not depend on the specific hysteresis operator.

An important contribution of the paper was the analysis of various system-theoretic properties for the infinite-dimensional, hysteretic system. The results were derived based on more specific assumptions of the function \(f(\cdot, \cdot, \cdot)\), yet most of these
assumptions had ground in physical problems. It was shown that the concepts of reachability, observability, input-output stability, and stability of equilibria can be investigated in a rigorous manner for the hysteretic system. The piecewise monotonicity of the Preisach operator turned out to play a key role, since often times one could leave out (or absorb) the term $\dot{z}(t)$ for qualitative analysis.

The existence of periodic solutions under periodic forcing was shown, which has practical implications. It could be an interesting problem to investigate under what conditions the periodic solution will be unique, and furthermore, be asymptotically stable.

Euler schemes were presented for numerical integration of the dynamical system with hysteresis. These methods are first-order accurate. It would be of interest to get higher-order schemes for such systems by adapting other ODE integration methods, e.g., the mid-point rule and other Runge-Kutta schemes. This, however, appears to be difficult because of the complicated implicit equations and the ambiguity of $\frac{dz}{dx}$ evaluated at intermediate values.

REFERENCES


