

Paper Title:

Nonclassical Filtering Problems in Quantum Electronics

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$$\text{and } c_{ij} = \begin{cases} E \{ (\text{Tr} \{ \rho(x_j) V_j \}) \rho(x_i) \}, & i \neq j \\ n_i V_i, & i = j \end{cases} \quad (8)$$

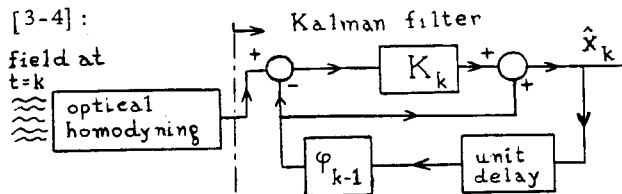
This general filter is not very satisfactory from practical considerations. The number of measuring devices needed for implementing the filter is large and  $\hat{V}_k$  may depend in a significant structural way on  $k$  and on new data. In addition if  $x_k$  satisfies a recursion like

$$x_{k+1} = \varphi_k x_k + w_k \quad (9)$$

where  $\varphi_k$  is a sequence of scalars and  $\{w_k\}$  is a sequence of independent, Gaussian random variables with zero mean and variance  $Q_k$ , a recursive filter is highly desirable. Additional assumptions on the problem produce highly simplified filter structures. A crucial result in this direction is the following "separation" theorem from [3, 4]:

**Theorem 2:** Suppose the signal process  $\{x_k\}$  is pairwise Gaussian, the measurements  $V_i, i=0, 1, \dots, k$ , are optimally chosen (according to Theorem 1) and that the outcomes  $\tau_i$  of the measurements represented by  $T_i$  have the property that  $(\tau_i, x_i)$  are jointly Gaussian for each  $i$ . Then the quantum mechanical linear filtered estimate of  $x_k$  (i.e.  $\hat{x}_k$ ) is equal to the classical linear MMSE estimate of  $x_k$  given the random variables  $\{\tau_i, i=0, 1, \dots, k\}$ .

Consider now the problem where  $x_k$  satisfies a recursion like (9) and is transmitted as the (real) amplitude of a laser (assumed monochromatic) and received, along with thermal noise, in a single mode cavity. Then the optimal filter becomes [3-4]:



$$K_k = P_k [P_k + (\frac{N}{2} + \frac{1}{4})]^{-1}, P_k = \varphi_k^2 [P_{k-1} [1 - K_{k-1}] + Q_{k-1}] \quad (10)$$

Next we describe our results when the signal process is an  $\mathbb{R}^N$  vector process. This case is more delicate and difficult because of the fundamental "compatibility" constraint in quantum mechanics. That is, only "compatible" measurements can be made simultaneously [5]. However by adjoining auxiliary apparatus to the original system [1, 7] one can perform "compatible" measurements on the augmented system which correspond statistically to "non compatible" measurements on the original system. The effect of this on our problem is that instead of considering measurements

as represented by projection valued measures (and thus self adjoint operators) we have to consider representations via positive operator valued measures (p.o.m.). Such measurements are called extended measurements [1]. Considering now a sequence of measurements represented by the p.o.m.'s  $M_i$ , we have that the probability measure characterizing the joint statistics of the vector measurement outcomes  $\{v_0, \dots, v_k\}$  is given by (1) with the appropriate changes from scalar to vector processes. The corresponding filtering problem is to find a p.o.m.  $M_k$  and  $N \times N$  matrices  $C_i(k), i=0, 1, \dots, k$  so as to minimize  $E \{ \|x(k) - \hat{x}(k)\|^2 \}_{\mathbb{R}^N}$

$$\text{where } \hat{x}_k = \sum_{i=0}^k C_i(k) v_i \quad (11)$$

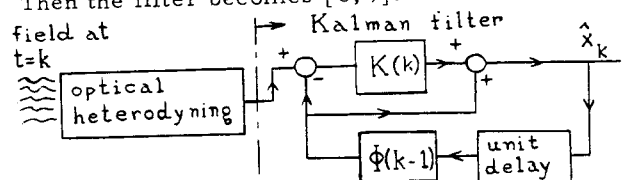
The existence of solution to this optimization problem has been established in [8, 9] and various necessary or sufficient conditions have been found. Here we have also a "separation" of the filter under additional assumptions as described in the sequel. We restrict to p.o.m.'s with a base [1], that is p.o.m.'s that can be expressed as

$$x(B) = \int_B P(u) \mu(du) \quad (12)$$

for some positive-operator valued function  $P$  and a measure  $\mu$  on  $\mathbb{R}^N$ . Let  $M_i, i=0, \dots, k$  be the optimal measurements when we use post processing (with outcomes  $v_i$ ) and let  $\hat{Z}_i, i=0, \dots, k$  be the optimal measurements (with outcomes  $z_i$ ) when we do not use postprocessing. Then we have [8, 9]:

**Theorem 3:** Suppose that the vector signal sequence  $\{x_i\}$  is pairwise Gaussian, and that  $(z_i, x_i)$  are jointly Gaussian for  $i=0, \dots, k$ . Then the quantum mechanical linear filtered estimate of  $x_k$  is equal to the classical linear MMSE estimate of  $x_k$  given the random variables  $\{z_i, i=0, 1, \dots, k\}$ .

As a multiparameter recursive filtering example consider the problem of estimating the two dimensional state  $x_k$  which is transmitted as the in-phase  $x_{1k}$  and quadrature  $x_{2k}$  amplitudes of a laser (assumed monochromatic) and received along with thermal noise, in a single mode cavity. Suppose that  $x_k$  satisfies the vector analog of (9). Then the filter becomes [8, 9]:



$$\text{where } K(k) = P(k) [P(k) + (\frac{n_0}{2} + \frac{1}{2}) I_2]^{-1}$$

$$P(k) = \Phi(k-1) [P(k-1) - K(k-1)P(k-1)] \Phi(k-1)^t + Q(k-1)$$

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