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Finite-Dimensional Methods for Computing the Information State in Nonlinear Robust Control[1]

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Abstract

In nonlinear output robust control and in nonlinear risk-sensitive partially observed stochastic control, the optimal control is a memoryless function of the information state. The information state dynamics are directly influenced by the control performance metric, thus displaying a direct linkage between control objectives and sufficient statistics for control. It has been observed in several examples that by modifying the control performance metric one can render the dynamics of the information state finite dimensional. In linear robust control, it is well known that the information state can be computed by a simple finite-dimensional formula. Using the Lie theory for transformations of this basic solution, we show how the information state for certain nonlinear control problems can also be obtained using finite-dimensional calculations, e.g. via the solution of systems of ODEs. This is explained using fundamental results on the invariance groups of the equations involved. An intuitive interpretation of the significance of the result is also provided.

1 Introduction

Recent developments in nonlinear output feedback robust control have demonstrated the equivalence of three problems under very general nonlinear models [4-10]: (a) The nonlinear output feedback robust control problem; (b) A partially observed nonlinear dynamical game; (c) A partially observed nonlinear stochastic control problem.

Furthermore, in these works the general structure of the solution was determined. It consists of a, forward in-time, nonlinear partial differential equation which describes the dynamics of the information state, and of a backward infinite dimensional dynamic programming equation. The dynamic programming equation can be solved off-line in principle. However, the equation for the information state has to be solved in real-time and this has been a major obstacle in the application of the results; one has to resort to various approximations.

In several recent papers [11-12] it has been observed that the two equations described above are coupled, in the sense that the control performance metric enters explicitly in the equation describing the information state. This is to be contrasted with the framework of ordinary stochastic control problems where the information state is the conditional probability measure of the state given the past of the observations. This later quantity satisfies the linear Zakai equation [13] or the nonlinear Kushner equation [13]. Neither depends explicitly on the control performance metric in the standard set-up. In the recent works cited [11-12] this coupling has been used to construct examples where for appropriate choice of the integrand in the performance metric, the infinite dimensional information state dynamics collapse to finite-dimensions [11-12].

It is the primary objective of this paper to explain why this is possible in a general setting. This is accomplished using the theory of Lie transformations on the information state dynamics.

The focus of the method is similar to earlier work by one of the authors on group invariance methods in nonlinear filtering [13]. The latter work was inspired by the so-called "similarity methods" for ordinary and partial differential equations [2,14-16]. In [13] we introduced the notion of equivalence between two nonlinear filtering problems if the solution of the one Zakai equation can be obtained from the solution of the other (i.e. computed) via the following three types of operations: (a) diffeomorphisms in time-space (i.e. change of coordinates and time rescaling); (b) scaling of the probability density by a variable factor; (c) solving a set of ordinary differential equations. We linked in [13] this equivalence to finitely computable nonlinear filtering problems and to the Lie-Bäcklund transformations [16]. These methods allowed us to prove then the converse to the well-known results of V. Beneš.

Here we use similar techniques to explain how the nonlinear p.d.e. associated with the information state can be solved by solving ODEs. Our results include earlier obtained results along this line of inquiry [11-12].

Finally we will provide an intuitive interpretation of our results, inspired by the interpretation of those transformations in the nonlinear filtering problem [13].

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2 Invariance properties of the Information State PDE

2.1 The Linear Control Problem

We shall omit the control parameter u from the notation, since it plays no part in the following calculations; in other words, instead of writing $A(u(t))$, we abbreviate to $A(t)$. The information state $p(t,x) \equiv p(t,x_1,\cdots,x_n)$ for the linear control problem satisfies the scalar PDE

$$\begin{aligned} &F(t,x,p_t,\nabla p) \equiv p_t + \nabla p\cdot(Ax+b) \\ &-|\nabla p|^2/2 + x^TGx/2 + h\cdot x + l = 0\,, \end{aligned} \tag{1}$$

where $G \equiv G(t)$ is a symmetric matrix, $A \equiv A(t)$ is a square matrix, $b \equiv b(t)$ and $h \equiv h(t)$ are n-vectors, and $l \equiv l(t)$ is a scalar function. James and Yuliar (1995) point out that there is a solution of the form

$$p(t,x) = -(x-r(t))^TW(t)(x-r(t))/2 + \phi(t), \tag{2}$$

with W symmetric. We call this the **solution of the linear control problem.** Taking the gradient of (2), substituting in (1) and equating coefficients of terms quadratic, linear, and constant in x we obtain the ODEs

$$\begin{aligned} \dot W &= -WA - A^TW - W^2 + G\,, \\ \dot r &= W^{-1}(-\dot W r + Wb - A^TWr - W^2r - h)\,, \\ \dot\phi &= r^T(W\dot r + \dot W r/2 - Wb + W^2r/2) - l\,. \end{aligned} \tag{3}$$

The last two equations can be rewritten as

$$\dot r = Ar + b - W^{-1}(Gr+h)\,, \tag{4}$$

$$\dot\phi = r^T(WA - A^TW - G)r/2 - r^Th - l\,. \tag{5}$$

Now (5), (6), and (7) can be solved in sequence for given initial values, and (2) follows.

2.2 Lie Transformation Theory for the Information State PDE

We shall now consider the invariance of (1) under an infinitesimal transformation given by a vector field of the form (note that we are not including a $\partial/\partial t$ term)

$$X \equiv \sum_{i=l}^{n} \xi^i(t,x)\frac{\partial}{\partial x_i} + \eta(t,x,p)\frac{\partial}{\partial p}\,. \tag{6}$$

According to Bluman and Kumei (1989), Theorem 4.1.1-1, the criterion for invariance is that

$$\begin{aligned} &X^{(1)}F(t,x,p_t,\nabla p) = 0 \text{ whenever} \\ &F(t,x,p_t,\nabla p) = 0\,, \end{aligned} \tag{7}$$

where $X^{(1)}$ is the first extended infinitesimal generator, namely

$$X^{(1)} \equiv \sum_{i=l}^{n} \xi^i \frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial p} + \eta_0^{(1)}\frac{\partial}{\partial p_t} + \sum_{i=1}^{n}\eta_i^{(1)}\frac{\partial}{\partial p_i}$$

where

$$p_i \equiv \frac{\partial}{\partial x_i}\,, \tag{8}$$

$$\eta_0^{(1)}(t,x,p,p_t,\nabla p) \equiv D_t\eta - \sum_{j=1}^{n}(D_t\xi^j)p_j\,, \tag{9}$$

$$\eta_i^{(1)}(t,x,p,p_t,\nabla p) \equiv D_i\eta - \sum_{j=1}^{n}(D_i\xi^j)p_j, \quad i=1,\ldots,n\,, \tag{10}$$

$$D_t \equiv \frac{\partial}{\partial t} + p_t\frac{\partial}{\partial p},\ D_i \equiv \frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial p}, \quad i=1,\ldots n\,. \tag{11}$$

Evaluating the entries of (6) term by term,

$$\sum_{i=1}^{n}\xi^i\frac{\partial F}{\partial x_i} = (A^T(\nabla p) + Gx + h)\cdot\xi\,,$$

$$\eta\frac{\partial F}{\partial p} = 0,$$

$$\eta_0^{(1)}\frac{\partial F}{\partial p_t} = \eta_t + p_t\eta_p - (\nabla p\cdot\xi_t)\,,$$

$$\begin{aligned} \sum_{i=1}^{n}\eta_i^{(1)}\frac{\partial F}{\partial p_i} &= \sum_{i=1}^{n}\left(D_i\eta - \sum_{j=1}^{n}(D_i\xi^j)p_j\right)(Ax+b-\nabla p)^i \\ &= (Ax+b-\nabla p)\cdot(\nabla\eta + \eta_p\nabla p - (\nabla p\cdot\nabla)\xi)\,. \end{aligned}$$

Adding up all these terms shows that (6) gives

$$\begin{aligned} &(A^T(\nabla p) + Gx + h)\cdot\xi + \eta_t + p_t\eta_p - (\nabla p\cdot\xi_t) \\ &= -(Ax+b-\nabla p)\cdot(\nabla\eta + \eta_p\nabla p - (\nabla p\cdot\nabla)\xi)\,. \end{aligned}$$

Grouping terms, we obtain:

2.2.1 Fundamental Transformation Relation:

$$\begin{aligned} &\eta_t + (Ax+b-\nabla p)\cdot\nabla\eta + (p_t + (Ax+b-\nabla p)\cdot\nabla p)\eta_p \\ = \ &\nabla p\cdot\xi_t - (A^T\nabla p + Gx + h)\cdot\xi + \\ &(Ax+b-\nabla p)\cdot(\nabla p\cdot\nabla)\xi\,, \end{aligned} \tag{12}$$

where p is given by (2). Note that only **linear** differential operators acting on ξ and η are involved, and ∇p and p_t are quadratic in x. Hence for any choice of ξ we may solve for η by the method of characteristics. That is given ξ, solving for η involves only the solution of an ODE.

2.3 How to Use the Fundamental Transformation Relation

Let $\varphi(\varepsilon;t,x,p)) \equiv (t,\bar x(\varepsilon,t,x),\bar p(\varepsilon,t,x,p))$ denote the flow of the vector field

$$X \equiv \sum_{i=1}^{n}\xi^i(t,x)\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial p} \tag{13}$$

where $\bar{x}(\varepsilon, t, x)$ are the transformed state space coordinates and $\bar{p}(\varepsilon, t, x, p) \equiv \bar{p}(t, \bar{x})$ is the information state for the transformed problem. By definition of φ,

$$\frac{d\varphi}{d\varepsilon}(\varepsilon) = X\varphi(t, \bar{x}, \bar{p}),\ \varphi(0; (t, x, p)) = (t, x, p)\cdot \quad (14)$$

This breaks down into the system of ODEs

$$\frac{\partial \bar{x}}{\partial \varepsilon} = \xi(t, \bar{x}),\ \bar{x}(0, t, x) = x\,; \quad (15)$$

together with the scalar ODE

$$\frac{\partial \bar{p}}{\partial \varepsilon} = \eta(t, \bar{x}, \bar{p}),\ \bar{p}(0, t, x, p) = p\,. \quad (16)$$

What we have proved so far is:

2.3.1 Theorem *Assume p satisfies (1). Suppose $\varepsilon \to \bar{x}(\varepsilon, t, x)$ is a one-parameter family of transformations of the space variable x satisfying the system of ODE (15), for some choice of $\xi \equiv \xi(t, x)$, and that $\eta \equiv \eta(t, x, p)$ is chosen to satisfy the Fundamental Transformation Relation (12) in terms of ξ. Then the solution $\varepsilon \to \bar{p}(\varepsilon, t, x, p)$ to the ODE (16), if unique, is a one-parameter family of transformations of the information state variable p, so that (1) holds with (x, p) replaced by $(\bar{x}, \bar{p})$.*

3 A Case Permitting Explicit Computations

3.1 A Special Class of Infinitesimal Generators

The drawback of Theorem 2.3.1 is that it is too abstract to be of immediate practical use. Therefore we consider a more specialized situation admitting explicit computations. We shall constrain the choice of η so as to satisfy

$$\eta = -x^T W \xi\,. \quad (17)$$

This implies

$$\begin{aligned} \eta_t &= -x^T \dot{W} \xi - x^T W \xi_t,\ \eta_p = 0, \\ \nabla \eta &= -(Wx \cdot \nabla)\xi - W\xi\,. \end{aligned} \quad (18)$$

Now (14) says

$$\begin{aligned} &-x^T \dot{W}\xi - x^T W \xi_t - (Ax + b - \nabla p)\cdot((Wx \cdot \nabla)\xi + W\xi) \\ &= \nabla p \cdot \xi_t - (A^T \nabla p + Gx + h)\cdot \xi \\ &+ (Ax + b - \nabla p)\cdot(\nabla p \cdot \nabla)\xi\,, \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} &(A^T \nabla p + Gx + h - \dot{W}x - W(Ax + b - \nabla p))\cdot \xi \\ &\qquad - (\nabla p + Wx)\cdot \xi_t \\ &= (Ax + b - \nabla p)\cdot((\nabla p + Wx)\cdot \nabla)\xi\,, \end{aligned}$$

$$\begin{aligned} &(-A^T W(x - r) + Gx + h - \dot{W}x - \\ &W(Ax + b + W(x - r)))\cdot \xi - (Wr)\cdot \xi_t \end{aligned}$$

$$= (Ax + b + W(x - r))\cdot((Wr)\cdot \nabla)\xi\,.$$

The coefficient of x in the first bracket is $-A^T W + G - \dot{W} - W)(A + W) = 0$, by (3). Define the following vector functions in terms of quantities determined above:

$$\beta(t) \equiv Wr, \Gamma(t) \equiv A + W, \gamma(t) \equiv b - \beta\,, \quad (19)$$

$$\alpha(t) \equiv (A^T + W)Wr + h - Wb = -\beta_t\,, \quad (20)$$

where the last identity follows from (3) and (4), since

$$\begin{aligned} W\dot{r} + \dot{W}r &= WAr + Wb - Gr - h + \\ &\quad (-WA - A^T W - W^2 + G)r \end{aligned}$$

$$= -(A^T W)r - W^2 r - h + Wb\,.$$

Now the linear PDE which $\xi(t, x)$ must satisfy is:

$$\beta_t \cdot \xi - \beta \cdot \xi_t - (\Gamma x + \gamma)\cdot(\beta \cdot \nabla)\xi = 0\,. \quad (21)$$

This can be put in an even more concise form:

3.1.1 Constraint on η when we assume $\eta = -x^T W\xi$: $\zeta(t, x) \equiv \beta \cdot \xi = r^T W \xi = \nabla p \cdot \xi - \eta$ must satisfy the linear first order PDE

$$\zeta_t + ((\Gamma x + \gamma)\cdot \nabla)\zeta = 0\,. \quad (22)$$

Suppose ζ is a polynomial of order N in x, i.e.

$$\zeta(t, x) \equiv \sum_{k=0}^{N} \Xi^{(k)}(t)(x^{\otimes k})\,, \quad (23)$$

where $x^{\otimes k} \equiv x \otimes \cdots \otimes x$($k$ factors), and $\Xi^{(k)}(t)$ is a symmetric $(0, k)$-tensor. Then

$$\nabla \zeta(t, x) = \sum_{k=1}^{N} k\Xi^{(k)}(t)(\cdot \otimes x^{\otimes(k-1)})\,.$$

Now (22) becomes

$$\sum_{0}^{N} \Xi_t^{(k)} x^{\otimes k} + \sum_{1}^{N} k\Xi^{(k)}((\Gamma x + \gamma)\otimes x^{\otimes(k-1)}) = 0\,.$$

Equating coefficients for each power of x forces the $\{\Xi^{(k)}(t)\}$ to satisfy the following system of ODEs:

$$\Xi_t^{(N)} + N\Xi^{(N)}(\Gamma(\cdot)\otimes\cdot) = 0\,; \tag{24}$$

$$\Xi_t^{(k)} + k\Xi^{(k)}(\Gamma(\cdot)\otimes\cdot) = -(k+1)\Xi^{(k+1)}(\gamma\otimes\cdot) \quad \text{for } k = 1,2,\dots, N-1\,; \tag{25}$$

$$\Xi_t^{(0)} = \Xi^{(1)}(\gamma)\,. \tag{26}$$

Notice the structure of this system of ODEs. Suppose $\Xi^{(0)}(0),\cdots,\Xi^{(N)}(0)$ have been chosen: first we solve (24) for $\Xi^{(N)}(t)$; insert this solution in the right side of (25), and solve (25) for $\Xi^{(N-1)}(t)$; and so on, down to $\Xi^{(0)}(t)$. Let us summarize our results.

3.1.2 Fundamental Transformation Relation in the Case $\eta = -x^TW\xi = \nabla p\cdot\xi - \zeta$:

Assume that

$$\zeta(t,x) \equiv r^TW\xi \equiv \sum_{k=0}^{N} \Xi^{(k)}(t)(x^{\otimes k})\,. \tag{27}$$

Then $\zeta(t,x)$ is completely determined by the initial conditions $\Xi^{(0)}(0),\cdots,\Xi^{(N)}(0)$ and ODEs (24)-(26). In particular, when $n=1$ and Wr is never zero, $\xi(t,x)$ is uniquely determined by $\xi(0,x)$, assuming $\xi(t,x)$ is a polynomial of arbitrary degree in x with coefficients depending on t.

3.2 Procedure for Computation of the Transformed Information State

The starting-point is the solution p given by (2) to the linear control problem. Pick an initial condition $\Xi^{(0)}(0),\cdots\Xi^{(N)}(0)$, and solve for $\zeta(t,x)$ using (24)-(26) by solving for each of the $\{\Xi^{(k)}(t)\}$. Now pick

$$\xi(t,x) \equiv \sum_{k=1}^{N} \Theta^{(k)}(t)(x^{\otimes x}) \tag{28}$$

so that $r^TW\xi = \zeta$, in other words so that

$$\Theta^{(k)}(t) = W(t)r(t)\cdot\Xi^{(k)}(t)\,. \tag{29}$$

Now we repeat the steps described in Section 2.3 under the assumption $\eta = -x^TW\xi = \nabla p\cdot\xi - \zeta$. As before, we solve the system of ODEs

$$\frac{\partial\bar{x}}{\partial\varepsilon} = \xi(t,\bar{x}), \bar{x}(0,t,x) = x\,; \tag{30}$$

(derived from (15)) to determine $\bar{x}(\varepsilon,t,x,p)$. Thus $\xi(t,\bar{x})$ and $\zeta(t,\bar{x})$ are now explicitly computable. Finally we determine $\bar{p}(\varepsilon,t,x,p)$ by solving the following first order PDE (derived from (15)) by the method of characteristics (see Abraham et al. (1988), p. 287):

$$\frac{\partial\bar{p}}{\partial\varepsilon} = \nabla\bar{p}\cdot\xi(t,\bar{x}) - \zeta(t,\bar{x}), \bar{p}(0,t,x,p) = p\,, \tag{31}$$

which can be written out in full as

$$\frac{\partial\bar{p}}{\partial\varepsilon} = \sum_{k=0}^{N}(\nabla\bar{p}\cdot\Theta^{(k)}(t) - \Xi^{(k)}(t))(\bar{x}^{\otimes k})\,. \tag{32}$$

3.3 Detailed Calculation of the First Order Case

For the sake of illustration, let us restrict to the somewhat trivial case $N=1$, i.e. with coefficients $\Xi^{(0)}(t)\in R$ and $\Xi^{(1)}(t)\in R^n$. Thus (23) becomes

$$\zeta(t,x) \equiv r^TW\xi \equiv \Xi^{(0)} + \Xi^{(1)}x\,, \tag{33}$$

and (24)-(26) simplify to the pair of ODEs:

$$\Xi_t^{(1)} + \Xi^{(1)}(A+W) = 0, \Xi_t^{(0)} = \Xi^{(1)}(b - Wr)\,, \tag{34}$$

which are explicitly solvable, given the initial conditions. Now we can make any choice of

$$\xi(t,x) \equiv \Theta^{(0)} + \Theta^{(1)}x \tag{35}$$

subject to the constraints

$$r^TW\Theta^{(k)} = \Xi^{(k)}, k = 0,1\,. \tag{36}$$

Now (30) becomes the first order ODE

$$\frac{\partial\bar{x}}{\partial\varepsilon} = \Theta^{(0)} + \Theta^{(1)}\bar{x}, \bar{x}(0,t,w) = x\,,$$

with solution

$$\bar{x}(\varepsilon,t,x) = \exp\{\varepsilon\Theta^{(1)}\}(x+\rho) - \rho, \rho \equiv \Theta^{(1)-1}\Theta^{(0)}\,. \tag{37}$$

The PDE (32) becomes

$$\frac{\partial\bar{p}}{\partial\varepsilon} = \nabla\bar{p}\cdot(\Theta^{(0)} + \Theta^{(1)}\bar{x}) - \Xi^{(0)} - \Xi^{(1)}\bar{x}\,.$$

The characteristic vector field on R^{n+2} for this problem is:

$$Z = \frac{\partial}{\partial\varepsilon} + \sum_i(\Theta^{(0)} + \Theta^{(1)}\bar{x})^i\frac{\partial}{\partial\bar{x}_i} + (\Xi^{(0)} + \Xi^{(1)}\bar{x})\frac{\partial}{\partial\bar{p}}\,. \tag{38}$$

The initial condition is the n-dimensional submanifold in R^{n+2} given parametrically by

$$\Psi \equiv \{(\varepsilon,\bar{x},\bar{p}) : \varepsilon = 0, \bar{x} = s, \bar{p} = p(t,s)\}, \quad s \equiv (s_1,\dots,s_n)\,, \tag{39}$$

for $p(t,x)$ as in (2). On this submanifold the following $(n+2) \times (x+2)$ matrix always has rank $n+1$, and therefore the vector field Z is never tangent to Ψ:

$$\begin{bmatrix} 1 & \xi^1 & \dots & \xi^n & \zeta \\ \frac{\partial \varepsilon}{\partial s_1} & \frac{\partial \bar{x}_1}{\partial s_1} & \dots & \frac{\partial \bar{x}_n}{\partial s_1} & \frac{\partial \bar{p}}{\partial s_1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial \varepsilon}{\partial s_n} & \frac{\partial \bar{x}_1}{\partial s_n} & \dots & \frac{\partial \bar{x}_n}{\partial s_n} & \frac{\partial \bar{p}}{\partial s_n} \end{bmatrix} = \begin{bmatrix} 1 & \xi^1 & \dots & \xi^n & \zeta \\ 0 & 1 & \dots & 0 & \bar{p}_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \bar{p}_n \end{bmatrix}$$

The flow $\psi(u;(\varepsilon,x,p))$ of the vector field Z satisfies

$$(\psi^1(u),\dots,\psi^{n+2}(u)) = (\varepsilon+u,\bar{x}_1,\dots,\bar{x}_n,\bar{p})$$

where

$$\frac{d\bar{p}}{du} = \zeta(t,\bar{x}(u,t,x)) = \Xi^{(0)} + \Xi^{(1)}\bar{x}\,,$$

$$\bar{p} = p + \Xi^{(0)}u + \Xi^{(1)} \int_0^u \bar{x}(\nu,t,x)d\nu$$

$$= p + \Xi^{(0)}u + \Xi^{(1)} \int_0^u [\exp\{\nu\Theta^{(1)}\}[(x+\rho)-\rho]d\nu\,,$$

$$= p + [\Xi^{(0)} - \Xi^{(1)}\rho]u + \Xi^{(1)}\,[\exp\{u\Theta^{(1)}\}-1]\Theta^{(1)-1}(x+\rho)\,.$$

Thus the submanifold swept out by Ψ along $\psi(u)$ is given parametrically by

$$\varepsilon(u,s) = u, \bar{x}(u,t,s) = \exp\{u\Theta^{(1)}\}(x+\rho)-\rho$$

$$\bar{p} = p + [\Xi^{(0)} - \Xi^{(1)}\rho]u + \Xi^{(1)}\,[\exp\{u\Theta^{(1)}\} - I]\Theta^{(1)-1}(s+\rho) \quad (40)$$

We eliminate u and s using $u=\varepsilon, s = exp\{-u\Theta^{(1)}\}(\bar{x}+\rho)-\rho$, to conclude:

3.3.1 Explicit Transformation Formula in the Simplest Case: When $N=1, \xi(t,x) \equiv \Theta^{(0)}(t)+\Theta^{(1)}(t)x$ satisfies (34), and $\eta \equiv -x^TW\xi$, then there is a one-parameter family $(t,\bar{x}(\varepsilon,t,x),\bar{p}(\varepsilon,t,x,p))$, indexed by ε, of solutions to the PDE (1), given by:

$$\bar{x}(\varepsilon,t,x) = \exp\{\varepsilon\Theta^{(1)}\}(x+\rho)-\rho\,, \quad (41)$$

$$\bar{p} = p + [\Xi^{(0)} - \Xi^{(1)}\rho]\varepsilon + r^TW\,[\exp\{u\Theta^{(1)}\} - I](x+\rho)\,, \quad (42)$$

where $\rho \equiv \Theta^{(1)-1}\Theta^{(0)}$, and p, W, and r are given by (2), (3), (4), and (5).

4 Infinitesimal Transformation Linear in the Information State

Consider a transformation with infinitesimal generator $X \equiv \xi \cdot \nabla f + \eta\frac{\partial f}{\partial p}$, where

$$\eta(t,x,p) = \theta(t,x)p + \varphi(t,x)\,.$$

Then

$$\eta_t = p\theta_t + \theta p_t + \varphi_t,\ \nabla\eta = p\nabla\theta + \theta\nabla p + \nabla\varphi, \eta_p = \theta\,.$$

The first line of (12) becomes

$$p\theta_t + \theta p_t + \varphi_t + (Ax+b-\nabla p)\cdot (p\nabla\theta + \theta\nabla p + \nabla\varphi) + \theta(p_t + (Ax+b-\nabla p)\cdot\nabla p)$$

$$= 2\theta p_t + p\theta_t + \varphi_t + (Ax+b-\nabla p)\cdot(2\theta\nabla p) \cdot(2\theta\nabla p + p\nabla\theta + \nabla\varphi)\,. \quad (43)$$

Take $z(t,x) \equiv x - r(t)$, and recall that

$$p = -z^TWz/2 + \phi,\ \nabla p = -Wz\,,$$

$$p_t = z^T(W(Ar+b) - (Gr+h)) - z^T\dot{W}z/2 + \dot{\phi}\,.$$

Then (43) becomes

$$2\theta(z^T[W(Ar+b)-(Gr+h)] - z^T\dot{W}z/2+\dot{\phi})$$

$$+(-z^TWz/2+\phi)\theta_t + \varphi_t - 2\theta z^TW(Ar+b+(W+A)z)$$

$$+(Ar+b+(W+A)z)\cdot[(-z^TWz/2+\phi)\nabla\theta + \nabla\varphi]$$

Suppose

$$\varphi(t,x) \equiv \sum_{k=0}^{N} \Phi^{(k)}(t)(z^{\otimes k})\,, \quad (44)$$

$$\theta(t,x) \equiv \sum_{k=0}^{N-2} \Theta^{(k)}(t)(z^{\otimes k})\,, \quad (45)$$

where $z^{\otimes k} \equiv z\otimes\cdots\otimes z(k$ factors), and $\Phi^{(k)}(t)$, etc. are symmetric $(0,k)$-tensors. Now (44) becomes simply a polynomial of order N in z, with coefficients determined by (23) and (45), namely

$$2\sum_{k=0}^{N-2} \Theta^{(k)}(z^{\otimes k})(z^T(W(Ar+b) - (Gr+h)) - z^T\dot{W}z/2 + \dot{\phi})$$

$$+(-z^TWz/2+\phi)\sum_{k=0}^{N-2}\dot{\Theta}^{(k)}(z^{\otimes k})+\sum_{k=0}^{N}\dot{\Phi}^{(k)}(z^{\otimes k})$$

$$+\quad (Ar+b+(W+A)z)\cdot\Bigg(2\sum_{k=0}^{N-2}\Theta^{(k)}(z^{\otimes k})(-Wz)$$
$$+\quad \sum_{k=1}^{n}k\Phi^{(k)}(\cdot\otimes z^{\otimes(k-1)})+(-z^TWz/2+\phi)$$
$$\sum_{k=1}^{N-2}k\Theta^{(k)}(\cdot\otimes z^{\otimes(k-1)})\Bigg)$$

Terms of order N in z:

$$-\Theta^{(N-2)}\otimes\dot{W}-\dot{\Theta}^{(N-2)}\otimes W/2+\dot{\Phi}^{(k)}$$
$$-\Theta^{(N-2)}\otimes(W^2WA)+\ldots$$

According to (12),

$$2\theta p_t+p\theta_t+\varphi_t+(Ax+b-\nabla p)\cdot(2\theta\nabla p+p\nabla\theta+\nabla\varphi)$$

$$=\nabla p\cdot\xi_t-(A^T\nabla p+Gx+h)\cdot\xi$$
$$+(Ax+b-\nabla p)\cdot(\nabla p\cdot\nabla)\xi\,,$$

and we can then take

$$\xi(t,x)\equiv\sum_{k-0}^{N-1}\Xi^{(k)}(t)(z^{\otimes k})$$

and solve for the coefficients.

5 Conclusions

We have shown how Lie transformations applied to the information state for certain nonlinear robust control problems can reduce them solvable by ODEs. The theory and computations suggest that in some cases this is similar to a "gauge" transformation by the cost integrand. The results are more general however. They suggest that the cost integrand "weights" or "focuses" the sufficient statistic needed for control.

References

[1] R. Abraham, J.E. Marsden, and T. Ratiu (1988): *Manifolds, Tensor Analysis, and Applications*, 2nd ed., Springer, New York.

[2] G.W. Bluman and S. Kumei (1989): *Symmetries and Differential Equations*, Springer, New York.

[3] R.W.R. Darling (1994): *Differential Forms and Connections*, Cambridge University Press, New York.

[4] M.R. James, J.S. Baras and R.J. Elliott (1993): "Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems", 32nd IEEE CDC, San Antonio.

[5] M.R. James and S. Yuliar (1995): "A nonlinear partially observed differential game with a finite-dimensional information state", Systems and Control Letters, 1996.

[6] M.R. James, J.S. Baras and R.J. Elliott: "Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems", IEEE Trans. on Automatic Control, pp. 780-792, Vol. 39, No. 4, April 1994.

[7] M.R. James and J.S. Baras: "Robust $H\infty$ output feedback control for nonlinear systems" IEEE Trans. on Automatic Control, pp. 1007-1017, Vol. 40, No. 6, June 1995.

[8] M.R. James and J.S. Baras: "Partially observed differential games, infinite dimensional HJI Equations, and Nonlinear H_∞ Control" SIAM Journal on Control and Optimization, Vol. 34, No. 4, pp. 1342-1364, July 1996.

[9] J.S. Baras and M.R. James: "Robust and risk-sensitive output feedback control for finite state machines and hidden markov models" Journal of Mathematical Systems, Estimation, and Control, Vol. 7, No. 3, pp. 371-374, 1997.

[10] J.S. Baras and N. Patel: "Robust control of set-valued discrete time dynamical systems" IEEE Transactions on Automatic Control, Vol. 43, No. 1, pp. 61-75, January 1998.

[11] J.B. Moore and J.S. Baras: "Finite-dimensional optimal controllers for nonlinear plants", Systems and Control Letters 26 (1995), pp. 223-230.

[12] R. Elliott and A. Bensoussan: "General Finite-Dimensional Risk-Sensitive Problems and Small Noise Limits", IEEE Trans. on Automatic Control, Vol. 41, pp. 210-215, January 1996.

[13] J.S. Baras: "Group invariance methods in nonlinear filtering of diffusion processes", invited paper, Proceedings of the 19th IEEE Conference on Decision and Control, pp. 72-80, Albuquerque, New Mexico, December 1980.

[14] G.W. Bluman and J.C. Cole: *Similarity methods for differential equations*, Springer-Verlag 1974.

[15] S.I. Rosencrans: "Perturbation algebra of an elliptic operator", J. of Math. Anal. and Appl. 56, pp. 317-329, 1976.

[16] R.L. Anderson and N.H. Ibragimov, *Lie-Bäcklund Transformations in Applications*, SIAM Press, 1979.