ON MORPHOLOGICAL OPENINGS AND CLOSINGS OF SIGNALS IN SHAPED NOISE

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ABSTRACT

In recent work, we have shown that morphological openings and closings can be viewed as consistent MAP estimators of morphologically smooth binary image signals immersed in i.i.d. union (clutter) noise, or suffering from i.i.d. random dropouts. We revisit this viewpoint under a different set of assumptions, which allows the explicit incorporation of geometric and morphological constraints into the noise model, i.e., the noise may now exhibit *geometric structure*; surprisingly, it turns out that this affects neither the optimality nor the consistency of these filters.

1. INTRODUCTION

In recent work [1, 2], we have obtained proofs of MAP optimality and strong consistency of openings and closings, viewed as estimators of morphologically smooth binary image signals in i.i.d. noise. These results were made possible by casting the filtering problem within a general framework of Uniformly Bounded Discrete Random Set (or, Discrete Random Set (DRS), for short) theory [3, 4].

A DRS X can be formally defined as a measurable mapping from some probability space to a measurable space $(\Sigma(B), \Sigma(\Sigma(B)))$, where $\Sigma(B)$ is a complete lattice with a finite least upper bound (usually, the power set of some finite $B \subset \mathbf{Z}^2$), and $\Sigma(\Sigma(B))$ is a σ-field over $\Sigma(B)$ (usually, the power set of the power set of B). A DRS X induces an associated probability structure $P_X(\cdot)$ on $\Sigma(\Sigma(B))$. DRS's can be viewed as finite-alphabet random variables, taking values in a finite partially ordered set (poset). Thus, the basic difference with ordinary finite-alphabet random variables is that the DRS alphabet naturally possesses only a partial order relation, instead of a total order relation.

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The following Theorem summarizes some of our earlier results [1, 2]. In what follows $\circ, \bullet$ denote morphological opening, and closing, respectively, whereas $O_W(B), C_W(B)$ denote the set of root signals of opening by W, (i.e., the collection of all images (subsets of B) which are invariant under opening by W, i.e., spanned by unions of translates of W [6]), and the set of root signals of closing by W, respectively.

Theorem 1 *Assume we observe $\mathbf{Y}^{(M)} = [Y_1, \cdots, Y_M]$, where $Y_i = X \cup N_i$, $\{N_i\}_{i=1}^M$ is an i.i.d. sequence of noise DRS's, which is independent of X, and each N_i is i.i.d. of intensity $r \in [0,1)$ (i.e., each point $z \in B$ is included in N_i with probability r, independently of all other points). Let us further assume that X is uniformly distributed over a collection, $\Phi(B) \subseteq \Sigma(B)$, of all subsets K of B which are spanned by unions of translates of a family of structural elements, W_l, $l = 1, \cdots, L$, i.e., those $K \subseteq B$ which can be written as $K = \cup_{l=1}^L K_l$, $K_l \in O_{W_l}(B)$, $l = 1, \cdots, L$. Then $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) = \bigcup_{l=1}^L \left((\cap_{i=1}^M Y_i) \circ W_l \right)$ is the unique MAP estimator of X on the basis of $\mathbf{Y}^{(M)}$, $\forall r \in [0,1)$. In addition, under the foregoing assumptions, and $\forall r \in [0,1)$, $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) \longrightarrow X$, a.s. as $M \to \infty$, i.e., this MAP estimator is strongly consistent.*

The proof critically depends on the following assumptions: B is *finite*; and the noise process is i.i.d., both within a given observation (pixel-wise), and across a sequence of observations (sequence-wide). As a result, the Theorem does not apply when the noise process is "smooth"; e.g., one cannot accommodate a composite noise process resulting by taking the union of translated replicas of some noise "primitives". In effect, the Theorem cannot be applied in the case of "colored" noise. However, as it turns out, the pixel-wise i.i.d. assumption, as well as the sequence-wide assumption of identical distribution can both be removed, as long as the sequence-wide independence assumption is main-

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tained, and a uniformity condition (to be specified) is imposed. This is the subject of this paper. The net result is that we end up with a new set of optimality conditions, which neither implies, nor is implied by the previous set. The most interesting feature of this new set of conditions is that it allows the explicit incorporation of geometric and morphological constraints into the noise model, thus establishing optimality in a more flexible and interesting environment.

2. BACKGROUND

The fundamental elements of Mathematical Morphology have been developed by Matheron [5, 6], Serra [7, 8], and their collaborators. Morphological filtering is one of the most popular and successful branches of this theory[1]. One good reason for the widespread use of morphological filters is their excellent shape-preservation (syntactic) properties. Important characterizations (e.g., root signal structure, relations to other filter classes) are well developed and understood [10, 11, 12, 13]. Another aspect of filter behavior is revealed through statistical analysis. We are mostly interested in optimizing filter behavior with respect to some statistical measure of goodness [1, 2, 3, 4]. Dougherty et al. [14, 15, 16, 17, 18, 19], Schonfeld et al. [20, 21, 22], and Goutsias [23] have worked on several related problems, using different measures of optimality and/or families of filters. We concentrate on MAP optimality and strong consistency.

3. NEW RESULTS

We have the following results. Proofs can be found in [24].

Theorem 2 *(MAP Optimality) Assume we observe* $\mathbf{Y}^{(M)} = [Y_1, \cdots, Y_M]$, *where* $Y_i = X \cup N_i$, $\{N_i\}_{i=1}^{M}$ *is an independent* **but not necessarily identically distributed** *sequence of noise DRS's, which is independent of* X, *and each* N_i *is uniformly distributed over* **some arbitrary collection,** $\Psi_i(B) \subseteq \Sigma(B)$, *of subsets of the observation lattice* B. *Let us further assume that* X *is uniformly distributed over a collection,* $\Phi(B) \subseteq \Sigma(B)$, *of all subsets* K *of* B *which are spanned by unions of translates of a family of structural elements,* W_l, $l = 1, \cdots, L$, *i.e., those* $K \subseteq B$ *which can be written as* $K = \cup_{l=1}^{L} K_l$, $K_l \in O_{W_l}(B)$, $l = 1, \cdots, L$. *Then* $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) = \bigcup_{l=1}^{L} \left((\cap_{i=1}^{M} Y_i) \circ W_l \right)$ is a *MAP estimator of* X *on the basis of* $\mathbf{Y}^{(M)}$.

[1]See [9] for a recent survey of the status of morphological filtering

What does a uniform distribution model? We may think of it as modeling an "unbiased" or "fair" adversary. If the noise is "biased", then, depending on the particular type of probabilistic noise structure, and assuming we can uncover this structure, we might well be able to construct better estimators, or, we might not even be able to guarantee consistency.

Theorem 3 *(Strong Consistency) In addition, if* $\emptyset \in \Psi_i(B)$, $\forall\, i \geq 1$, *then, under the foregoing assumptions,* $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) \longrightarrow X$, *a.s. as* $M \to \infty$, *i.e., this MAP estimator is strongly consistent.*

We now present two more theorems. They can both be established by appealing to duality (note that closing is the dual of opening with respect to lattice complementation).

Theorem 4 *(MAP Optimality Dual) Assume we observe* $\mathbf{Y}^{(M)} = [Y_1, \cdots, Y_M]$, *where* $Y_i = X \cap N_i$, $\{N_i\}_{i=1}^{M}$ *is an independent* **but not necessarily identically distributed** *sequence of noise DRS's, which is independent of* X, *and each* N_i *is uniformly distributed over* **some arbitrary collection,** $\Psi_i(B) \subseteq \Sigma(B)$, *of subsets of the observation lattice* B. *Let us further assume that* X *is uniformly distributed over a collection,* $\Phi(B) \subseteq \Sigma(B)$, *of all subsets* K *of* B *which can be written as* $K = \cap_{l=1}^{L} K_l$, $K_l \in C_{W_l}(B)$, $l = 1, \cdots, L$. *Then* $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) = \bigcap_{l=1}^{L} \left((\cup_{i=1}^{M} Y_i) \bullet W_l \right)$ is a *MAP estimator of* X *on the basis of* $\mathbf{Y}^{(M)}$.

Theorem 5 *(Strong Consistency Dual) In addition, if* $B \in \Psi_i(B)$, $\forall\, i \geq 1$, *then, under the foregoing assumptions* $\widehat{X}_{MAP}(\mathbf{Y}^{(M)}) \longrightarrow X$, *a.s. as* $M \to \infty$, *i.e., this MAP estimator is strongly consistent.*

4. DISCUSSION

A little reflection on the above results is in order. The discussion will focus on Theorems 2,3, but the remarks are equally applicable to the case of Theorems 4,5.

The first remark is that proof of both theorems crucially depends on B being finite. We view this as further evidence of the utility of this restriction. The second remark is that the results are fairly general: apart from the mild condition $\emptyset \in \Psi_i(B)$, $\forall\, i \geq 1$, which is needed for consistency, we have imposed absolutely no other restrictions on the sequence of range spaces $\{\Psi_i(B)\}$ of the noise DRS's $\{N_i\}$.

In general, we cannot derive analytical formulas for some standard measures of estimator performance, such as bias and variance, without specifying the sequence of range spaces $\{\Psi_i(B)\}$ of the noise DRS's $\{N_i\}$; this is obvious, since these measures strongly depend on the structure of this sequence. Based on our experience in [2], our feeling is that these derivations are going to be nasty, except in some limited cases. However, it should be noted that the MAP principle leads to optimal estimators in a particular Bayesian sense: it minimizes the total probability of error, P_e [25]. In other words, even though the MAP estimator may not be unbiased and/or minimize the error variance (as a MMSE estimator typically does) it is optimal in the sense that for each and every M, it minimizes the total probability of error. This is just an alternative concept of optimality.

Let us now consider two special cases.

• $\Psi_i(B) = \Sigma(B)$, $\forall\ i \geq 1$: The noise DRS's are identically distributed, each noise DRS is uniformly distributed over the power set of B. This is in fact the only nontrivial noise distribution compatible both with our earlier results in [2], and with our results herein. This corresponds to the case of an i.i.d. sequence of i.i.d. DRS's, each being a Bernoulli lattice process of constant intensity $\lambda = \frac{1}{2}$. In addition to MAP optimality and strong consistency, compatibility with [2] buys *uniqueness* of the functional form of the MAP estimator, and a handle on the bias.

• $\Psi_i(B) = \Psi(B)$, $\forall\ i \geq 1$, where $\Psi(B) \subseteq \Sigma(B)$, is the collection of all subsets K of B which are spanned by unions of translates of a family of structural elements, V_l, $l = 1, \cdots, \Lambda$ i.e., those $K \subseteq B$ which can be written as $K = \cup_{l=1}^{\Lambda} K_l$, $K_l \in O_{V_l}(B)$, $l = 1, \cdots \Lambda$. The noise is now a system of overlapping particles of several different types, i.e., constrained to be smooth with respect to a union of openings by an appropriately chosen family of structural elements. Noise particles overlap with signal particles. Regardless of the degree of overlap and the particular types of signal and noise particles, we can claim optimality and strong consistency[2]. However, small sample behavior will be governed by the interplay between the two families of structural elements which span the signal and noise DRS's ($\{W_m\}$,$\{V_l\}$, respectively). For example, if $|V_l| < |W_m|$, $\forall m = 1, \cdots, L$ then application of the $M = 1$ MAP filter will eliminate all isolated instances of V_l noise patterns. This may well be the case in applications, where the signal is usually associated with the more prominent image structures.

5. CONCLUSIONS

We have revisited the problem of estimating realizations of random sets immersed in random clutter, or suffering from random dropouts, under a new, and, in a sense, more appealing set of assumptions, which allows the explicit incorporation of geometric and morphological constraints into the noise model, i.e., the noise may now exhibit *geometric structure*; Surprisingly, it turns out that this affects neither the optimality nor the consistency of these filters.

[2]Observe that the consistency condition is satisfied, since $\emptyset \circ V = \emptyset$, $\forall V$.

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