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FREQUENCY DOMAIN DESIGN OF LINEAR DISTRIBUTED SYSTEMS*

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Abstract

Using techniques from the theory of operators and the Hardy spaces of analytic functions we extend to certain classes of linear distributed systems a well known design technique for lumped parameter systems: dynamic-pole-relocation.

1. Introduction

In the last decade we have witnessed a resurgence of interest in frequency domain design methods for linear multivariable systems. These methods based on properties of polynomials and polynomial matrices not only complement the state space methods, but on several occasions have been shown to be superior for design purposes. This subject has been treated extensively and still new interesting problems are generated[1-3].

During the same period, the theory of linear multivariable distributed systems has been developed in parallel lines. Representative papers are [4-9] and I hope the forthcoming monograph [10] will make these developments more widely known. In this theory, which applies to a specific class of distributed systems all the fundamental ingredients of the finite dimensional theory have been captured, and there exists to date a complete theory that provides the analog of the transfer function theory of the lumped case. Unfortunately these developments have not been applied to design problems todate, and this may explain the fact that they are not widely known. I have indicated elsewhere that because of the uniformity of this theory and the finite dimensional theory, the setting for approximation in design is very well set up. It is the purpose of the present paper to indicate how a well known frequency domain design technique for linear multivariable lumped systems extends to this class of linear multivariable distributed systems. Furthermore we indicate that other techniques can be extended using these methods, again due to the uniformity that exists between the two theories. Finally by isolating the basic results from the theory that are seen to be fundamental in design applications we suggest what new theories need to be developed and what basic results they should contain, in order to be useful in design. The reader can find the fundamental results and the needed mathematical background in the references quoted. These results will be assumed known here and we will give only the necessary notation to carry out our calculations. We note that a similar effort has been undertaken by Callier and Desoer[16] using different techniques (which result in a different class of distributed systems) but very much akin to our spirit.

2. A Simple Design Problem

Let us briefly describe the design method we want to investigate. We start with the well known lumped case[2-3], and we treat the scalar input-scalar output case. Consider a given open loop proper transfer function factored as a ratio of two coprime polynomials:

$$T_0(s) = \frac{n_0(s)}{d_0(s)} \tag{2.1}$$

where deg d_0=n, deg $n_0 \leq n$. The objective is to design input and output compensator transfer functions H_i, H_0, proper and stable, connected as in figure 1 below, so that the closed loop transfer function is

$$T_c(s) = \frac{n_0(s)}{d_c(s)}$$

where deg d_c=n, but d_c is otherwise arbitrary.

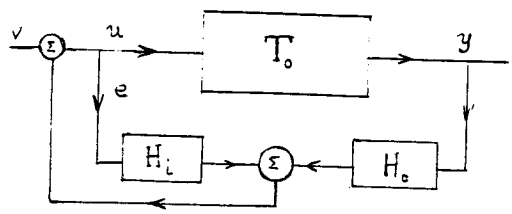

Fig. 1. Illustrating dynamic pole-relocation design method.

This is the well known dynamic pole-relocation design method and its state space equivalent is a compensator consisting from an observer (where the dynamics come from) and linear memoryless feedback from the observer output. The transfer function (or frequency domain) solution is brief and simple. It can actually be done with

$$H_i = \frac{k_i}{q} \quad , \quad H_0 = \frac{k_0}{q} \tag{2.2}$$

i.e. with the same denominator. From the diagram

$$e(s) = \frac{1}{q(s)}[k_i(s)u(s)+k_0(s)y(s)] \tag{2.3}$$

$$u(s) = e(s) + r(s) \tag{2.4}$$

$$y(s) = \frac{n_0(s)}{d_0(s)}\, u(s). \tag{2.5}$$

From these by elementary calculations we get

$$y(s) = \frac{n_0(s)\, q(s)}{q(s)d_0(s)-k_i(s)d_0(s)-k_0(s)n(s)}\, v(s). \tag{2.6}$$

A degree counting shows that the transfer function is proper and if we want it to look like n_0/d_c we have to solve the polynomial Diofantive equation (PDE)

$$q(s)(d_0(s)-d_c(s)) = k_i(s)d_0(s)+k_0(s)n_0(s) \ , \tag{2.7}$$

given d_0, n_0 (coprime and n_0/d_0 proper), d_0, d_c (of degree n) for k_i, k_0, q so that q is stable and k_i/q, k_0/q are proper. It is well known that this can be done quite straightforwardly if (as they are assumed) n_0, d_0 are coprime and that the minimum feasible degree of q is n-1 (which represents the needed augmentation of the dynamics). Central to these constructions is the polynomial Bezout identity (PBI) which states that given any two coprime polynomials n_0, d_0 as above there exist polynomials x,y such that

$$xn_0 + yd_0 = 1 \tag{2.8}$$

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A strengthened version states that if deg $n_0=m$ and deg $d_0=n$, $n\geq m$ then this can be done satisfying the degree constraints deg $x<n$, deg $y<m$, in which case x,y are unique[1-3]. The actual construction employs the Sylvester resultant or Bezout resultant test for coprimeness[1-3]. These results are valid verbatum for discrete time systems by replacing s with z.

We want to motivate a little the approach that has been taken in [4-10] for the class of distributed systems we want to analyze. It is much easier to work first in discrete time. We take a completely frequency domain point of view. Now the transfer function of a lumped discrete time system is given as a Laurent series

$$T(z)=T_0+T_1z^{-1}+T_2z^{-2}+\ldots, \tag{2.9}$$

since we are discussing only proper transfer functions. Strictly proper transfer functions have $T_0=0$ in (2.9). Now the power series (2.9) will converge uniformly for $|z|>\rho$ for some ρ. Since we are given the information that it represents a rational function we can analytically continue it inside the disk $\mathbb{D}_\rho=\{z\in\mathbb{C},|z|<\rho\}$ and represent it there as the ratio of two polynomials

$$T(z)=\frac{n(z)}{d(z)} \quad . \tag{2.10}$$

Now suppose we do not have the information that we have a rational T, but nevertheless we try to write the ratio (2.10) by formally using power series (think of them as infinite degree polynomials). The notion to abstract is this: we were given a function analytic outside a disk and we represented it in (2.10) as the ratio of two functions analytic inside the disk.

Now then we have

$$T(z)=T_0+T_1z^{-1}+\ldots = \frac{\sum_{i=0}^{\infty} n_iz^i}{\sum_{i=0}^{\infty} d_iz^i} \tag{2.11}$$

Clearly we are not going to make much progress unless we put some growth constraint on the coefficients n_i,d_i. Where does this constraint come from? (You see in the finite dimensional case this question does not arise since you have a finite series only). It is coming from growth constraints on the input and output spaces in the definition of the systems. This and continuity or growth conditions on the input-output map will induce growth conditions on the coefficients $\{n_i\}$, $\{d_i\}$. Here we first take the case that input and output spaces are $\ell_2(Z^+)$ and all maps are continuous. This naturally induces that $T\in K^\infty(\mathbb{D})$, and similarly for the functions

$$\left.\begin{aligned} n(z)&=\textstyle\sum_{i=0}^{\infty} n_iz^i \\ d(z)&=\textstyle\sum_{i=0}^{\infty} d_iz^i. \end{aligned}\right\} \in H^\infty(\mathbb{D}) \tag{2.12}$$

Here $H^\infty(\mathbb{D})$ is the Banach algebra of uniformly bounded analytic functions on the unit disk $\mathbb{D}=\{z\in\mathbb{C};|z|<1\}$[13-15]. We will frequently identify $H^\infty(\mathbb{D})$ with the boundary value algebra $H^\infty(\mathbb{T})$ as usual, where $\mathbb{T}=\{z\in\mathbb{C};|z|=1\}$. It seems thus, that other similar theories can be developed by defining an input function space $\mathcal{U}$, an output function space $\mathcal{Y}$ and some restriction on the singularities of T and then analyzing the properties of the Laplace transforms of the algebra of convoluteurs between $\mathcal{U}$ and $\mathcal{Y}$ that satisfy the singularity constraints. We shall return to this point later.

So the theory we have aluded to[4-10] treats transfer functions that belong to the following algebra

$$H_m^\infty(\mathbb{D})=\{f\in K^\infty(\mathbb{D}) \ ; \ \exists \ n,d\in H^\infty(\mathbb{D}) \text{ and } \lim_{|z|\to 1+} f(z) = \lim_{|z|\to 1-}\frac{n(z)}{d(z)} \ \} \quad . \tag{2.13}$$

Here $K^\infty(\mathbb{D})$ is the algebra of functions which are analytic and uniformly bounded on the exterior of the unit disk. A complete list of the notation can be found in [6]. The condition in (2.13) specifies that f has a meromorphic pseudocontinuation of bounded type inside $\mathbb{D}$. Note this is a weakening of (2.11). These are the so called noncyclic, strictly noncyclic, roomy functions of the new theory[4-10]. This condition is an example of the singularity constraint we discussed earlier. It turns out that the algebras H^∞, K^∞, H_m^∞ have a rich structure which has been developed in full detail by mathematicians[13-15], and the forthcoming monograph[10]. The point we want to make is that the transfer functions in the algebra $H_m^\infty(\mathbb{D})$ are appropriate generalizations of proper rational functions. Let us denote K_0^∞ the subset of K^∞ which consists of functions that vanish at infinity. Then strictly proper corresponds to $f\in H_m^\infty(\mathbb{D})\cap K_0^\infty(\mathbb{D})$.

What other features should the algebra have, useful for design purposes? First we should be able to pull zeroes out of these functions. It turns out that this can be done here very nicely. Indeed every $f\in H^\infty(\mathbb{D})$ has a factorization

$$f=f_{in}\cdot f_{out} \tag{2.14}$$

where f_{in} is the inner part and has the property that $|f(e^{i\theta})|=1$ a.e, while f_{out} is the outer part and has no zeroes in $\mathbb{D}$. So f_{in} carries all the "zeroes" of f via its Blaschke product part and via its singular measure on $\mathbb{T}$ [13-15]. This is identical to the "all pass", "minimum phase" factorization of network theory. Moreover since the inner part f_{in} is uniquely specified, once the zeroes are given, it provides a global parametrization of the zeroes of f.

Next we would like to have a coprimeness condition and its consequences. Here we have two coprimeness notions[10]. First we can define coprimeness in terms of no common zeroes which in view of (2.14) means no common inner divisor. Second we can define strong coprimeness as implying

$$\inf_{z\in\mathbb{D}} (|f(z)|+|g(z)|)\geq\delta>0 \ . \tag{2.15}$$

Notice that if f,g have a common zero (2.15) is violated. But (2.15) is stronger because it does not allow for a common zeroing sequence i.e. $\nexists$ sequence of $z_n\in\mathbb{D}$ converging to a point in $\mathbb{T}$ s.t.

$$\lim_{|z_n|\to 1-} f(z_n) = \lim_{|z_n|\to 1-} g(z_n)=0, \text{ (see}^{16}\text{ for a similar}$$

notion). It turns out that strong coprimeness is strongly related to the famous Carleson corona theorem[17] which implies that (2.15) holds for $f,g\in H^\infty(\mathbb{D})$ iff $\exists$ $x,y\in H^\infty(\mathbb{D})$ such that

$$x(z)f(z)+y(z)g(z)=1, \ z\in\mathbb{D} \ . \tag{2.16}$$

We call this the strong scalar Bezout identity (SSBI) for H^∞ (compare with (2.8)). On the other hand f,g have no common inner divisors if and only if $\exists$ sequences of

functions $\{x_i\}\{y_i\}$ in H^∞ such that[6,7]

$$\lim_{i\to\infty} (x_i f+y_i g) = 1 , \tag{2.17}$$

where convergence is in L^2 sense on $\mathbb{T}$ or uniformly on compact subsets of $\mathbb{D}$.

It is worth emphasizing that the state space theory of transfer functions in $H_m^\infty(\mathbb{D})$ has been worked out in full detail and completeness. So there exist state space isomorphism theorem, explicit realizations, degree theory, McMillan Smith form, pole-zero theory etc. (see [4]-[9] and in particular [10] for details).

Now let us pose the analog of the dynamic pole-relocation design method in this setting. Referring again to figure 1 we are given the open loop transfer function $T_0\varepsilon H_m^\infty(\mathbb{D})$,

$$T_0(z) = \frac{n_0(z)}{d_0(z)} , \; z\varepsilon\mathbb{T} \text{ a.e} , \tag{2.18}$$

where d_0 is inner and n_0, d_0 strongly coprime. It turns out that if we have a representation for T_0 inside $\mathbb{D}$ as a ratio of two functions in H^∞ we can always choose the "denominator" to be inner. This is a convenient normalization of the singularities of T_0. Now we have two distinct cases: we can write T_0 in an H^∞ fraction where the two functions are either coprime or strongly coprime. Here we start with the strongly coprime case because its similarity with the lumped case (and thus polynomials) is more striking. We then want to choose H_0, H_i in $H_m^\infty(\mathbb{D})$ so that the closed loop transfer function $T_c\varepsilon H_m^\infty(\mathbb{D})$

$$T_c(z) = \frac{n_0(z)}{d_c(z)} , \; z\varepsilon\mathbb{T} \text{ a.e} , \tag{2.19}$$

where d_c is in $H^\infty(\mathbb{D})$ and is arbitrary modulo the constraint:

$$\frac{d_c(z)}{d_0(z)} = g_c(z), \; z\varepsilon\mathbb{T} \text{ a.e, for some } g_c\varepsilon K^\infty(\mathbb{D}) . \tag{2.20}$$

That is we assume d_c/d_0 is the pseudocontinuation in $\mathbb{D}$ of some transfer function in $H_m^\infty(\mathbb{D})$. We note that we shall write freely equations like (2.18)(2.19)(2.20) and we always interpret them in the nontangential limit sense specified in (2.13). We shall show as in the polynomial case that this can be done actually with $H_0=k_0/q$, $H_i=k_i/q$ and q such that

$$\frac{d_0(z)}{zq(z)} = f(z), \; z\varepsilon\mathbb{T} \text{ a.e, for some } f\varepsilon K^\infty(\mathbb{D}) \tag{2.21}$$

The tricky part is to be careful interpreting the various transfer function relationships. We note that (2.20) is the analog of the requirement deg d_c=deg d_0 which is equivalent to saying that d_c/d_0 is proper in the polynomial setting. Indeed (2.20) means according to the correspondence established earlier (see (2.13) and comments) that $(d_c/d_0)\varepsilon H_m^\infty$, i.e. it is proper! This is a good point to emphasize again the intuitive meaning of H_m^∞ and the way it is used in this paper. Since we are dealing with irrational functions we cannot count "degree" as in the rational case: all systems here are infinite dimensional. However we can easily count finite "degree" differences. For example (2.20) means equal "degree", while (2.21) means q has "degree" one less than that of d_0. We thus use the generalized notion of proper embodied in the definition of H_m^∞ (i.e. (2.13)) as a means to generalize finite degree differences

We represent signals by their z-transforms so $u(z)=\sum_{i=0}^{\infty} u_i z^i$. Since $\{u_i\}\varepsilon\ell_2(Z^+)$ then $u(z)\varepsilon K^2(\mathbb{D})$. Similarly for y(z). Then from the diagram

$$e(z)=H_i(z)u(z)+H_0(z)y(z) \tag{2.22}$$

$$u(z)=e(z)+v(z) \tag{2.23}$$

$$y(z)=T_0(z)u(z) \tag{2.24}$$

In view of the well known (see [13]-[15]) one-to-one relationship between functions in the Hardy spaces of the disk $\mathbb{D}$ and their nontangential limits, we can rewrite (2.22)-(2.24) with $z=e^{i\theta}$. In doing so we can also use the fact that all transfer functions are in H_m^∞, to give a fractional description of each system, which is akin to the polynomial descriptions emphasized by Rosenbrock[1]. Thus we shall see that it is advantageous to describe the open loop system by (instead of (2.24))

$$\begin{aligned} d_0(e^{i\theta})\zeta(e^{i\theta}) &= u(e^{i\theta}) \\ y(e^{i\theta}) &= n_0(e^{i\theta})\zeta(e^{i\theta}) \end{aligned} \tag{2.25}$$

Here d_0, $n_0\varepsilon H^\infty$ while, $\zeta, u, y\varepsilon K^2$. Indeed (2.25) is an appropriate generalization of the well known polynomial description. In the time domain the first of (2.25) defines the "generalized state" ζ as the solution of a convolution equation. Note that since d_0 is inner

$$\zeta(e^{i\theta}) = \overline{d_0(e^{i\theta})}\, u(e^{i\theta})$$

and therefore $\zeta\varepsilon K^2$ for any $u\varepsilon K^2$; i.e. (2.25) is always well posed. We then rewrite all systems appearing in (2.22)-(2.24) in the fractional description introduced in (2.25). Thus (2.22)-(2.24) are completely equivalent to

$$\left.\begin{aligned} d_0(e^{i\theta})\zeta(e^{i\theta}) &= u(e^{i\theta}) \\ y(e^{i\theta}) &= n_0(e^{i\theta})\zeta(e^{i\theta}) \\ q(e^{i\theta})e(e^{i\theta}) &= k_i(e^{i\theta})u(e^{i\theta}) + k_0(e^{i\theta})y(e^{i\theta}) \\ u(e^{i\theta}) &= v(e^{i\theta}) + e(e^{i\theta}) \end{aligned}\right\} \tag{2.26}$$

Solving the fourth of (2.26) for e, substituting for u and y from the first and second in the third there results

$$[q\, d_0 - k_i\, d_0 - k_0\, n_0]\zeta = q\, v . \tag{2.27}$$

We emphasize again that (2.27) is viewed as an equality of functions on $\mathbb{T}$. Suppose we could solve the following factorization problem on $\mathbb{T}$: Given n_0, d_0, d_c as above, find functions q, k_i, k_0 such that (2.20)(2.21) hold and k_i/q, k_0/q have pseudocontinuations in $K^\infty(\mathbb{D})$ and

$$q(d_0-d_c) = k_i\, d_0 + k_0\, n_0 \tag{2.28}$$

is satisfied a.e on $\mathbb{T}$. Then in view of (2.28), (2.27) can be rewritten as

$$q\, d_c\, \zeta = q\, v$$

which since $q \neq 0$ implies

$$d_c(e^{i\theta})\zeta(e^{i\theta}) = v(e^{i\theta}) . \tag{2.29}$$

But (2.29) together with the second of (2.26) provide a fractional description of the closed loop system, to which we can associate the tranfer function $(n_0/d_c)\varepsilon H_m^\infty$. Clearly then to solve the dynamic pole relocation problem in this setting, is equivalent to solving the factorization problem (2.28). The solution is provided by the following theorem.

Theorem 2.1: Given n_0, $d_0\,\varepsilon\, H^\infty(\mathbb{D})$, strongly coprime with

d_0 inner and $(n_0/d_0) = T_0 \varepsilon H_m^\infty(\mathbb{D})$; d_c in $H^\infty(\mathbb{D})$ such that $(d_c/d_0)\varepsilon H_m^\infty(\mathbb{D})$. There exist k_i, k_0 $q\varepsilon H^\infty(\mathbb{D})$ such that k_i/q, k_0/q, $d_0/zq\varepsilon H_m^\infty(\mathbb{D})$ and

$$q(d_0-d_c) = k_i d_0 + k_0 n_0 \quad . \tag{2.30}$$

Proof: Strong coprimeness of n_0, d_0 imply

$$\exists\ x,y\varepsilon H^\infty(\mathbb{D}) \quad \text{s.t.}$$
$$x\, d_0 + y\, n_0 = 1,$$

and therefore

$$q(d_0-d_c) = q(d_0-d_c)x d_0 + q(d_0-d_c)y\, n_0 \quad . \tag{2.31}$$

Now let

$$\left.\begin{aligned} p &= q(d_0-d_c)\, x \\ r &= q(d_0-d_c)\, y \ . \end{aligned}\right\} \tag{2.32}$$

Then $p,r\varepsilon H^\infty(\mathbb{D})$. To proceed further in a manner identical to the polynomial case we need a device to reduce "degree". This is done in the polynomial case via the Euclidean division algorithm. For the kind of ring H^∞ is, there does not exist a Euclidean division algorithm. We can prove though a property which is the appropriate generalization in our setting. We restate first the Euclidean division algorithm as follows. Given two polynomials α,β with β/α strictly proper there exist polynomials κ and ρ such that ρ/β is strictly proper and

$$\alpha = \kappa\beta + \rho \tag{2.33}.$$

Employing the notion of proper (transfer) functions implied by the definition of $H_m^\infty(\mathbb{D})$ and discussed earlier, it can be shown that the following "alternative" to the Euclidean division algorithm holds for H^∞ (see[18] for details). Given two functions $\alpha\varepsilon H^\infty(\mathbb{D})$, $\beta\varepsilon H^\infty(\mathbb{D})$ and inner there exist κ, $\rho\varepsilon H^\infty(\mathbb{D})$ such that $\rho/\beta\varepsilon H_m^\infty(\mathbb{D}) \cap K_0^\infty(\mathbb{D})$ (i.e. it is strictly proper) and

$$\alpha = \kappa\beta + \rho \quad . \tag{2.34}$$

Now r, as introduced in the second of (2.32) is in $H^\infty(\mathbb{D})$, $d_0\varepsilon H^\infty(\mathbb{D})$ is inner and by (2.34) we can write r as

$$r = r' d_0 + k_0 \tag{2.35}$$

where $(\frac{k_0}{d_0})\varepsilon H_m^\infty(\mathbb{D}) \cap K_0^\infty(\mathbb{D})$. (2.36)

Then (2.31) gives

$$q(d_0-d_c) = (p + r'\, n_0)\, d_0 + k_0 n_0 \quad . \tag{2.37}$$

Now $\dfrac{k_0(z)}{q(z)} = \dfrac{zk_0(z)}{d_0(z)} \cdot \dfrac{d_0(z)}{zq(z)}$, and therefore from (2.36) and the condition (2.21) $\dfrac{k_0}{q} \varepsilon H_m^\infty(\mathbb{D})$. Let $k_i = p + r' n_0$. Then from (2.37)

$$1 - \frac{d_c}{d_0} = \frac{k_i}{q} + \frac{k_0}{q}\frac{n_0}{d_0} \quad .$$

In view of the hypotheses and what we already proved, $k_i/q\varepsilon H_m^\infty(\mathbb{D})$. The only point that remains is to show that the set $Q = \{q\varepsilon H^\infty;\ \dfrac{d_0(z)}{zq(z)} = \ell(z)$ for some $\ell\varepsilon K^\infty(\mathbb{D})\}$ is nonempty, which is obvious.

According to our previous discussion then we have

Corollary 2.2: Under the hypothesis of theorem 2.1 the dynamic pole-relocation problem is solvable for the class of transfer functions $H_m^\infty(\mathbb{D})$.

There are several interesting questions that can be addressed now with respect to the "size" of the augmented dynamics, state space interpretations and interpretation of the conditions in terms of spectra. These will be addressed elsewhere[18].

3. Extensions and Modifications

As in the finite dimensional case the theory of linear distributed systems developed in [4]-[10] extends easily from single variable to multivariable systems and from discrete time to continuous time systems. We just state the corresponding results here; details will be given in [18].

For continuous time the appropriate algebra that has been analyzed is

$$H_m^\infty(\pi_+) = \{f\varepsilon H^\infty(\pi_+)\ ;\ \text{there exist } n,d\varepsilon H^\infty(\pi_-)$$
$$\text{s.t.} \quad \lim_{\mathrm{Re}s\to 0+} f(s) = \lim_{\mathrm{Re}s\to 0-} \frac{n(s)}{d(s)} \quad \text{a.e} \quad . \tag{3.1}$$

That is, here we work with the right half plane π_+ and left half plane π_- instead of the disks. Then we have

Theorem 3.1: Under hypotheses identical to theorem 3.1, i.e. exchange $\mathbb{D}$ with π_+, $K^\infty(\mathbb{D})$ with $H^\infty(\pi_-)$, the factorization problem (2.26) is solvable in $H_m^\infty(\pi_+)$ and therefore the dynamic-pole-relocation design problem is also solvable.

Similarly, for the multivariable case, suppose we are given the open loop matrix transfer function for a discrete time system as its right-coprime fraction[4-10].

$$T_0(z) = N(z)\ D^{-1}(z) \ , \ \text{a.e. on } \mathbb{T} \ , \tag{3.2}$$

where T_0 is pxm and is originally given in $K_{pxm}^\infty(\mathbb{D})$ and $D\varepsilon H_{mxm}^\infty(\mathbb{D})$ and is inner. Again in the diagram in figure 1 we have to find compensators H_i, H_0 to change the "denominator matrix" of the closed loop system. The appropriate algebra here is

$$_mH_{pxm}^\infty(\mathbb{D}) = \{A\varepsilon K_{pxm}^\infty(\mathbb{D})\ ;\ \text{there exists a meromorphic pseudocontinuation of bounded type in } \mathbb{D}\}. \tag{3.3}$$

Then there is an extension of the coprimeness conditions from the scalar case[10]: two pxm matrix functions in $H_{pxm}^\infty(\mathbb{D})$, A,B are left (right) coprime if they do not have any left pxp (right mxm) matrix inner divisor. This is again equivalent to the existence (for right coprimeness) of sequences of matrices $\{X_i\},\{Y_i\}$ s.t. (matrix weak Bezout identity on H^∞)

$$\lim_{i\to\infty} (X_i\, A + Y_i\, B) = I \quad . \tag{3.4}$$

For $A,B\varepsilon H_{pxm}^\infty(\mathbb{D})$ we say they are strongly right (left) coprime[10] if

$$\inf_{\substack{z\varepsilon D \\ \|x\| = 1}} \|A(z)\, x\| + \|B(z)\, x\| \geq \delta > 0. \tag{3.5}$$

By the matrix version of the Carleson Corona Theorem[10] this is equivalent to (matrix strong Bezout identity on H^∞): $X,Y\varepsilon H_{mxp}^\infty(\mathbb{D})$

$$X(z)A(z) + Y(z)B(z) = I \quad . \tag{3.6}$$

Then in a fashion similar to the scalar case we can show

Theorem 3.2: Given $T_0 = N_0 D_0^{-1}$ in ${}_m H^\infty_{pxm}(\mathbb{D})$, where D_0 inner, N_0, D_0 strongly right coprime, D_c inner in $H^\infty_{mxm}(\mathbb{D})$ s.t. $D_c D_0^{-1}$ has a meromorphic pseudo continuation outside $\mathbb{D}$. There exist K_i, K_0, Q in $H^\infty_{mxm}(\mathbb{D})$, $H^\infty_{mxp}(\mathbb{D})$, $H^\infty_{mxm}(\mathbb{D})$ so that $Q^{-1}K_i$, $Q^{-1}K_0\, z^{-1}Q^{-1}D_0$ have meromorphic pseudo continuations outside $\mathbb{D}$, and such that

$$Q(z)(D_0(z) - D_c(z)) = K_i(z)D_0(z) + K_0(z)N_0(z) \quad . \quad (3.7)$$

Moreover the dynamic-pole-relocation design problem is solvable.

Going from the disk to the plane we obtain also for ${}_m H^\infty_{pxm}(\pi_+)$

Theorem 3.3: Under conditions similar to Theorem 3.2 (exchange $\mathbb{D}$ with π_+) the dynamic-pole-relocation problem is solvable for ${}_m H^\infty_{pxm}(\pi_+)$.

Finally we would like to note that there are weak versions of all of these results which ascertain approximate changes in the denominators. These will appear in [18].

A more substantial extension which is particularly important for design considerations, is to consider (say for the scalar case, continuous time) transfer functions in

$$\pi^\infty = \{f \text{ , } f \text{ analytic and bounded in some half-plane } \mathrm{Res} > \eta, \text{ with a meromorphic pseudo continuation of bounded type to the corresponding left half plane}\}. \quad (3.7)$$

Much of the theory can be lifted to these algebras which contain unstable elements. Then the compensators should still be stable. This introduces some complications in the design procedure. These will appear elsewhere[18].

4. Conclusions

We have seen that using the methods developed in [4]-[10] for transfer function and state space analysis of linear distributed systems in specific classes, progress can be achieved in some design problems. More analysis is needed to clarify some of the constructions. Coming back to an earlier point it appears that one can analyze by similar methods, problems where singularities are restricted in parts of $\mathbb{C}$. This may be of particular importance for systems say with analytic semigroups (then a wedge in $\mathbb{C}$ is appropriate), or hyperbolic systems (a strip in $\mathbb{C}$ is appropriate). These promise to be interesting and useful extensions of similar constructs in lumped systems[19].

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