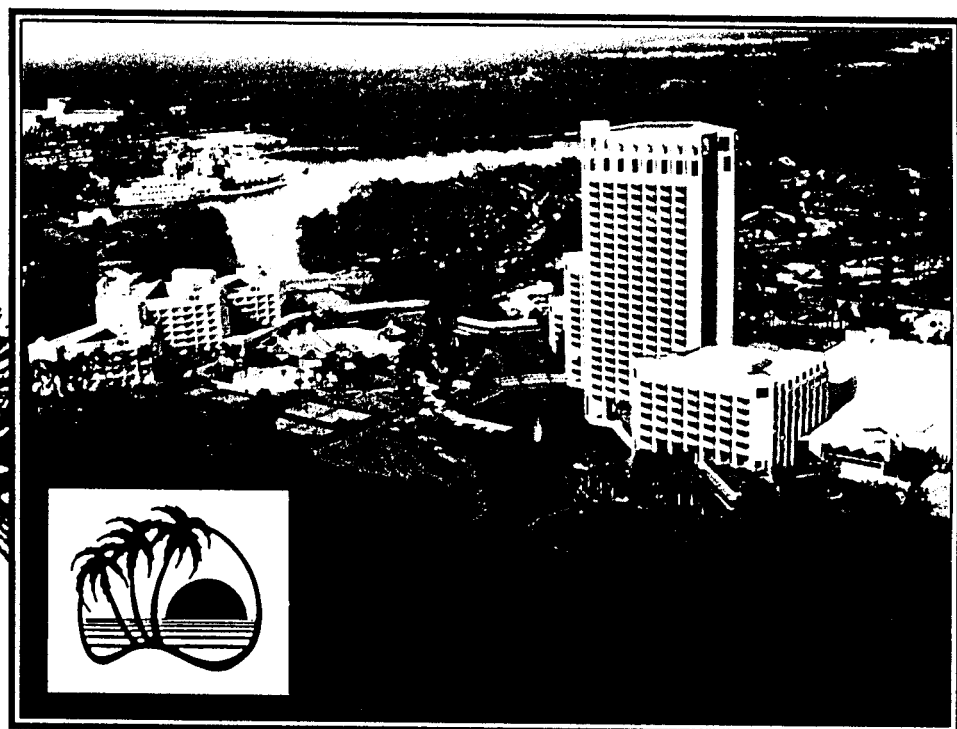94θ2

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ROBUST H^∞ OUTPUT FEEDBACK CONTROL FOR BILINEAR SYSTEMS*

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Abstract

In this paper we present a solution to the robust H_∞ output feedback control problem for bilinear systems over a finite time interval. The solution is obtained through the use of the information state method. We discuss the relation of the information state controller with the certainty equivalence controller under smoothness and uniqueness assumptions. Finally, numerical examples are provided to illustrate implementation issues of the information state feedback controller.

1. Introduction

The study of robust H_∞ output feedback control for nonlinear systems has attracted increasing interest over the last few years. As a generalization of the results from linear theory, the solution to the output feedback problem has been postulated to involve a nonlinear observer combined with a controlled dissipation inequality for an augmented system. By postulating such a structure and solving an augmented dynamic game problem, several researchers [1], [5], [12] have established results yielding sufficient conditions for the existence of a solution to the robust H_∞ output feedback control problem.

In [6], a solution to the partially observed risk sensitive stochastic control for nonlinear systems is obtained and a close connection between it and a partially observed dynamic game is discovered.

Motivated by the results for the risk sensitive control, the authors in [7], [11], and [8] obtain a solution to the robust H_∞ output feedback control problem for nonlinear systems which is both necessary and sufficient. The novelty of the approach developed in these papers is the use of an information state to convert the original output feedback control problem into a new equivalent one with full state. The information state is, in general, an infinite dimensional state obeying a dynamic programming equation (DPE) which evolves forward in time. In the equivalent full state problem, the information state serves as the appropriate state, and the problem is solved using dynamic programming methods, in which a value function is defined in terms of the information state. The difficulty associated with this approach is that, in general, the solution to the problem is, in a sense, doubly infinite dimensional. For convenience, we shall call this method the *information state approach*.

In this paper we address the problem of finite time horizon, robust H_∞ output feedback control of bilinear systems. For our purposes bilinear systems are of particular interest since, as we shall see, they belong to the class of systems with finite dimensional information state [9]. This fact allows us to use dynamic programming methods for finite dimensional systems. This paper is organized as follows. A statement of the problem we consider is given in §2, and a reformulation of the problem in terms of a dynamic game is given in §3. In §4, we present the *information state* solution to the problem. In §5 the relation between the information state solution and the certainty equivalence controller proposed in [2], [3], is explored. In §6 numerical examples

are presented to show that the above-mentioned value function has nontrivial domain, and to demonstrate the stabilizing and robustness properties of the information state controller.

2. Problem Statement

We consider the class of bilinear systems, denoted by Σ^u, described by the following state space equation

$$
\begin{aligned}
\dot{x}(t) &= A_u(t)x(t) + Bu(t) + w(t), \\
y(t) &= Cx(t) + v(t), \\
z(t) &= Cx(t) + Du(t),
\end{aligned} \quad (1)
$$

with $x(0) = x_0 \in \mathbf{R}^n$, where $A_u(t) \triangleq A_0 + \sum_{i=1}^m A_i u_i(t)$, and where we assume $D'[C \ D] = [0 \ R]$, $R > 0$, and write $C'C = Q$. We assume that the state $x(t) \in \mathbf{R}^n$ is not directly measured and that the initial condition $x_0 \in \mathbf{R}^n$ is unknown. At time $t \geq t_0$, knowledge of the state is obtained only through the observations history $y(s) \in \mathbf{R}^p, s \in [t_0, t]$. The additional output $z(t) \in \mathbf{R}^{n+m}$ is a performance measure. The admissible controls $u = [u_1, ... u_m]'$ take values in $U = \mathbf{R}^m$ and are restricted to be non-anticipating functions of the observation path y. The class of such controllers is denoted by $\mathcal{O}$. The disturbances w and v are assumed to be finite energy signals on the interval $[t_0, t_f]$, i.e., $w \in L_2([t_0, t_f], \mathbf{R}^n)$ and $v \in L_2([t_0, t_f], \mathbf{R}^p)$. For $x \in \mathbf{R}^n, Q \in \mathbf{R}^{n \times n}$, we write $|x|_Q^2 = x'Qx$.

Problem 2.1 Finite Time Horizon Robust H_∞ Output Feedback Control
Given $\gamma > 0$ and finite time interval $[t_0, t_f]$, find a control $u \in \mathcal{O}_{t_0, t_f}$ such that Σ^u has finite L_2 gain strictly less than γ, which means that for each initial condition $x_0 \in \mathbf{R}^n$, there exists $\beta_{t_f}^u(x_0) \geq 0$ finite, with $\beta_{t_f}^u(0) = 0$, such that

$$
\int_{t_0}^{t_f} |z(s)|^2 ds \leq (\gamma^2 - \epsilon) \int_{t_0}^{t_f} (|w(s)|^2 + |v(s)|^2) \\
ds + \beta_{t_f}^u(x_0),
$$

for all $(w, v) \in L_2([t_0, t_f], \mathbf{R}^{n+p})$, for some $\epsilon > 0$.

3. Dynamic Game Formulation

The solution to the robust H_∞ output feedback control problem can be expressed in terms of the solution to a related zero-sum dynamic game problem. We define the function space $\mathcal{E} = \{p : \mathbf{R}^n \to \mathbf{R}^*\}$, where $\mathbf{R}^*$ denotes the extended real line. Consider the cost functional

$$
\begin{aligned}
J_{p,t_f}(u) = \sup_{w,v} \sup_{x_0} \{ & p(x_0) + \tfrac{1}{2} \int_{t_0}^{t_f} (|x(s)|_Q^2 \\
& + |u(s)|_R^2 - \gamma^2(|w(s)|^2 + |v(s)|^2))ds \},
\end{aligned} \quad (2)
$$

where $p \in \mathcal{E}$. The robust control problem can be expressed in terms of $J_{p,t_f}(u)$.

Lemma 3.2 *The system Σ^u has finite gain less than $\gamma > 0$ on $[t_0, t_f]$ if and only if there exists some finite quantity $\beta(x) \geq 0$, with $\beta(0) = 0$, such that $J_{-\beta,t_f}(u) \leq 0$.*

We are interested in the set of functions $p \in \mathcal{E}$ for which the finite time cost $J_{p,t_f}(u)$ is finite. Define the "sup pairing" $(\cdot, \cdot)$ by $(p, q) \triangleq \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}$.

Lemma 3.3 *(c.f. [7]) Suppose Σ^u is finite gain on $[t_0, t_f]$. Then*

$$
(p, 0) \leq J_{p,t_f}(u) \leq (p, \beta_{t_f}^u)
$$

Thus, if we define

$$
\text{dom } \Sigma_{t_f}^u = \{p \in \mathcal{E} : (p, 0), (p, \beta_{t_f}^u) \text{ finite}\},
$$

then $J_{p,t_f}(u)$ is finite on dom $\Sigma_{t_f}^u$. We see that a solution to Problem 2.1 can be obtained by minimizing $J_{p,t_f}(u)$ over $\mathcal{O}_{0,t_f}$ on dom$\Sigma_{t_f}^u$.

4. Information State Controller

The information state can be thought of as a deterministic *sufficient statistic* in that it contains all the information needed to control the system with respect to the given performance measure. The information state p_t is given by (see [6], [7], [9])

$$
\begin{aligned}
p_t(x) = \sup_w \sup_{x_0} \{ & p(x_0) + \tfrac{1}{2} \int_{t_0}^t (|x(s)|_Q^2 + \\
& |u(s)|_R^2 - \gamma^2(|w(s)|^2 + \\
& |Cx(s) - y(s)|^2))ds : x(t) = x\},
\end{aligned}
$$

in which past observations and controls $\{u(s), y(s) : s \in [t_0, t]\}$ are known. Dynamic programming

methods imply that p_t satisfies the Hamilton-Jacobi equation

$$\frac{\partial p_t}{\partial t} = -\sup_w\{\nabla_x p \cdot (A_u x + Bu + w) \\ + \frac{1}{2}(\gamma^2(|w|^2 + |Cx - y|^2) \\ -(|x|_Q^2 + |u|_R^2))\}, \quad p_0 = p.$$

We have the following representation result.

Theorem 4.4 *For all $u \in \mathcal{O}$, we have*

$$J_{p,t_f}(u) = \sup_{y \in L_2([t_0,t_f],\mathbf{R}^p)} \{(p_{t_f},0) : p_{t_0} = p\}. \quad (4)$$

This key representation result (4) allows us to view the original output feedback problem as an equivalent full state one, in which the information state serves as the appropriate state.

In general the information state is infinite dimensional, i.e., p_t cannot be identified by finite dimensional quantities. However, as we shall see, bilinear systems belong to the class of systems with finite dimensional information state described in [9].

Theorem 4.5 *Assume $p_0(x) = \phi - \frac{\gamma^2}{2}|x - \hat{x}|_{Y^{-1}}^2$, for some $\phi \in \mathbf{R}, 0 < Y' = Y \in \mathbf{R}^{n \times n}, \hat{x} \in \mathbf{R}^n$. Then we have*

$$p_t(x) = \phi(t) - \frac{\gamma^2}{2}|x(t) - \hat{x}(t)|_{Y^{-1}(t)}^2 \quad (5)$$

where $\hat{x}(t), Y(t) = Y'(t) > 0$ and $\phi(t)$ satisfy the ODE's

$$\dot{\hat{x}}(t) = (A_u(t) + \gamma^{-2}Y(t)Q)\hat{x}(t) + Bu(t) \\ + Y(t)C'\hat{v}(t),$$

$$\dot{Y}(t) = Y(t)A_u(t)' + A_u(t)Y(t) - \\ Y(t)(C'C - \gamma^{-2}Q)Y(t) + I,$$

$$\dot{\phi}(t) = \frac{1}{2}(|\hat{x}(t)|_Q^2 + |u(t)|_R^2 - \gamma^2|\hat{v}(t)|^2),$$

$$(6)$$

with $\hat{x}(0) = \hat{x}, Y(0) = Y, \phi(0) = \phi, \hat{v}(t) = y(t) - C\hat{x}(t)$.

Theorem 4.5 implies that in the case at hand the information state can be identified with the finite dimensional quantity $\rho = (\hat{x}, Y, \phi)$. We denote the quadratic information state by p_ρ, i.e.,

$p_\rho = \phi - \frac{\gamma^2}{2}|x - \hat{x}|_{Y^{-1}}^2$. Since $(p_t, 0) = \phi(t)$, the representation (4) becomes

$$J_{p_\rho,t_f}(u) = \sup_y\{\frac{1}{2}\int_{t_0}^{t_f}(|\hat{x}(s)|_Q^2 + |u(s)|_R^2 - \\ \gamma^2(|y(s) - C\hat{x}(s)|^2))ds : \rho(0) = \rho\}. \quad (7)$$

Thus, the output feedback robust H_∞ control problem is equivalent to a new state feedback game with the finite dimension state ρ obeying the state equation $\dot{\rho}(t) = \hat{f}(\rho(t), u(t), y(t))$, $\rho(0) = \rho$, in which $\hat{f}(\rho, u, y)$ is given in (6). The value function for the problem is

$$W(\rho, t) = \inf_u J_{p_\rho,t_f}(u),$$

which satisfies the Hamilton-Jacobi-Isaacs (HJI) equation (see also [9])

$$\frac{\partial W}{\partial t}(\rho, t) + \sup_y \inf_u\{\nabla_\rho W \cdot \hat{f}(\rho, u, y)\} = 0,$$

$$W(\rho, t_f) = \phi. \quad (8)$$

where y plays the role of a competing disturbance. Since Isaacs condition is satisfied in (8), the order in which the inf and sup are applied is inconsequential. Note that from (7) one can write $W(\hat{x}, P, \phi) = W^*(\hat{x}, P) + \phi$.

An interesting and novel feature of this solution is that the value function need not be finite for all values of ρ, t. In the linear case, this is closely related to the coupling condition [9]. Let us denote $\mathcal{D}$ as the set of points for which $W(\rho, t)$ is finite

$$\mathcal{D} = \{(\hat{x}, P, \phi, t) : W(\hat{x}, P, \phi, t) \text{ is finite }\}, \quad (9)$$

where $\mathcal{S}^n$ is the space of real symmetric positive definite matrices. In general, $\mathcal{D}$ is a nontrivial subset of $\mathbf{R}^n \times \mathcal{S}^n \times \mathbf{R} \times [t_0, t_f]$; see the numerical examples in §6.

The dynamic programming equation (8) provides a means to solve the new game problem as stated in the following theorem ([4], [9]).

Theorem 4.6 (Verification) *Assume there exists a smooth solution $W \in C^1(\mathcal{D})$ of the Hamilton-Jacobi-Isaacs equation (8). Then the control $u^*(\rho, t)$ which attains the infimum in (8) defines an optimal controller $u^* \in \mathcal{O}_{t_0, t_f}$ which minimizes the cost functional (7). In particular, we have*

$$u^*(\rho, t) = -R^{-1}(k(\rho, t) + B'\nabla_{\hat{x}}W)$$

where k is an $m \times 1$ vector with

$$k_i(\rho, t) \triangleq \nabla_{\hat{x}} W(\rho, t) \cdot A_i \hat{x} + \nabla_Y W(\rho, t) \cdot$$
$$(Y A_i) + \nabla_Y W(\rho, t) \cdot (A_i Y),$$

$i = 1, \ldots, m$, and the optimal control at time t is $u_t^* = u^*(\rho(t), t)$.

Remark 4.7 In general, the value function need not be C^1, and equation (8) must be interpreted in the viscosity sense [4], [9]. This is typically the case in optimal control and game theory.

5. Relation with Certainty Equivalence

In this section we relate the information state feedback controller obtained in §4 with the certainty equivalence controller of Başar and Bernhard [2], [3].

Consider the value function, $V(x, t)$, of a full state feedback zero-sum dynamic game problem which satisfies the dynamic programming equation

$$\frac{\partial V}{\partial t} = -\inf_u \sup_w \{\nabla_x V \cdot (A_u x + B u + w)$$
$$+ \tfrac{1}{2}(|x|_Q + |u|_R - \gamma^2 |w|^2)\}, \quad V_{t_f} = 0,$$
$$\tag{10}$$

and suppose that $\bar{u}(x, t)$ is the value of u achieving the minimum in (10). The *minimum stress estimate* of the state is defined by [2], [3]

$$\bar{x}(\rho, t) = \arg\max_x \{p_\rho(x) + V_t(x)\}. \tag{11}$$

Note that in general, $\bar{x}$ is set valued. The CE controller is defined by

$$u_{CE}(\rho, t) = \bar{u}(\bar{x}(\rho, t), t). \tag{12}$$

Theorem 5.8 *If (i) the minimum stress estimate $\bar{x}(\rho, t)$ is unique for all $(\rho, t) \in \mathcal{D}$, and (ii) the full state information value function V_t satisfying (10) is continuously differentiable, then the function $W_{CE}(\rho, t) = (p_\rho, V_t)$ is a solution to the dynamic programming equation (8), and the optimal controller is given by*

$$u^*(\rho, t) = -R^{-1}(k(\rho, t) + B' \gamma^2 (\bar{x} - \hat{x})' Y^{-1}),$$

where $k_i(\rho, t) = \gamma^2 (\bar{x} - \hat{x})' Y^{-1} A_i \bar{x}$, $i = 1, \cdots m$, which is precisely the CE controller of [3].

The conditions given in Theorem 5.8 are essentially those of [3], and are difficult to verify in general. The key difficulty is the uniqueness of the minimum stress estimate. When the CE principle is valid the resulting controller agrees with the information state feedback controller presented in §4.

6. Examples

In this section we present two numerical examples to illustrate the optimal information state solution described in §4. We employ finite difference scheme similar to those presented in [8] to solve the HJI equation (8). In both examples we illustrate the domain of W, and in the second one we show the stabilizing and noise attenuation properties of the controller.

Example 1

Consider a linear system with the state space model $\dot{x}(t) = -0.5x(t) + u(t) + w(t)$, $y(t) = x(t) + v(t)$, $|z(t)|^2 = 4|x(t)|^2 + |u(t)|^2$. Using the standard linear H_∞ control theory we know that the system has L_2 gain less than γ for all $\gamma > 1.789$ on $[0, \infty)$. In this case, the stationary value function $W(\hat{x}, Y, \phi)$, which can be computed explicitly, is given by [9]

$$W(\hat{x}, Y, \phi) = \frac{1}{2}\hat{x}' X (I - \gamma^{-2} Y X)^{-1} \hat{x} + \phi,$$

where X is the minimum solution of the algebraic Riccati equation

$$0 = A_0' X + X A_0 - X(B R^{-1} B' - \gamma^{-2} I) X + Q,$$

and the domain of W can be expressed as

$$\mathcal{D} = \left\{ (\hat{x}, Y, \phi) \ : \ XY < \gamma^2 I, \text{ if } \hat{x} \neq 0 \right\}.$$

For the linear system, $X = 1.77$. Note that the only restriction is on Y. As indicated in §4 we can write $W(\hat{x}, Y, \phi, t) = W^*(\hat{x}, Y, t) + \phi$. Plot of the stationary value, $W^*(\hat{x}, Y)$, for $\hat{x} \in [-0.6, 0.6]$, $Y \in (0, 2]$, with $\gamma = 1.85$, is shown in Figure 1 (top). As seen in the figure, $W^*(\hat{x}, Y)$ blows up at $Y \approx 1.55$. We shall use the value of Y, at $\hat{x} = 0$, for which $W^*(0, Y)$ starts to blow up as a measure of the size of the domain $\mathcal{D}$. We denote this measure by d. Plot of d versus the number of iterations, denoted by k, for $\gamma = 1.85$ and $\gamma = 1.7$ is depicted in Figure 1 (bottom). As seen, the size of the domain decreases to 0 for $\gamma = 1.7$ and converges to ≈ 1.55 (prediction using the linear theory yields $d = \gamma^2 / X = 1.93$) for $\gamma = 1.85$. Therefore, we conclude that the minimum value of γ, denoted by γ^*, for which the robust H_∞ output feedback control problem admits a solution lies in $(1.7, 1.85]$. This result agrees with the prediction using the linear theory.

Example 2

Consider now an open loop ($u = 0$) unstable bilinear system with the state space model $\dot{x}(t) = (0.5 + u(t))x(t) + u(t) + w(t)$, $y(t) = x(t) + v(t)$,

$|z(t)|^2 = 4|x(t)|^2 + |u(t)|^2$. We computed $W^*(\cdot, \cdot)$ on $[-0.4, 0.4] \times (0, 5.0]$. Plot of the stationary value of $W^*(\cdot, \cdot)$ for $\gamma = 6.0$ is illustrated in Figure 2 (top). As shown in Figure 2 (middle), for $\gamma = 5.65$, the size of the domain of W^* decreases to zero, while for $\gamma = 6.0$ the size of the domain converges to ≈ 1.99. Thus, we conclude that the minimum γ^* lies in $(5.65, 6]$. The stabilizing and noise attenuation properties of the resulting controller for $\gamma = 6.0$ is shown in Figure 2 (bottom).

Remark 6.9 Truncation of control and disturbance spaces in the numerical computation results in the finiteness of $W^*(\hat{x}, Y, t)$ on the region outside its domain and inaccuracy in the determination of the size of the domain. Further numerical analysis is still required to study the effect of this truncation.

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value function W*(xhat,Y) ; gamma = 1.85

size of domain (d) vs. iterations

gamma = 1.85

gamma = 1.70

value function W*(xhat,Y) ; gamma = 6.00

size of domain (d) vs. iterations

gamma = 6.00

gamma = 5.65

state and noise trajectories ; gamma = 6.0
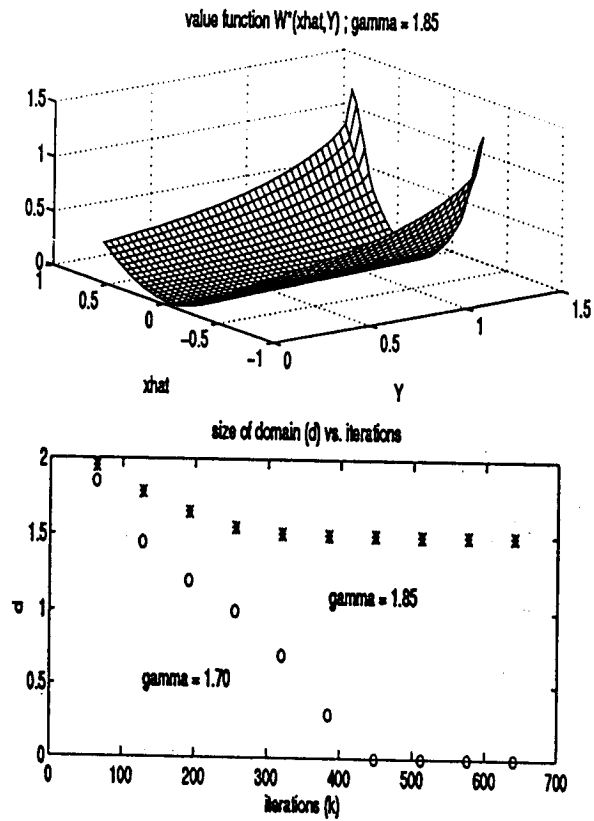
____ : x(t)

---- : w(t)

...... : v(t)

Figure 1: Example 1 (linear system) : stationary value function for $\gamma = 1.85$ (top); domain of value function for $\gamma = 1.85$, and $\gamma = 1.7$ (bottom).
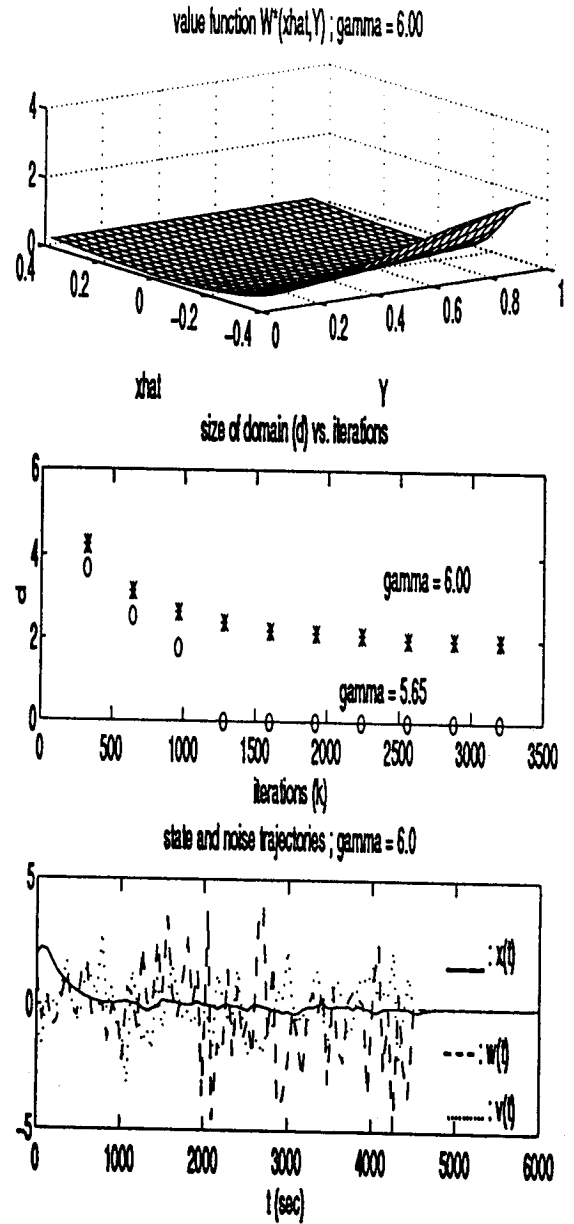
Figure 2: Example 2 (bilinear system) : stationary value function for $\gamma = 6.0$ (top); domain of value function for $\gamma = 6.0$, and $\gamma = 5.65$ (middle); state and noise trajectories for $\gamma = 6.0$ (bottom).