OPTIMAL SENSOR SCHEDULING IN NONLINEAR FILTERING OF DIFFUSION PROCESSES*

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Abstract. The nonlinear filtering problem of a vector diffusion process is considered when several noisy vector observations with possibly different dimension of their range space are available. At each time any number of these observations (or sensors) can be used in the signal processing performed by the nonlinear filter. The problem considered is the optimal selection of a schedule of these sensors from the available set, so as to optimally estimate a function of the state at the final time. Optimality is measured by a combined performance measure that allocates penalties for errors in estimation, for switching between sensor schedules, and for running a sensor. The solution is obtained in the form of a system of quasi-variational inequalities in the space of solutions of certain Zakai equations.

1. Introduction.

1.1. Motivation and preliminaries.
The problem of nonlinear filtering of diffusion processes has received considerable attention in recent years; see the anthologies [1]–[3] for a review of important developments. In current studies, as well as in related analyses of the partially observed stochastic control problem with such models [4], [5], a key role is played by the linear stochastic partial differential equation describing the evolution of the unnormalized conditional probability measure of the state process given the past of the observations, the so-called Zakai equation.

A significant byproduct of these advances is the feasibility of analyzing complex signal processing problems, including adaptive and sensitivity studies, in an integrated, systematic manner, without heuristic or ad hoc assumptions. A problem of interest in this area is the so-called *sensor scheduling problem.* Roughly speaking, this problem is concerned with the simultaneous selection (according to some performance measure) of a signal processing scheme *together* with the sensors that collect the data to be processed. Particular applications include multiple sensor platforms, distributed sensor networks, and large-scale systems. For example, in a multiple sensor platform, there is definite need for coordinating the data obtained from the various sensors, which may include radar, infrared, or sonar. The data obtained from different sensors are of varying quality and a systematic way is needed for allocating confidence or basing decisions on data collected from different types of sensors. For example, radar sensors are more accurate than infrared sensors for long-range tracking while the opposite is true for short-range tracking. In sensor networks we need to coordinate data collected from a large number of sensors distributed over a large geographical area. Conflicts should be resolved and a preferred set of sensors must be selected over finite (short) time intervals, and used in detection, estimation, or control decisions. Similarly, large-scale systems typically involve an attached information network with the objective of collecting data, processing it, and making the results available to the many control

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agents for their decisions (actions). Again the need for coordinating this information in a systematic way is critical.

In such sensor scheduling problems, the systematic utilization of sensors should be the result of optimizing reasonably defined performance measures. Clearly these performance measures will include terms allocating penalties for errors in detection and/or estimation. But more importantly, they must include terms for costs associated with turning sensors on or off, and for switching from one sensor to another. Examples of such costs arising in practice abound. Turning on a radar sensor increases the detectability of the platform (since radars are active sensors) and this should be reflected as a switching cost. Deciding to use a more accurate, albeit more complex, sensor will require higher bandwidth communications and often more computational power allocated to that sensor. In distributed sensor networks it may mean the physical movement of a sensor carrying platform (such as a helicopter or airplane) to a particular geographical location. In large-scale systems the use of several sensors (often hundreds) for decision making may provide better average performance, but it certainly reduces the response speed of the system to changing conditions and increases computational and communication costs both in terms of hardware and software. The latter are obviously evident in large computer/communication networks. These running and switching costs will depend often on the part of the state space occupied by the state vector, i.e., they will be functions of the state as well. For example, sensors have different accuracy or noise characteristics when the state process takes values in different areas of the state space. There is additional cost associated with handling the transfer of information, or tracking record, when there are changes in the set of sensors used; these costs often depend on the state process.

It is not our intent to provide an extensive description of applications here. Detailed descriptions of some of these problems can be found elsewhere; see for example [6], [7]. The underlying thread in all these problem areas is the existence of a variety of sensors, which provide data (for processing), including information of widely varying quality about parameters or variables of interest, for control, detection, estimation, etc. Due to the complexity of these problems it is important to develop systematic conceptual, analytical, and numerical methods for their study and to reduce reliance on ad hoc, heuristic methods as much as possible. The present paper is offered as a contribution in this direction. It provides a general methodology to this problem by reducing it to the analysis of a system of quasi-variational inequalities (see § 3 for details). Numerical methods will be described elsewhere [13].

The sensor scheduling problem is considered here in the context of nonlinear filtering of diffusion processes, and is therefore applicable to detection problems with the same signal models. Modifications of the results apply to other situations including control. In the next section we present a somewhat heuristic definition of the problem, intended to describe the problem clearly, at an intuitive level. The intricacies of establishing this model in a rigorous mathematical fashion are given in § 2 and constitute one of the main contributions of the paper.

1.2. Preliminary description of the problem. The problem considered is as follows. A signal (or state) process $x(\cdot)$ is given, modeled by the diffusion

$$dx(t) = f(x(t))\, dt + g(x(t))\, dw(t),$$

(1.1)

$$x(0) = \xi$$

in $\mathbb{R}^n$. We further consider M noisy observations of $x(\cdot)$, described by

$$dy^i(t) = h^i(x(t))\, dt + R_i^{1/2}\, dv^i(t),$$

(1.2)

$$y^i(0) = 0$$

with values in $\mathbb{R}^{d_i}$. Here $w(\cdot)$, $v^i(\cdot)$ are independent, standard, Wiener processes in $\mathbb{R}^n$, $\mathbb{R}^{d_i}$, respectively, and $R_i = R_i^T > 0$ are $d_i \times d_i$ matrices. Further mathematical details on the system (1.1), (1.2) will be given in § 2. Let us consider a finite time horizon $[O, T]$. To formulate the problem of determining an *optimal utilization schedule* for the available sensors, so as to *simultaneously minimize* the cost of errors in estimating a function of $x(\cdot)$ and the costs of using as well as of switching between various sensors, we need to specify these costs. To this end, let $c_i(x)$ denote the cost per unit time when using sensor i, and the state of the system is x; $k_{io}(x)$, $k_{oi}(x)$ denote the cost for turning off, respectively on, the ith sensor when the state of the system is x. The objective of the performed signal processing is to compute, at time T, an estimate $\hat{\phi}(T)$ of a given function $\phi(x(T))$ of the state. Penalties for errors in estimation are assessed according to the cost function

(1.3) $$E\{c_e(\phi(x(T)) - \hat{\phi}(T))\} := E\{|\phi(x(T)) - \hat{\phi}(T)|^2\}.$$

We shall comment briefly on more general estimation problems in § 4 of this paper. In particular, the consideration of a quadratic $c_e(\cdot)$ is not a serious restriction.

Next we consider the set of all possible *sensor activation configurations*, denoted here by $\mathcal{N}$. An element $\nu \in \mathcal{N}$ is a *word* of length M from the alphabet $\{0, 1\}$. If the lth position is occupied by a 1, the lth sensor is activated (used); if by a zero, the lth sensor is off. There are $N = 2^M$ elements in $\mathcal{N}$. A *schedule of sensors is then a piecewise constant function* $u(\cdot):[O, T] \to \mathcal{N}$. We let $\tau_j \in [O, T]$ denote the instants of changing schedule; i.e., the moments when at least one sensor is turned on or off. At such a switching moment, suppose the schedule before is characterized by $\nu \in \mathcal{N}$, and after by $\nu' \in \mathcal{N}$. Then the *switching cost associated with such a scheduling change* is

(1.4) $$\mathbf{k}_{\nu\nu'}(x) := \sum_{\{i \in \nu\}\{i \notin \nu'\}} k_{io}(x) + \sum_{\{j \notin \nu\}\{j \in \nu'\}} k_{oj}(x).$$

The *total running cost, associated* with schedule $\nu \in \mathcal{N}$ is

(1.5) $$\mathbf{c}_\nu(x) := \sum_{\{j \in \nu\}} c_j(x).$$

In (1.4), (1.5), *the symbol* $\{i \in \nu\}$ *denotes the set of all indices (from the set* $\{1, 2, \cdots, M\}$*) that are occupied by a* 1 *in* ν (i.e., the indices corresponding to the sensors that are on); similarly, the symbol $\{i \notin \nu\}$ denotes the set of indices corresponding to sensors that are off.

Using the above notation, the available observations, *under sensor schedule* $u(\cdot)$, are described by

(1.6) $$dy(t, u(t)) := h(x(t), u(t)) \, dt + r(u(t)) \, dv(t),$$

where it is apparent that the available observations depend explicitly on the sensor schedule $u(\cdot)$. In (1.6), for $x \in \mathbb{R}^n$, $\nu \in \mathcal{N}$,

(1.7) $$h(x, \nu) := \begin{bmatrix} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix},$$

a block column vector, where in standard notation

(1.8) $$\chi_{\{\nu\}}(i) := \begin{cases} 1 & \text{if the } i\text{th position in the word } \nu \text{ is occupied by a 1} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for $\nu \in \mathcal{N}$,

(1.9) $r(\nu) := \text{block diagonal } \{R_i^{1/2}\chi_{\{\nu\}}(i)\},$

where R_i are the symmetric, positive matrices defined above. Finally

(1.10) $v(t) := \begin{bmatrix} v^1(t) \\ \vdots \\ v^M(t) \end{bmatrix}$

is a higher-dimensional standard Wiener process. In view of (1.7), for all $\nu \in \mathcal{N}$

(1.11) $h(\cdot, \nu): \mathbb{R}^n \to \mathbb{R}^D,$

while

(1.12) $r(\nu): \mathbb{R}^D \to \mathbb{R}^D$

where

(1.13) $D = d_1 + d_2 + \cdots + d_M.$

To make the notation clearer, consider the case $M = 2$, $N = 4$. Then $\mathcal{N} = \{00, 01, 10, 11\}$ and

(1.14)
$$h(x, 00) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad h(x, 01) = \begin{bmatrix} 0 \\ h^2(x) \end{bmatrix},$$
$$h(x, 10) = \begin{bmatrix} h^1(x) \\ 0 \end{bmatrix}, \qquad h(x, 11) = \begin{bmatrix} h^1(x) \\ h^2(x) \end{bmatrix},$$

while

(1.15)
$$r(00) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad r(10) = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix},$$
$$r(01) = \begin{bmatrix} 0 & 0 \\ 0 & R_2^{1/2} \end{bmatrix}, \qquad r(11) = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & R_2^{1/2} \end{bmatrix}.$$

Clearly the dimension of the range space of $y(\cdot, \nu)$ is

(1.16) $D_\nu := \sum_{i=1}^{M} d_i\chi_{\{\nu\}}(i).$

Of course for all ν, $y(t, \nu) \in \mathbb{R}^D$.

Following established terminology (cf. [9]), we see that a sensor scheduling strategy is defined by an increasing sequence of switching times $\tau_j \in [O, T]$ and the corresponding sequence $\nu_j \in \mathcal{N}$ of sensor activation configurations. We shall denote such a strategy by $u(\cdot)$, where

(1.17) $u(t) = \nu_j, \quad t \in [\tau_j, \tau_{j+1}), \quad j = 1, 2, \cdots.$

As stated earlier we are interested in the *simultaneous* minimization of costs due to estimation errors as well as sensor scheduling. We shall therefore consider *joint estimation and sensor scheduling strategies*. Such a strategy consists of two parts: the

sensor scheduling strategy u (see (1.17)) and the estimator $\hat{\phi}$. The set of admissible strategies U_{ad} is the customary set of strategies adapted to the sequence of σ-algebras

$$(1.18) \qquad \mathscr{F}_t^{y(\cdot, u(\cdot))} := \sigma\{y(s, u(\cdot)), s \leq t\}.$$

That is, we consider *strict sense* admissible controls in the sense of [4]. For the problem under investigation, this last statement must be interpreted very carefully. First, we have indicated in (1.18) that the available past observation data σ-algebra depends (as is evident from (1.6)-(1.9)) very strongly on the sensor schedule $u(\cdot)$. This dependence is nonstandard, as here the dimension of the observation vector and the noise covariance change drastically at each switching time τ_i. In standard stochastic control formulations [4], [5], the dependence of y on $u(\cdot)$ is much more implicit. This is a difficult part of the formulation here, since it prevents us from using Girsanov transformations in a straightforward manner. Second, (1.18) means that the switching times τ_i and the variables ν_i, which define $u(\cdot)$, must be adapted to the filtration $\mathscr{F}_t^{y(\cdot, u(\cdot))}$, *which depends essentially* on the values of τ_i and ν_i! Finally (1.18) also means that $\hat{\phi}(T)$ must be measurable with respect to $\mathscr{F}_t^{y(\cdot, u(\cdot))}$. We describe a rigorous mathematical construction of such a model in § 2.

Given such a strategy, the corresponding cost is

$$(1.19) \qquad J(u(\cdot), \hat{\phi}) := E\Bigg\{ |\phi(x(T)) - \hat{\phi}(T)|^2$$

$$(1.20) \qquad\qquad\qquad + \int_0^T c(x(t), u(t))\, dt$$

$$(1.21) \qquad\qquad\qquad + \sum_j k(x(t), u(\tau_{j-1}), u(\tau_j)) \Bigg\}.$$

Here for $x \in \mathbb{R}^n$, $\nu, \nu' \in \mathcal{N}$

$$(1.22) \qquad c(x, \nu) := \mathbf{c}_\nu(x),$$

(cf. (1.5)), and

$$(1.23) \qquad k(x, \nu, \nu') = \mathbf{k}_{\nu, \nu'}(x),$$

(cf. (1.4)).

The optimal sensor scheduling in nonlinear filtering is thus formulated as the determination of a strategy achieving

$$(1.24) \qquad \inf_{u(\cdot), \hat{\phi}} J(u(\cdot), \hat{\phi})$$

among all admissible strategies.

To somewhat simplify the notation, let us order the elements of $\mathcal{N}$ according to the numbers they represent in binary form. For example in the case $M = 2$, $N = 4$ we replace $\mathcal{N} = \{00, 01, 10, 11\}$ by the set of integers $\{1, 2, 3, 4\}$. That is, the one-to-one correspondence between $\mathcal{N}$ and $\{1, 2, \cdots, N\}$ is described by

$$(1.25) \qquad \begin{aligned} &\nu \mapsto (\text{integer represented by } \nu) + 1, \\ &k \mapsto \text{binary representation of } (k-1). \end{aligned}$$

So in the sequel of the paper we replace all the ν, ν' in (1.4)-(1.23) by the corresponding integers from $\{1, 2, \cdots, N\}$.

The structure of the paper is as follows. In § 2 a precise mathematical formulation is given and the corresponding stochastic control problem is precisely defined. In § 3 the set of quasi-variational inequalities solving the problem is derived. In § 4 we offer

some comments and discussion for extensions, further developments, and computational methods.

2. The stochastic control formulation.

2.1. Setting of the model. Let $(\Omega, \mathscr{A}, P)$ be a complete probability space, on which a filtration $\mathscr{F}_t$ is given, $\mathscr{A} = \mathscr{F}_\infty$. Let $w(\cdot)$ and $z(\cdot)$ be two independent, standard $\mathscr{F}_t$-Wiener processes with values in $\mathbb{R}^n$ and $\mathbb{R}^D$, respectively, carried by this probability space. On the same space we consider also an $\mathbb{R}^n$-valued random variable ξ, independent of $w(\cdot), z(\cdot)$, and with probability distribution function π_0.

We consider the Itô equation (1.1), where $f(\cdot)$ is $\mathbb{R}^n$-valued, bounded, and Lipschitz, while $g(\cdot)$ is $\mathbb{R}^{n \times n}$-valued, bounded, and Lipschitz. Letting $a = \frac{1}{2}gg^T$, we assume $a > \alpha I_n$, where $\alpha > 0$ and I_n is the $n \times n$ identity matrix. The Lipschitz property is unnecessary and can be easily removed using Girsanov's transformation (i.e., consider weak solutions of (1.1)) [8]. It is assumed here to simplify the technicalities not related with the main issues of the paper. Under these assumptions (1.1) has a strong solution with well-known properties [8]. Note that *under P, $z(\cdot)$ is independent of $x(\cdot)$.*

Next consider functions $h^i(\cdot)$, $i = 1, \cdots, M$, from $\mathbb{R}^n$ into $\mathbb{R}^{d_i}$ that are bounded and Hölder continuous. We shall denote by L the infinitesimal generator of the Markov process $x(\cdot)$

$$(2.1) \qquad L := \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$$

or in divergence form

$$(2.1a) \qquad L := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j} - \sum_{i=1}^n a_i(x)\frac{\partial}{\partial x_i},$$

where

$$(2.1b) \qquad a_i(x) := -f_i(x) + \sum_{j=1}^n \frac{\partial a_{ij}(x)}{\partial x_j}.$$

Let us next consider an *impulsive control* defined as follows. There is a sequence $\tau_1 < \tau_2 \cdots < \tau_k < \cdots$ of increasing $\mathscr{F}_t$-stopping times. To each time τ_i we attach an $\mathscr{F}_{\tau_i}$-measurable random variable u_i with values in the set of integers $\{1, 2, \cdots, N\}$.[1] We define

$$(2.2) \qquad u(t) = u_i. \quad \tau_i \leqq t < \tau_{i+1}, \quad i = 0, 1, 2, \cdots$$

and set $\tau_0 = 0$. We require that

$$(2.3) \qquad \tau_i \uparrow T \quad \text{as } i \uparrow \infty,$$

while $\tau_k = T$ is possible for some finite k.

Let ν_i be the element of $\mathcal{N}$, corresponding to u_i via (1.25).

Then define

$$(2.4) \qquad h(x, u(t)) := h(x, \nu_i), \quad \tau_i \leqq t < \tau_{i+1}$$

where $h(x, \nu)$ is defined by (1.7), in terms of the given functions $h^i(\cdot)$. Clearly $h(\cdot, u(t))$

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[1] Recall that $N = 2^M$ and the binary representation of each integer $1, 2, \cdots, N$ determines a sensor activation configuration by (1.25).

maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. Define also

$$(2.5) \qquad r(u(t)) := r(\nu_i), \qquad \tau_i \leqq t < \tau_{i+1},$$

where $r(\cdot)$ is defined by (1.9), in terms of the given matrices R_i, $i = 1, 2, \cdots, M$. Clearly $r(u(t))$ maps $\mathbb{R}^D$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ but is *singular*. Next we define $\tilde{h}(x, \nu)$ to be the vector-valued function

$$(2.6) \qquad \tilde{h}(x, \nu) := \begin{bmatrix} R_1^{-1/2} h^1(x) \chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1/2} h^i(x) \chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1/2} h^M(x) \chi_{\{\nu\}}(M) \end{bmatrix}$$

with $\chi_{\{\nu\}}(i)$ defined as in (1.8). Let

$$(2.7) \qquad \tilde{h}(x, u(t)) := \tilde{h}(x, \nu_i), \qquad \tau_i \leqq t < \tau_{i+1}.$$

Clearly $\tilde{h}(\cdot, u(t))$ maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. We shall refer to $u(\cdot)$ as the *impulsive control*. As we shall see, it describes essentially the decision to select at a sequence of decision times one of the functions $h(\cdot, k)$, $k \in \{1, 2, \cdots, N\}$. This is the precise mathematical implementation of the sensor selection decision described in the Introduction.

To see that indeed this is the case, we can, with the above preparation, use Girsanov's measure transformation method. Let us then consider the process

$$(2.8) \qquad \zeta(t) = \exp \left\{ \int_0^t \tilde{h}(x(s), u(s))^T dz(s) - \frac{1}{2} \int_0^t \|\tilde{h}(x(s), u(s))\|^2 ds \right\},$$

where T denotes transpose and $\|\cdot\|$ is the $\mathbb{R}^D$ norm. Note that the process $u(t)$ is adapted to $\mathscr{F}_t$. Then since $x(\cdot)$ is adapted to $\mathscr{F}_t^w \subset \mathscr{F}_t$ and $u(\cdot)$ is cadlag [8], (2.8) is well defined. Moreover, since $\tilde{h}$ is bounded, by Girsanov's theorem [8], [14], $\zeta(\cdot)$ is an $\mathscr{F}_t$-martingale. We can thus define a change of probability measure

$$(2.9) \qquad \frac{dP^{u(\cdot)}}{dP} \bigg|_{\mathscr{F}_t} = \zeta(t)$$

and consider the process

$$(2.10) \qquad v(t) = z(t) - \int_0^t \tilde{h}(x(s), u(s))\, ds.$$

By Girsanov's theorem [8], [14], under the probability measure $P^{u(\cdot)}$ on $(\Omega, \mathscr{A})$, $v(\cdot)$ is a standard $\mathscr{F}_t$-Wiener process with values in $\mathbb{R}^D$. Furthermore, by the independence of $w(\cdot)$ and $z(\cdot)$, $w(\cdot)$ remains a standard $\mathbb{R}^n$-valued, $\mathscr{F}_t$-Wiener process that is independent of $v(\cdot)$. Finally, ξ remains independent of $w(\cdot)$, $v(\cdot)$ while keeping its probability law, denoted by π_0. Thus $x(\cdot)$ also retains its probability law under $P^{u(\cdot)}$.

To relate this construction, i.e., (2.2)-(2.10), to the M noisy observations (sensors) loosely described in the Introduction (cf. in particular (1.6)), observe that (2.10) can be written as

$$(2.11) \qquad r(u(t))\, dz(t) = h(x(t), u(t))\, dt + r(u(t))\, dv(t)$$

in view of (1.7), (1.9), (2.4), (2.5), (2.6), and (2.7). Indeed,

(2.12)
$$r(u(t))\tilde{h}(x,(u(t))) = \begin{bmatrix} R_1^{1/2}\chi_{\{\nu_i\}}(1) & 0 & 0 \\ 0 & R_2^{1/2}\chi_{\{\nu_i\}}(2) & 0 \\ & & \ddots \\ 0 & 0 & R_M^{1/2}\chi_{\{\nu_i\}}(M) \end{bmatrix}$$
$$\cdot \begin{bmatrix} R_1^{-1/2}h^1(x)\chi_{\{\nu_i\}}(1) \\ R_2^{-1/2}h^2(x)\chi_{\{\nu_i\}}(2) \\ \vdots \\ R_M^{-1/2}h^M(x)\chi_{\{\nu_i\}}(M) \end{bmatrix}$$
$$= h(x,\nu_i), \qquad \tau_i \leqq t < \tau_{i+1}.$$

To give a precise meaning to (1.2), or (1.6), let us introduce the continuous path process in $\mathbb{R}^D$:

$$(2.13) \qquad y(t,u(t)) := y^{\nu_i}(t), \qquad \tau_i \leqq t < \tau_{i+1}$$

where

$$(2.14) \qquad dy^{\nu_i}(t) := r(\nu_i)\, dz(t) = h(x(t), \nu_i)\, dt + r(\nu_i)\, dv(t).$$

In other words, in the integration from (2.14) to (2.13), we use the left limits of $y(\ , u(\cdot))$ to initialize. As a consequence, when a sensor is not used, the corresponding components of $y(t, u(t))$ will remain constant, a convention without any consequences. It is clear that if we select $u(t) = \nu$ for all t, where ν has zero everywhere except for one 1 in the ith location, then (1.2) results. It is also rather evident that $dy^\nu(t) \in \mathbb{R}^{D_\nu}$ and that in this case the Wiener process $r(\nu)v(\cdot)$ is also D_ν-dimensional (see (1.16) for the definition of D_ν). The process $dy^{\nu_i}(t)$ represents exactly the observation available in $[\tau_i, \tau_{i+1})$.

The next issue we wish to clarify relates to the measurability question we discussed in § 1.2, after (1.18). For any $u(\cdot)$, given the construction of $y(\cdot, u(\cdot))$ above, we can now consider $\mathcal{F}_t^{y(\cdot, u(\cdot))}$ as defined by (1.18). We shall say that $u(\cdot)$ is *admissible*, denoted $u \in U_{ad}$, if $u(t)$ is $\mathcal{F}_t^{y(\cdot, u(\cdot))}$ measurable, $t > 0$, where $\mathcal{F}_t^{y(\cdot, u(\cdot))}$ is constructed as above. More precisely, this means that the τ_i are $\mathcal{F}_t^{y(\cdot, u(\cdot))}$-stopping times or that

$$(2.15) \qquad \{\tau_i < t\} \subset \mathcal{F}_t^{y(\cdot, u(\cdot))}$$

and that

$$(2.16) \qquad \nu_i \in \mathcal{F}_{\tau_i}^{y(\cdot, u(\cdot))}.$$

Note that since $\mathcal{F}_t^{y(\cdot, u(\cdot))} \subset \mathcal{F}_t$ for any sensor schedule $u(\cdot)$ adapted to $\mathcal{F}_t^{y(\cdot, u(\cdot))}$, if τ_i are $\mathcal{F}_t^{y(\cdot, u(\cdot))}$-stopping times they are also $\mathcal{F}_t$-stopping times, and the above construction (2.8)–(2.14) is still valid. The implication of (2.15), (2.16) is that we should check *that an optimizing strategy, obtained by some procedure, must satisfy the admissibility conditions.* Clearly U_{ad} is nonempty as strategies $u(t) = \nu, t \in [O, T]$, obviously are admissible. Also strategies with fixed switchings are admissible. Note that for an admissible control $\mathcal{F}_t^{y(\cdot, u(\cdot))} \subset \mathcal{F}_t^z$. This can be shown in a straightforward manner by proving by induction that $\mathcal{F}_{\tau_i \vee (t \wedge \tau_{i+1})}^{y(\cdot, u(\cdot))} \subset \mathcal{F}_{\tau_i \vee (t \wedge \tau_{i+1})}^z$ using (2.14) and the convention employed in constructing (2.13) from (2.14).

We have thus established in this section the precise mathematical models of nonlinear filtering problems where selection of sensors is possible. In particular we have succeeded in circumventing the subtleties associated with the definition of admissible sensor schedules discussed in § 1.2.[2]

2.2. The optimization problem. For the dynamical system described in § 2.1, we consider now the cost functional (1.19) where the underlying probability measure is $P^{u(\cdot)}$. As indicated in the Introduction, the general problem where the function ϕ will be in a nice class, e.g., bounded C^2, or polynomial, or C^∞ can be treated along identical lines. To simplify the notation we have chosen to formulate the problem for $\phi(x) = x$. The technical difficulties for this case are identical to the ones in the more general cases discussed above, particularly since this $\phi(\cdot)$ is unbounded on $\mathbb{R}^n$. For this choice the selection of the optimal estimator $\hat{\phi}(T)$ is the conditional mean

$$(2.17) \qquad \hat{\phi}(T) = E^{u(\cdot)}\{x(T) \,|\, \mathscr{F}_T^{y(\cdot, u(\cdot))}\}$$

where $E^{u(\cdot)}$ denotes expectation with respect to $P^{u(\cdot)}$. Let $\mu(u, t)$ denote the conditional probability measure of $x(t)$, given $\mathscr{F}_t^{y(\cdot, u(\cdot))}$, on $\mathbb{R}^n$. It is convenient to express (2.17) as a vector valued functional of $\mu(u, t)$:

$$(2.18) \qquad \hat{\phi}(T) = \Phi(\mu(u, T)) = \int_{\mathbb{R}^n} x \, d\mu(u, T).$$

We shall further assume that the running and switching cost functions $c_i(\cdot)$, $k_{ij}(\cdot)$, $i, j \in \{1, \cdots, N\}$, introduced in (1.4) and (1.5) have the following regularity:

$$(2.19) \qquad c_i(\cdot), k_{ij}(\cdot) \quad \text{are in } C_b(\mathbb{R}^n) \text{ (i.e., bounded and continuous)}.$$

As a result of this simple transformation we can rewrite the cost as a function of the impulsive control $u(\cdot)$ only (i.e., the selection of $\hat{\phi}(\cdot)$ has been eliminated):

$$(2.20) \qquad \begin{aligned} J(u(\cdot)) = E^{u(\cdot)}\bigg\{ &\|x(T) - \Phi(\mu(u, T))\|^2 + \int_0^T c(x(t), u(t)) \, dt \\ &+ \sum_{j=1}^\infty k(x(\tau_j), u(\tau_{j-1}), u(\tau_j))\chi_{\tau_j < T} \bigg\} \end{aligned}$$

where $\chi_{\tau_i < T}$ is the characteristic function of the Ω-set $\{\omega; \tau_i(\omega) < T\}$. We further assume that the switching costs are uniformly bounded below

$$(2.21) \qquad k(x, i, j) \geqq k_o, \quad x \in \mathbb{R}^n, \quad i, j \in \{1, \cdots, N\}$$

with k_0 a positive constant. Note that as a consequence of (2.20) if for some admissible $u(\cdot)$ with positive probability, the number of times $\tau_i < T$ is infinite, then the cost $J(u(\cdot))$ will be infinite. Therefore for T finite the optimal policy will exhibit a finite number of sensor switchings.

The optimal sensor selection problem can now be stated precisely as the optimization problem:

$\mathscr{P}$: Find an admissible impulsive control $u^*(\cdot)$ such that

$$(2.22) \qquad J(u^*(\cdot)) = \inf_{u(\cdot) \in U_{\text{ad}}} J(u(\cdot))$$

---

[2] Since $r(u(t))$ is a singular matrix, this stage is more delicate than in standard stochastic control theory, where $\mathscr{F}_t^z$ would suffice.

where U_{ad} are all impulsive control strategies adapted to $\mathscr{F}^{y(\cdot, u(\cdot))}$, or equivalently, satisfying (2.15), (2.16). Problem $\mathscr{P}$ is a *nonstandard* stochastic control problem of a partially observed diffusion.

2.3. The equivalent fully-observed problem. In this section we transform the problem of § 2.2 into a fully-observed stochastic control problem by introducing appropriate Zakai equations. As is customary in the theory of nonlinear filtering [1]-[4], we introduce the operator

$$(2.23) \qquad p(u(\cdot), t)(\psi) = E\{\zeta(t)\psi(x(t)) \mid \mathscr{F}^{y(\cdot, u(\cdot))}_t\}$$

for each impulsive control $u(\cdot)$. The notation is chosen so as to emphasize the dependence on $u(\cdot)$, which is due to the dependence of $\zeta(\cdot)$ on $u(\cdot)$ as introduced in (2.8).[3] The operator (2.23) maps the set of Borel bounded functions on $\mathbb{R}^n$, into the set of real-valued stochastic processes adapted to $\mathscr{F}^{y(\cdot, u(\cdot))}_t$. Note that $p(u(\cdot), t)$ can be viewed as a positive finite measure on $\mathbb{R}^n$. It is the *unnormalized conditional probability measure* of $x(t)$ given $\mathscr{F}^{y(\cdot, u(\cdot))}_t$ [1], [2].

With the help of these measures we can rewrite the various cost terms in (2.20) as follows:

$$(2.24) \qquad \begin{aligned} E^{u(\cdot)}\{\|x(T) - \Phi(\mu(u, T))\|^2\} &= E\{\zeta(T)\|x(T) - \Phi(\mu(u, T))\|^2\} \\ &= E\{p(u(\cdot), T)(\theta)\} \end{aligned}$$

where

$$(2.25) \qquad \theta(x) := \left\| x - \frac{p(u(\cdot), T)(\chi)}{p(u(\cdot), T)(\mathbb{1})} \right\|^2,$$

with χ representing the function $\chi(x) := x$ and $\mathbb{1}$ the function $\mathbb{1}(x) := 1$, $x \in \mathbb{R}^n$. A straightforward computation implies that

$$(2.26) \qquad E^{u(\cdot)}\{\|x(T) - \Phi(\mu(u, T))\|^2\} = E\{\Psi(p(u(\cdot), T))\}$$

where Ψ is the functional on finite measures of $\mathbb{R}^n$ defined by

$$(2.27) \qquad \Psi(\mu) = \mu(\chi^2) - \frac{\|\mu(\chi)\|^2}{\mu(\mathbb{1})}$$

where $\chi^2(x) = \|x\|^2$, $x \in \mathbb{R}^n$, and μ is any finite measure on $\mathbb{R}^n$ such that the quantities $\mu(\chi^2)$ and $\mu(\chi)$ make sense.

Next, we have

$$(2.28) \qquad \begin{aligned} E^{u(\cdot)}\left\{\int_0^T c(x(t), u(t))\, dt\right\} &= E\left\{\zeta(T)\int_0^T c(x, (t), u(t))\, dt\right\} \\ &= E\left\{\int_0^T E\{\zeta(T)c(x(t), u(t)) \mid \mathscr{F}_t\}\, dt\right\} \\ &= E\left\{\int_0^T E\{\zeta(T) \mid \mathscr{F}_t\}c(x(t), u(t))\, dt\right\} \\ &= E\left\{\int_0^T \zeta(t)c(x(t), u(t))\, dt\right\} \end{aligned}$$

---

[3] But the expectation is with respect to P and not $P^{u(\cdot)}$.

because $x(t), u(t)$ are measurable with respect to $\mathcal{F}_t$ and $\zeta(\cdot)$ is an $\mathcal{F}_t$-martingale. Now define a map C with values in $C_b(\mathbb{R}^n)$ via

$$(2.29) \qquad\qquad C(u_i) := \mathbf{c}_{u_i}(\cdot), \qquad u_i \in \{1, 2, \cdots, N\}.$$

Then in view of (2.29), (2.23), we can rewrite (2.28) as

$$(2.30)\quad
\begin{aligned}
E^{u(\cdot)}\left\{\int_0^T c(x(t), u(t))\, dt\right\} &= E\left\{\int_0^T E\{\zeta(t)c(x(t), u(t)) \mid \mathcal{F}_t^{y(\cdot, u(\cdot))}\}\, dt\right\} \\
&= E\left\{\int_0^T p(u(\cdot), t)(C(u(t)))\, dt\right\}.
\end{aligned}$$

Finally,

$$E^{u(\cdot)}\{k(x(\tau_i), u(\tau_{i-1}), u(\tau_i))\chi_{\tau_i < T}\} = E\{\zeta(\tau_i)k(x(\tau_i), u(\tau_{i-1}), u(\tau_i))\chi_{\tau_i < T}\}$$

$$(2.31)$$

$$= E\{E\{\zeta(\tau_i)k(x(\tau_i), u(\tau_{i-1}), u(\tau_i))\chi_{\tau_i < T} \mid \mathcal{F}_{\tau_i}^{y(\cdot, u(\cdot))}\}\}$$

$$= E\{p(u(\cdot), \tau_i)(K(u(\tau_{i-1}), u(\tau_i)))\chi_{\tau_i < T}\}.$$

Here we have introduced the function K with values in $C_b(\mathbb{R}^n)$ via

$$(2.32) \qquad\qquad K(u_i, u_j) = \mathbf{k}_{u_i, u_j}(\cdot), \qquad u_u, u_j \in \{1, 2, \cdots, N\},$$

and we have used the admissibility of $u(\cdot)$. Note that in the simpler case, where $\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot), i, j \in \{1, 2, \cdots, N\}$ are constant independent of x, (2.30) simplifies to

$$(2.33) \qquad E^{u(\cdot)}\left\{\int_0^T c(x(t), u(t))\, dt\right\} = E\left\{\int_0^T p(u(\cdot), t)(\mathbb{1})\mathbf{c}_{u(t)}\, dt\right\}$$

and (2.31) simplifies to

$$(2.34) \qquad E^{u(\cdot)}\{k(x(\tau_i), u(\tau_{i-1}), u(\tau_i))\chi_{\tau_i < T}\} = E\{\mathbf{k}_{u_{i-1}, u_i}\chi_{\tau_i < T}\, p(u(\cdot), \tau_i)(\mathbb{1})\}.$$

Utilizing (2.26), (2.30), (2.31), we can rewrite the cost corresponding to policy $u(\cdot)$, given in (2.20), as follows:

$$(2.35)\quad
\begin{aligned}
J(u(\cdot)) = E\Big\{ &\Psi(p(u(\cdot), T)) + \int_0^T p(u(\cdot), t)(C(u(t)))\, dt \\
&+ \sum_{i=1}^\infty p(u(\cdot), \tau_i)(K(u_{i-1}, u_i))\chi_{\tau_i < T}\Big\}.
\end{aligned}$$

In (2.35) we have succeeded in displaying the cost as a functional of the unnormalized conditional measure $p(u(\cdot), \cdot)$, which is the "information" state of the equivalent fully-observed stochastic control problem. To complete this transformation we need to derive the evolution equation for $p(u(\cdot), \cdot)$, i.e., the Zakai equation. We turn this problem next and derive a weak form of the Zakai equation for $p(u(\cdot), \cdot)$ in the following lemma. Here $C_b^{2,1}$ denotes the space of all functions $\psi(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$ that are bounded, continuous together with their first and second derivatives with respect to x, and first derivatives with respect to t.

LEMMA 2.1. *For any $\psi \in C_b^{2,1}$ we have the relation*

$$(2.36)\quad
\begin{aligned}
p(u(\cdot), t)(\tilde{\psi}(t)) = \pi_0(\tilde{\psi}(0)) &+ \int_0^t p(u(\cdot), s)\left(\frac{\partial\tilde{\psi}}{\partial s} + L\tilde{\psi}\right) ds \\
&+ \int_0^t \sum_{i=1}^D p(u(\cdot), s)(\tilde{H}_i(u(s))\tilde{\psi}(s))\, dz_i(s)
\end{aligned}$$

where

(2.37)
$$[\tilde{H}_i(u(s))\phi](x) := \tilde{h}_i(x, u(s))\phi(x), \qquad i = 1, 2, \cdots, D, \quad \phi \in C_b^2,$$

$$\tilde{\psi}(s)(x) := \psi(x, s),$$

and $\tilde{h}_i$ is the ith component of $\tilde{h}$ (see (2.6)).

Proof. Let $\beta(\cdot) \in L^\infty(0, T; \mathbb{R}^D)$ given and consider the $\mathscr{F}_t$-martingale $\rho(t)$, defined by

(2.38)
$$d\rho(t) = \rho(t)\beta(t)^T dz(t), \qquad \rho(0) = 1.$$

Recall that by definition of $\zeta(t)$ (cf. eq. (2.8))

(2.39)
$$d\zeta(t) = \zeta(t)\tilde{h}(z(t), u(t))^T dz(t), \qquad \zeta(0) = 1.$$

Therefore by Itô's rule [8]

(2.40)
$$d(\zeta(t)\rho(t)) = \zeta(t)\rho(t)[(\tilde{h}(x(t), u(t)) + \beta(t))^T dz(t) + \tilde{h}^T(x(t), u(t))\beta(t) \, dt]$$

$$\zeta(0)\rho(0) = 1.$$

and since $\psi \in C_b^{2,1}$

(2.41)
$$d\psi(x(t), t) = \left(\frac{\partial \psi(x(t), t)}{\partial t} + L\psi(x(t), t)\right) dt + [\nabla \psi(x(t), t)]^T g(x(t)) \, dw(t)$$

where L is given in (2.1). Therefore, with some arguments suppressed for ease of notation

(2.42)
$$d[\psi(x(t), t)\zeta(t)\rho(t)] = \zeta(t)\rho(t)\left[\left(\frac{\partial \psi}{\partial t} + L\psi + \tilde{h}^T\beta\psi\right) dt + \nabla\psi^T g \, dw(t) + \psi(\tilde{h} + \beta)^T dz(t)\right].$$

In (2.41), (2.42) we use the notation $\nabla \psi = (\partial \psi/\partial x_1, \cdots, \partial \psi/\partial x_n)^T$. Integrating (2.42), and taking expectations, we deduce

(2.43)
$$E\{\psi(x(t), t)\zeta(t)\rho(t)\} = \pi_0(\tilde{\psi}(0)) + E\left\{\int_0^t \zeta(s)\rho(s)\left[\frac{\partial \psi}{\partial s} + L\psi + \tilde{h}^T\beta\psi\right] ds\right\}.$$

We can then write

(2.44)
$$E\left\{\int_0^t \zeta(s)\rho(s)\left[\frac{\partial \psi}{\partial s} + L\psi\right] ds\right\} = E\left\{\int_0^t E\left\{\rho(s)\zeta(s)\left(\frac{\partial \psi}{\partial s} + L\psi\right) \Big| \mathscr{F}_s^{y(\cdot, u(\cdot))}\right\} ds\right\}$$

$$= E\left\{\int_0^t \rho(s)p(u(\cdot), s)\left(\frac{\partial \psi}{\partial s} + L\psi\right) ds\right\}$$

$$= E\left\{\rho(t)\int_0^t p(u(\cdot), s)\left(\frac{\partial \psi}{\partial s} + L\psi\right) ds\right\}$$

by virtue of the $\mathscr{F}_t$-martingale property of $\rho(\cdot)$. Similarly,

(2.45)
$$E\left\{\int_0^t \zeta(s)\rho(s)\tilde{h}(x(s), u(s))^T\beta(s)\psi(x(s), s) \, ds\right\}$$

$$= E\left\{\rho(t)\int_0^t \zeta(s)\psi(x(s), s)\tilde{h}(x(s), u(s))^T dz(s)\right\}$$

$$= E\left\{\rho(t)\int_0^t \sum_{i=1}^D p(u(\cdot), s)(\tilde{h}_i(\cdot, u(s))\psi(\cdot, s)) \, dz_i(s)\right\}$$

where in the first equality we have used the representation $\rho(t) = 1 + \int_0^t \rho(s)\beta(s)^T dz(s)$, and the well-known isomorphism between Itô stochastic integrals and L^2 [8]. Finally,

$$(2.46) \qquad E\{\psi(x(t), t)\zeta(t)\rho(t)\} = E\{\rho(t)p(u(\cdot), t)(\tilde{\psi}(t))\}.$$

Using (2.44), (2.45), (2.46) in (2.43), we obtain

$$(2.47) \qquad E\left\{\rho(t)\left[p(u(\cdot), t)(\tilde{\psi}(t)) - \pi_0(\tilde{\psi}(0)) - \int_0^t p(u(\cdot), s)\left(\frac{\partial\psi}{\partial s} + L\psi\right) ds \right.\right.$$
$$\left.\left. - \int_0^t \sum_{i=1}^D p(u(\cdot), s)(\tilde{H}_i(u(s))\tilde{\psi}(s)) \, dz_i(s)\right]\right\} = 0.$$

We can replace $\rho(t)$ in (2.47) by a linear combination of such variables, with different β. The set of corresponding variables is dense in $L^2(\Omega, \mathcal{F}_t^z, P)$. However, the random variable in the brackets in the right-hand side of (2.47) is clearly in $L^2(\Omega, \mathcal{F}_t^{y(\cdot, u(\cdot))}, P)$ and therefore in $L^2(\Omega, \mathcal{F}_t^z, P)$, since $\mathcal{F}_t^{y(\cdot, u(\cdot))} \subset \mathcal{F}_t^z$. Then (2.47) implies the result of the lemma (2.36).

Remark. Note that the assumed nondegeneracy of $x(\cdot)$ implies that the solution of (2.36) is unique. In general this can be proved under our working hypotheses for solutions that are measure-valued processes. Here we outline such a proof for the case when these conditional measures are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$, i.e., in the case unnormalized conditional densities exist. For this we need to assume in addition that

$$(2.48) \qquad \pi_0 \text{ has a density } p_0 \text{ with respect to Lebesgue measure; } p_0 \in L^2(\mathbb{R}^n).$$

We denote by L^* the formal adjoint of L (see (2.1), (2.1a), (2.1b)):

$$(2.49) \qquad L^* = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j} + \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i,$$

and consider the Hilbert space form of the Zakai equation [10]

$$(2.50) \qquad \begin{aligned} dp &= L^*p \, dt + p\tilde{h}(\cdot, u(t))^T dz(t), \\ p(0) &= p_0. \end{aligned}$$

The function space in which the solution is sought is

$$(2.51) \qquad L^2(\Omega, \mathcal{A}, P; C(0, T; L^2(\mathbb{R}^n))) \cap L^2_{\mathcal{F}^{y(\cdot, u(\cdot))}}(0, T; H^1(\mathbb{R}^n)).$$

Here H^1 is the usual Sobolev space on $\mathbb{R}^n$ [11], and the subindex $\mathcal{F}^{y(\cdot, u(\cdot))}$ in the second L^2 space denotes that the solution is adapted to the filtration $\mathcal{F}_t^{y(\cdot, u(\cdot))}$, $t \geq 0$. It follows from the results of Pardoux [11] that a unique solution of (2.49) exists in the function space (2.50) under the assumptions made here. We can then establish the following.

LEMMA 2.2. *The following property holds:*

$$(2.52) \qquad p(u(\cdot), t)(\psi) = (p(u(\cdot), t), \psi)$$

for all ψ in $L^2(\mathbb{R}^n)$ and bounded, where $(\cdot, \cdot)$ denotes inner product in $L^2(\mathbb{R}^n)$.

Proof. By slight abuse of notation we use the same symbol to denote the conditional unnormalized measure *and* density (whenever the latter exists). Let us prove inductively

that

$$(2.53) \qquad p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1}))(\psi) = (p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1})), \psi),$$

where the left-hand notation refers to the measure appearing in (2.36) and the right-hand notation to the solution of (2.50), which is uniquely defined. The induction is necessary because the right-hand side of (2.55) is discontinuous and so we can only examine (2.55) on the intervals $(\tau_i, \tau_i \vee (t \wedge \tau_{i+1}))$. Suppose then that (2.53) holds for $i - 1$, and therefore, in particular,

$$(2.54) \qquad p(u(\,\cdot\,), \tau_i)(\psi) = (p(u(\,\cdot\,), \tau_i), \psi) \quad \forall \psi.$$

Now consider the solution η of

$$(2.55) \qquad \begin{aligned} \frac{\partial \eta}{\partial s} + L\eta &= -\eta \tilde{h}(\,\cdot\,, u(s))^T \beta(s), \qquad s \in (\tau_i, \tau_i \vee (t \wedge \tau_{i+1})), \\ \eta(x, \tau_i \vee (t \wedge \tau_{i+1})) &= \psi(x) \end{aligned}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ and β is a smooth deterministic function with values in $\mathbb{R}^D$. From the assumptions on f, g and h^i (it is here that we use the assumed Hölder continuity of h^i), we can assert that the solution of (2.55) belongs to $C_b^{2,1}(\mathbb{R}^n \times (\tau_i, \tau_i \vee (t \wedge \tau_{i+1})))$, for any sample ω [11]. Therefore (2.36) implies (using (2.55))

$$(2.56) \qquad \begin{aligned} p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1}))(\psi) &= p(u(\,\cdot\,), \tau_i)(\tilde{\eta}(\tau_i)) \\ &\quad - \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \sum_{j=1}^D p(u(\,\cdot\,), s)(\tilde{H}_j(u(s))\tilde{\eta}(s))\beta_j(s)\, ds \\ &\quad + \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \sum_{j=1}^D p(u(\,\cdot\,), s)(\tilde{H}_j(u(s))\tilde{\eta}(s))\, dz_j(s) \end{aligned}$$

where $\tilde{H}_j$ is as defined in Lemma 2.1, and $\tilde{\eta}(s)(x) := \eta(x, s)$. Therefore, by Itô's rule and recalling that $\rho(t)$ is the martingale associated with $\beta(t)$, we have

(2.57)

$$\begin{aligned} &p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1}))(\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1})) \\ &= p(u(\,\cdot\,), \tau_i)(\tilde{\eta}(\tau_i))\rho(\tau_i) + \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \rho(s) \sum_{j=1}^D p(u(\,\cdot\,), s)(\tilde{H}_j(u(s))\tilde{\eta}(s))\, dz_j(s) \\ &\quad + \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \rho(s) \sum_{j=1}^D p(u(\,\cdot\,), s)(\tilde{H}_j(u(s))\tilde{\eta}(s))\beta_j(s)\, dz_j(s). \end{aligned}$$

Hence

$$(2.58) \quad E\{p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1}))(\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1}))\} = E\{p(u(\,\cdot\,), \tau_i)(\tilde{\eta}(\tau_i))\rho(\tau_i)\}.$$

On the other hand, from (2.50) and (2.55) we obtain

$$(2.59) \qquad \begin{aligned} (p(u(\,\cdot\,), \tau_i \vee (t \wedge \tau_{i+1})), \psi) &= (p(u(\,\cdot\,), \tau_i), \tilde{\eta}(\tau_i)) \\ &\quad + \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \sum_{j=1}^D (p(u(\,\cdot\,), s)\tilde{h}_j(\,\cdot\,, u(s)), \tilde{\eta}(s))\, dz_j(s) \\ &\quad - \int_{\tau_i}^{\tau_i \vee (t \wedge \tau_{i+1})} \sum_{j=1}^D (p(u(\,\cdot\,), s), \tilde{H}_j(u(s))\tilde{\eta}(s))\beta_j(s)\, ds, \end{aligned}$$

and thus also

$$(2.60) \quad E\{(p(u(\cdot), \tau_i \vee (t \wedge \tau_{i+1})), \psi)\rho(\tau_i \vee (t \wedge \tau_{i+1}))\} = E\{(p(u(\cdot), \tau_i), \tilde{\eta}(\tau_i))\rho(\tau_i)\}.$$

But from the inductive hypothesis (2.54), the right-hand sides of (2.58) and (2.60) are equal. Hence the left-hand sides coincide. Varying β, we easily deduce that (2.53) holds, at least for $\psi \in C_0^\infty(\mathbb{R}^n)$, which is sufficient to conclude the proof of the lemma.

With this result we can rewrite the cost (2.35) as follows:

$$(2.61) \quad J(u(\cdot)) = E\left\{ \Psi(p(u(\cdot), T)) + \int_0^T (p(u(\cdot), t), C(u(t)))\, dt \right.$$
$$\left. + \sum_{i=1}^\infty \chi_{\tau_i < T}(p(u(\cdot), \tau_i), K(u_{i-1}, u_i)) \right\}$$

where (see (2.27))

$$(2.62) \quad \Psi(p(u(\cdot), T)) = (p(u(\cdot), T), \chi^2) - \frac{\|(p(u(\cdot), T), \chi)\|^2}{(p(u(\cdot), T), \mathbb{1})}.$$

Since (2.62) involves unbounded functions we must show that it makes sense.

At this point it is useful to introduce a weighted Hilbert space to express $\Psi(p(u(\cdot), T))$ in a more convenient form. To this end let

$$(2.63) \quad \mu(x) = 1 + \|x\|^4$$

and $L^2(\mathbb{R}^n; \mu)$ denote the space of functions φ such that $\varphi\mu \in L^2(\mathbb{R}^n)$. Define in a similar way the space $L^1(\mathbb{R}^n; \mu)$. From the discussion of existence and uniqueness of solutions of (2.50) in the functional space (2.51) and if

$$p_0 \in L^2(\mathbb{R}^n; \mu) \cap L^1(\mathbb{R}^n; \mu),$$

it is easy to check that (2.50), under the assumptions made in § 2.1, has a unique solution in the space

$$(2.64) \quad L^2(\Omega, \mathcal{A}, P; C(0, T; L^2(\mathbb{R}^n; \mu) \cap L^1(\mathbb{R}^n; \mu))) \cap L^2(0, T; H^1(\mathbb{R}^n; \mu))$$

where $H^1(\mathbb{R}^n; \mu)$ is the obvious modification of $H^1(\mathbb{R}^n)$. This justifies that the quantities arising in (2.62) have a meaning.

We note that $J(u(\cdot))$ is indexed implicitly (we do not include this in our notation) by π_0 (or p_0) and $u(0) = j$, $j \in \{1, \cdots, N\}$, which is deterministic since it is $\mathcal{F}_0^z$-measurable, by construction.

We close this section by rewriting the dynamics (2.50), in terms of the originally given observation nonlinearities h^i, and with forcing inputs the processes $y^i(\cdot)$ introduced in (2.13), (2.14). In view of (2.5), (2.6), (2.7), (2.13), (2.14), we have

$$\tilde{h}(\cdot, u(t))^T dz(t) = \sum_{j=1}^M h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j)R_j^{-1/2}\, dz_j(t), \qquad \tau_i \leqq t < \tau_{i+1}$$

(where we have written $z = [z_1, z_2, \cdots z_M]^T$)

$$= \sum_{j=1}^M h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j)R_j^{-1}R_j^{1/2}\chi_{\{\nu_i\}}(j)\, dz_j(t), \qquad \tau_i \leqq t < \tau_{i+1}$$

$$= \delta(\cdot, \nu_i)^T\, dy(t, \nu_i), \qquad \tau_i \leqq t < \tau_{i+1}$$

$$=: \delta(\cdot, u(t))^T\, dy(t, u(t))$$

where

$$(2.65) \qquad \delta(x, \nu) = \begin{bmatrix} R_1^{-1} h^1(x) \chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1} h^i(x) \chi_{\{\nu\}}(u) \\ \vdots \\ R_M^{-1} h^M(x) \chi_{\{\nu\}}(M) \end{bmatrix}.$$

Therefore the system dynamics (2.50) can be written equivalently:

$$(2.66) \qquad \begin{aligned} dp(u(\,\cdot\,), t) &= L^* p(u(\,\cdot\,), t)\, dt + p(u(\,\cdot\,), t) \delta(\,\cdot\,, u(t))^T dy(t, u(\,\cdot\,)), \\ p(u(\,\cdot\,), 0) &= p_0, \end{aligned}$$

where $y(t, u(t))$ is defined in (2.13), (2.14). This makes precise the construction of a Zakai equation driven by "controlled" observations alluded to in the Introduction. It also now becomes clear that the spaces described by (2.51), (2.64) are the appropriate ones as far as solutions of (2.50) or (2.66) are concerned.

3. The solution of the optimization problem.

3.1. Setting up a system of quasi-variational inequalities.
Let us consider the Banach space $H = L^2(\mathbb{R}^n; \mu) \cap L^1(\mathbb{R}^n; \mu)$ and the metric space H^+ of positive elements of H. Let

$$(3.1) \qquad \begin{aligned} \mathscr{B} &:= \text{space of Borel measurable, bounded functions on } H^+, \\ \mathscr{C} &:= \text{space of uniformly continuous, bounded functions on } H^+. \end{aligned}$$

Let us now define semigroups $\Phi_j(t)$ on $\mathscr{B}$ or $\mathscr{C}$ as follows. Consider (2.50) with fixed schedule $u(t) = j$, and let p_j denote the corresponding density $p(\,\cdot\,, j)$. Then for $j \in \{1, 2, \cdots, N\}$

$$(3.2) \qquad dp_j = L^* p_j\, dt + p_j \tilde{h}^{j^T} dz(t), \qquad p_j(0) = \pi$$

where

$$(3.3) \qquad \tilde{h}^j := \tilde{h}(\,\cdot\,, j).$$

We set

$$(3.4) \qquad \Phi_j(t)(F)(\pi) = E\{F(p_{j,\pi}(t))\}, \qquad F \in \mathscr{B} \text{ or } \mathscr{C},$$

where $p_{j,\pi}$ indicates the solution of (3.2) with initial value π. It is easy to see that Φ_j is a semigroup since $p_j(t)$ is a Markov process with values in H^+. It is also useful to introduce the subspaces $\mathscr{B}_1$ and $\mathscr{C}_1$ of functions such that

$$(3.5) \qquad \|F\|_1 = \sup_{\pi \in H^+} \frac{|F(\pi)|}{1 + \|\pi\|_\mu} < \infty$$

where $\|\pi\|_\mu = \|\pi\|_{L^1(\mathbb{R}^n; \mu)}$. The spaces $\mathscr{B}_1$ and $\mathscr{C}_1$ are also Banach spaces. They are needed because we shall encounter functionals with linear growth in the cost function (2.61). To simplify the statement and analysis of the quasi-variational inequalities that solve the optimization problem considered here, we give the details for the case $N = 2$ only in the sequel. We shall insert remarks to indicate how the results should be modified for the general case. Let us introduce the notation

$$(3.6) \qquad \begin{aligned} C_i &:= C(i, \cdot), \qquad i = 1, 2, \\ K_1 &:= K(1, 2), \qquad K_2 := K(2, 1). \end{aligned}$$

Since C_1, C_2, K_1, K_2 are bounded functions, we can use them to define elements of $\mathscr{C}_1$ via (for example)

$$(3.7) \qquad C_1(\pi) = (C_1, \pi)$$

where a slight abuse of notation, in denoting the functional and the function by the same symbol, has been allowed. Similarly the functional on H^+:

$$(3.8) \qquad \Psi(\pi) = (\pi, \chi^2) - \frac{\|(\pi, \chi)\|^2}{(\pi, 1)}$$

belongs to $\mathscr{C}_1$ since it is positive and

$$(3.9) \qquad \Psi(\pi) \le (\pi, \chi^2) \le \|\pi\|_\mu.$$

Consider now the set of functionals $U_1(\pi, t)$, $U_2(\pi, t)$ such that

$$U_1, U_2 \in C(0, T; \mathscr{C}_1),$$

$$U_1(\cdot, t) \ge 0, \qquad U_2(\cdot, t) \ge 0,$$

$$U_1(\pi, T) = U_2(\pi, T) = \Psi(\pi),$$

$$(3.10) \qquad U_1(\pi, t) \le \Phi_1(s-t) U_1(\pi, s) + \int_t^s \Phi_1(\lambda - t) C_1(\pi)\, d\lambda,$$

$$U_2(\pi, t) \le \Phi_2(s-t) U_2(\pi, s) + \int_t^s \Phi_2(\lambda - t) C_2(\pi)\, d\lambda \quad \forall s \ge t,$$

$$U_1(\pi, t) \le K_1(\pi) + U_2(\pi, t),$$

$$U_2(\pi, t) \le K_2(\pi) + U_1(\pi, t).$$

In what follows we occasionally use the notation $U_i(s)(\pi) = U_i(\pi, s)$, $i = 1, 2$.

3.2. Existence of a maximum element. We shall refer to (3.10) as the system of quasi-variational inequalities (QVI). Our first objective is to prove the following.

THEOREM 3.1. *We assume that the conditions on the data f, g, h^i introduced in § 2.1 hold. Then the set of functionals U_1, U_2 satisfying (3.10) is nonempty and has a maximum element, in the sense that if $\tilde{U}_1$, $\tilde{U}_2$ denotes this maximum element and U_1, U_2 satisfies (3.10), then*

$$\tilde{U}_1 \ge U_1, \qquad \tilde{U}_2 \ge U_2.$$

The proof will be carried out in several steps. In fact there is some difficulty due to the functional $\Psi(\pi)$. We shall modify it to assume that

$$(3.11) \qquad 0 \le \Psi(\pi) \le \bar{\Psi}(\pi, 1)$$

where $\bar{\Psi}$ is a constant. We shall prove the theorem with the additional assumption (3.11) and prove the probabilistic interpretation, i.e., the connection with the infimum of (2.61). The probabilistic formula will be used next in an approximation procedure. We can approximate, for instance, the functional Ψ defined by (3.8) in the following way. Set

$$(3.12) \qquad \Psi_n(\pi) = \int \frac{\pi \|x\|^2}{1 + (\|x\|^2/n)}\, dx - \frac{\|\int (\pi x/((1 + \|x\|^2/n)^{1/2}))\, dx\|^2}{\int \pi\, dx},$$

which clearly satisfies (3.11) with $\bar{\Psi} = n$.

Proof of Theorem 3.1 *under assumption* (3.11). The set of functionals satisfying (3.10) is a subset of $\mathcal{B}_1$ or $\mathcal{C}_1$ defined in (3.5). However for this subset the norm (3.5) is unnecessarily restrictive. For those functionals it is sufficient to set

(3.13)
$$\tilde{H} = L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$
$$\tilde{H}^+ = \text{set of positive elements of } \tilde{H}$$

and to consider $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{C}}_1$ the space of Borel or continuous functionals on $\tilde{H}^+$ such that

(3.14)
$$\|F\|_1 = \sup_{\pi \in \tilde{H}^+} \frac{|F(\pi)|}{1 + (\pi, \mathbb{1})} < \infty.$$

We shall then study the system (3.10) with $\mathcal{C}_1$ replaced by $\tilde{\mathcal{C}}_1$. Let us note that
$$H^+ \subset \tilde{H}^+,$$

and if we consider a functional F in $\tilde{\mathcal{B}}_1$ or $\tilde{\mathcal{C}}_1$, its restriction to H^+ belongs to $\mathcal{B}_1$ or $\mathcal{C}_1$; the injection

$$F \to \text{restriction of } F \text{ to } H^+$$

is continuous from $\tilde{\mathcal{B}}_1$ or $\tilde{\mathcal{C}}_1$ to $\mathcal{B}_1$ or $\mathcal{C}_1$. Therefore replacing $\mathcal{C}_1$ by $\tilde{\mathcal{C}}_1$ in (3.10) gives a stronger result.

In the proof we shall omit the symbol $\tilde{}$ and write $\mathcal{B}_1$, $\mathcal{C}_1$ instead of $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{C}}_1$, H^+ instead of $\tilde{H}^+$; the norm $\| \ \|_1$ is then given by (3.14).

The proof is then an adaptation of the methods of Bensoussan and Lions [9] to the present case to take into account the fact that we use $\mathcal{C}_1$ instead of $\mathcal{C}$.

First note that

(3.15)
$$\|\Phi_1(t)\|_{\mathcal{L}(\mathcal{C}_1; \mathcal{C}_1)} \leq 1$$

where $\mathcal{L}(\mathcal{C}_1; \mathcal{C}_1)$ is the space of linear continuous operators from $\mathcal{C}_1$ into itself. Indeed we have

$$\frac{|\Phi_1(t)(F)(\pi)|}{1 + (\pi, \mathbb{1})} = \frac{|E\{F(p_{1,\pi}(t))\}|}{1 + (\pi, \mathbb{1})}$$
$$\leq \|F\|_1 \frac{(1 + E(p_{1,\pi}(t), \mathbb{1}))}{1 + (\pi, \mathbb{1})}$$
$$= \|F\|_1$$

since from (3.2)

(3.16)
$$E(p_{1,\pi}(t), \mathbb{1}) = (\pi, \mathbb{1}).$$

Therefore

(3.17)
$$\|\Phi_1(t)(F)\|_1 \leq \|F\|_1,$$

which implies (3.15).

Note also that a solution of (3.10) will satisfy

(3.18)
$$U_1(\pi, t) \leq \Phi_1(T - t) U_1(\pi, T) + \int_t^T \Phi_1(\lambda - t) C_1(\pi) \, d\lambda$$

and due to positivity, we also have

(3.19)
$$\|U_1(t)\|_1 \leq \|U_1(T)\|_1 + \|C_1\|_1 (T - t) \leq \bar{\Psi} + \|C_1\|(T - t)$$

where $\|C_1\| = \sup_x C_1(x)$.

As it is customary in the study of QVI, we begin with the corresponding obstacle problem:

(3.20)

$$U_1, U_2 \in C(0, T; \mathscr{C}_1),$$

$$U_1(\cdot, t) \geqq 0, U_2(\cdot, t) \geqq 0,$$

$$U_1(\pi, T) = U_2(\pi, T) = \Psi(\pi),$$

$$U_1(\pi, t) \leqq \Phi_1(s - t) U_1(\pi, s) + \int_t^s \Phi_1(\lambda - t) C_1(\pi) \, d\lambda$$

$$U_2(\pi, t) \leqq \Phi_2(s - t) U_2(\pi, s) + \int_t^s \Phi_2(\lambda - t) C_2(\pi) \, d\lambda \quad \forall s \geqq t,$$

$$U_1(\pi, t) \leqq K_1(\pi) + \zeta_2(\pi, t)$$

$$U_2(\pi, t) \leqq K_2(\pi) + \zeta_1(\pi, t)$$

where we assume that

(3.21)

$$\zeta_1, \zeta_2 \in C(0, T; \mathscr{C}_1),$$

$$\zeta_1(\pi, t) \geqq 0, \qquad \zeta_2(\pi, t) \geqq 0,$$

$$\zeta_1(\pi, T), \zeta_2(\pi, T) \geqq \Psi(\pi).$$

We then have the following.

PROPOSITION 3.1. *For* ζ_1, ζ_2 *as in* (3.21), *the set of* U_1, U_2 *satisfying* (3.20) *is not empty and has a maximum element.*

It is clear that for ζ_1, ζ_2 given, the system of inequalities (3.20) can be decoupled and U_1, U_2 can be considered separately. Let us then omit indices momentarily and consider

(3.22)

$$U \in C(0, T; \mathscr{C}_1),$$

$$U(\cdot, t) \geqq 0,$$

$$U(\pi, T) = \Psi(\pi),$$

$$U(\pi, t) \leqq \Phi(s - t) U(\pi, s) + \int_t^s \Phi(\lambda - t) C(\pi) \, d\lambda \quad \forall s \geqq t,$$

$$U(\pi, t) \leqq \zeta(t)$$

where ζ stands, for instance, for $K_1(\pi) + \zeta_2(\pi, t)$. To prove Proposition 3.1, it suffices to show that (3.22) has a maximum element. This can be done by the penalty method. So we look for U_ε solving

(3.23)

$$U_\varepsilon(t) = \Phi(s - t) U_\varepsilon(s) + \int_t^s \Phi(\lambda - t) \left[C(\pi) - \frac{1}{\varepsilon} (U_\varepsilon(\lambda) - \zeta(\lambda))^+ \right] d\lambda \quad \text{for } t \leqq s \leqq T,$$

$$U_\varepsilon(T)(\pi) = \Psi(\pi),$$

$$U_\varepsilon \in C(0, T; \mathscr{C}_1),$$

$$U_\varepsilon(\cdot, t) \geqq 0.$$

We can then assert the following lemma.

LEMMA 3.1. *There is a unique solution of* (3.23).

Proof. Note that (3.23) is equivalent to

$$(3.24) \quad U_\varepsilon(t) = \Phi(T-t)U_\varepsilon(T) + \int_t^T \Phi(\lambda-t)\left[C(\pi) - \frac{1}{\varepsilon}(U_\varepsilon(\lambda) - \zeta(\lambda))^+ \right] d\lambda$$

and also to

$$(3.25) \quad \begin{aligned} U_\varepsilon(t) &= e^{-1/\varepsilon(T-t)}\Phi(T-t)\Psi(\pi) + \int_t^T e^{-1/\varepsilon(\lambda-t)}\Phi(\lambda-t) \\ &\quad \cdot \left[C(\pi) + \frac{1}{\varepsilon}U_\varepsilon(\lambda) - \frac{1}{\varepsilon}(U_\varepsilon(\lambda) - \zeta(\lambda))^+ \right] d\lambda. \end{aligned}$$

Let us define the transformation T_ε of $C(0, T; \mathscr{C}_1)$ into itself using the right-hand side of (3.25). Then the latter can be written as a fixed-point equation:

$$(3.26) \quad U_\varepsilon = T_\varepsilon U_\varepsilon.$$

Using (3.11) and (3.15), we can show precisely, as in Bensoussan and Lions [9, p. 488], that some power of T_ε is a contraction. Hence the result of the lemma follows.

We then can also prove, as in [9, pp. 489–490], that if $\varepsilon \le \varepsilon'$, $\|U_\varepsilon\|_1 \le K$, then $0 \le U_\varepsilon \le U_{\varepsilon'}$. As in [9, pp. 494–495] we then show that as $\varepsilon \downarrow 0$, $U_\varepsilon \downarrow U$, which is the maximum element of (3.22). The convergence takes place in $C(0, T; \mathscr{C}_1)$. This establishes Proposition 3.1.

We can then proceed with the proof of Theorem 3.1.

Proof of Theorem 3.1 (*continuation*). Let us consider the map H mapping $C(0, T; \mathscr{C}_1) \times C(0, T; \mathscr{C}_1)$ into itself defined by

$$(3.27) \quad H(\zeta_1, \zeta_2) = (U_1, U_2)$$

where the right-hand side represents the maximum element of (3.20). Now let

$$(3.28) \quad \begin{aligned} U_1^0(\pi, t) &= \Phi_1(T-t)\Psi(\pi) + \int_t^T \Phi_1(\lambda-t)C_1(\pi) \, d\lambda, \\ U_2^0(\pi, t) &= \Phi_2(T-t)\Psi(\pi) + \int_t^T \Phi_2(\lambda-t)C_2(\pi) \, d\lambda. \end{aligned}$$

Consider $\zeta_i(t)$, $\xi_i(t)$, $i = 1, 2$ such that

$$(3.29) \quad 0 \le \zeta_i(t) \le \xi_i(t) \le U_i^o(t), \qquad i = 1, 2,$$

$$(3.30) \quad \xi_i(t) - \zeta_i(t) \le \gamma \xi_i(t), \qquad \gamma \in [0, 1].$$

Then we have

$$(3.31) \quad 0 \le H(\xi_1, \xi_2) - H(\zeta_1, \zeta_2) \le \gamma(1-\gamma')H(\xi_1, \xi_2),$$

where

$$(3.32) \quad \gamma' \le \frac{k_0}{k_0 + \bar{\Psi} + \max(\|C_1\|, \|C_2\|)T}.$$

Indeed, setting

$$(3.33) \quad \kappa = 1 - \gamma(1-\gamma'),$$

we have to prove that

$$(3.34) \quad \kappa H(\xi_1, \xi_2) \le H(\zeta_1, \zeta_2).$$

Let us set

$$(3.35) \qquad (U_1, U_2) = H(\zeta_1, \zeta_2), \qquad (\tilde{U}_1, \tilde{U}_2) = H(\xi_1, \xi_2).$$

We need then to show that

$$(3.36) \qquad \kappa \tilde{U}_1 \leqq U_1, \qquad \kappa \tilde{U}_2 \leqq U_2.$$

If we can establish that

$$(3.37) \qquad \begin{aligned} \kappa K_1(\pi) + \kappa \xi_2(\pi, t) &\leqq K_1(\pi) + \zeta_2(\pi, t), \\ \kappa K_2(\pi) + \kappa \xi_1(\pi, t) &\leqq K_2(\pi) + \zeta_1(\pi, t), \end{aligned}$$

then (3.36) is implied by the monotonicity properties of variational inequalities. But

$$(3.38) \qquad \xi_2(\pi, t)(1 - \gamma) \leqq \zeta_2(\pi, t);$$

hence, it is enough to establish that

$$(3.39) \qquad \begin{aligned} \kappa K_1(\pi) + \kappa \xi_2(\pi, t) &\leqq K_1(\pi) + (1 - \gamma)\xi_2(\pi, t), \\ \kappa K_2(\pi) + \kappa \xi_1(\pi, t) &\leqq K_2(\pi) + (1 - \gamma)\xi_1(\pi, t). \end{aligned}$$

The first of (3.39) will be satisfied if

$$(3.40) \qquad [\kappa - (1 - \gamma)]\xi_2(\pi, t) \leqq (1 - \kappa)K_1(\pi)$$

or if

$$(3.41) \qquad \gamma'\xi_2(\pi, t) \leqq (1 - \gamma')K_1(\pi).$$

But observe that

$$\xi_2(\pi, t) \leqq U_2^0(\pi, t) \leqq (\bar{\Psi} + \|C_2\|T)(\pi, 1).$$

So it is enough to choose γ' so that

$$(3.42) \qquad \gamma'(\bar{\Psi} + \|C_2\|T)(\pi, 1) \leqq (1 - \gamma')k_0(\pi, 1)$$

where k_0 is the uniform lower bound (2.21), since $K_1(\pi) \geqq k_0(\pi, 1)$. This last inequality requires

$$(3.43) \qquad \gamma' \leqq \frac{k_0}{k_0 + \bar{\Psi} + \|C_2\|T}.$$

In an identical fashion, the second part of (3.39) will be satisfied if

$$(3.44) \qquad \gamma' \leqq \frac{k_0}{k_0 + \bar{\Psi} + \|C_1\|T}.$$

So both parts of (3.39) will be satisfied if we choose γ' according to (3.32). The proof of the theorem then proceeds via the standard iteration

$$(3.45) \qquad (U_1^{n+1}, U_2^{n+1}) = H(U_1^n, U_2^n)$$

as in [9, pp. 512-514].

   Remark. The extension of this result to the general case $N \neq 2$ is straightforward. The system (3.10) has N functionals $U_1, \cdots, U_N$. Everything in (3.10) is the same except for the last two inequalities, which are replaced by

$$(3.46) \qquad U_i(\pi, t) \leqq \min_{\substack{j \neq i \\ j = 1, \cdots, N}} (K_{ij}(\pi) + U_j(\pi, t)), \qquad i = 1, \cdots, N.$$

We again introduce the system (3.20), where the last two inequalities are replaced by

$$(3.47) \qquad U_i(\pi, t) \leqq \min_{\substack{j \neq i \\ j=1,\cdots,N}} (K_{ij}(\pi) + \zeta_j(\pi, t)), \qquad i = 1, \cdots, N$$

where $\zeta_i \in C(0, T; \mathscr{C}_1)$, and satisfy the remainder of (3.21). We then establish the analogue of Proposition 3.1 by penalization. The analogue of Theorem 3.1 is established by introducing a map H mapping $C(0, T; \mathscr{C}_1)^N$ into itself defined by

$$H(\zeta_1, \zeta_2, \cdots, \zeta_N) = (U_1, U_2, \cdots, U_N)$$

where the right-hand side is the maximum element of the analogue of (3.20).

3.3. Existence of an admissible sensor schedule. Our objective in this section is to show that the maximum element U_1, U_2 of the QVI (3.10) provides the value function for the optimization problem (2.61), (2.66) when assumption (3.11) holds. Furthermore, we want to show how an admissible optimal sensor schedule is determined once the pair U_1, U_2 is known.

We shall prove that

$$(3.48) \qquad U_i(\pi, 0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J(u(\cdot)), \qquad i = 1, 2$$

where $\pi \in H^+$ satisfies $(\pi, \mathbb{1}) = 1$. An optimal schedule will be constructed as follows. To fix ideas, suppose that $i = 1$. Then define

$$(3.49) \qquad \tau_1^* = \inf_{t \leqq T} \{U_1(p_1(t), t) = K_1(p_1(t)) + U_2(p_1(t), t)\}$$

where again $p_i(t)$ is the solution of (3.2). We write

$$(3.50) \qquad p^*(t) = p_1(t), \qquad t \in [0, \tau_1^*].$$

Next we define

$$(3.51) \qquad \tau_2^* = \inf_{\tau_1^* \leqq t \leqq T} \{U_2(p_2(t), t) = K_2(p_2(t)) + U_1(p_2(t), t)\}.$$

In (3.51), it must be kept in mind that $p_2(t)$ represents the solution of (3.2) with $j = 2$, starting at τ_1^* with value $p_1(\tau_1^*)$. We then define

$$(3.52) \qquad p^*(t) = p_2(t), \qquad t \in [\tau_1^*, \tau_2^*].$$

Note that, unless $\tau_1^* = T$,

$$(3.53) \qquad\qquad\qquad \tau_2^* > \tau_1^*;$$

otherwise

$$(3.54) \quad \begin{aligned} U_1(p_1(\tau_1^*), \tau_1^*) &= K_1(p_1(\tau_1^*)) + U_2(p_1(\tau_1^*), \tau_1^*), \\ U_2(p_1(\tau_1^*), \tau_1^*) &= K_2(p_1(\tau_1^*)) + U_1(p_1(\tau_1^*), \tau_1^*), \end{aligned}$$

which is impossible since

$$(3.55) \qquad K_1(p_1(\tau_1^*)) > 0, \qquad K_2(p_1(\tau_1^*)) > 0 \quad \text{a.s.}$$

Similarly we proceed to construct a sequence of $\tau_1^* < \tau_2^* < \tau_3^* < \cdots$ and the process $p^*(\cdot)$. We can then prove the following.

THEOREM 3.2. *With the same assumptions as in Theorem* 3.1, *and in addition, assuming that* (3.11) *holds, the sequence of stopping times* $\tau_1^*, \tau_2^*, \cdots$ *defines an optimal admissible sensor schedule.*

Proof. Considering (3.10) as a VI with obstacle ζ_2, ζ_1, we can write from the definition of τ_1^*:

$$(3.56) \qquad U_1(\pi, 0) = E\left\{ U_1(p_1(\tau_1^*), \tau_1^*) + \int_0^{\tau_1^*} C_1(p_1(\lambda))\, d\lambda \right\}.$$

This can be established by using the penalization (3.23), along lines similar to those of [9, pp. 578–587]. Then

$$E\{U_1(p_1(\tau_1^*), \tau_1^*)\} = E\{U_1(p^*(\tau_1^*)), \tau_1^*\}$$
$$= E\{\Psi(p^*(T))\chi_{\tau_1^*=T}\} + E\{U_1(p^*(\tau_1^*), \tau_1^*)\chi_{\tau_1^*<T}\}.$$

Substituting back in (3.56) and using the definition of τ_1^* in (3.49), we obtain

$$(3.57) \qquad U_1(\pi, 0) = E\left\{ \Psi(p^*(T))\chi_{\tau_1^*=T} + \int_0^{\tau_1^*} C_1(p^*(\lambda))\, d\lambda \right.$$
$$\left. + K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} + U_2(p^*(\tau_1^*), \tau_1^*)\chi_{\tau_1^*<T} \right\}.$$

Furthermore, again by employing penalization, we can show that

$$(3.58) \quad E\{U_2(p^*(\tau_1), \tau_1^*)\} = E\{U_2(p_2(\tau_1^*), \tau_1^*)\} = E\left\{ U_2(p_2(\tau_2^*), \tau_2^*) + \int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))\, d\lambda \right\}.$$

This implies

$$(3.59) \quad E\{U_2(p_2(\tau_1^*), \tau_1^*)\chi_{\tau_1^*<T}\} = E\left\{ U_2(p_2(\tau_2^*), \tau_2^*)\chi_{\tau_1^*<T} + \chi_{\tau_1^*<T} \int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))\, d\lambda \right\}.$$

Next

$$E\{U_2(p_2(\tau_2^*), \tau_2^*)\chi_{\tau_1^*<T}\} = E\{\Psi(p^*(T))\chi_{\tau_1^*<T,\tau_2^*=T}\} + E\{U_2(p^*(\tau_2^*), \tau_2^*)\chi_{\tau_2^*<T}\}.$$

Substituting back in (3.57) and using the definition of τ_2^* in (3.51), we obtain

$$(3.60) \qquad U_1(\pi, 0) = E\left\{ \Psi(p^*(T))\chi_{\tau_2^*=T} + K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} + K_2(p^*(\tau_2^*))\chi_{\tau_2^*<T} \right.$$
$$\left. + \int_0^{\tau_1^*} C_1(p^*(\lambda))\, d\lambda + \int_{\tau_1^*}^{\tau_2^*} C_2(p^*(\lambda))\, d\lambda + U_1(p^*(\tau_2^*), \tau_2^*)\chi_{\tau_2^*<T} \right\}.$$

Proceeding in a similar fashion and collecting results we can write:

$$(3.61) \qquad U_1(\pi, 0) = E\left\{ \Psi(p^*(T))\chi_{\tau_n^*=T} + \sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T} \right.$$
$$\left. + \sum_{i=0}^{n-1} \chi_{\tau_{i+1}^*<T} \int_{\tau_i^*}^{\tau_{i+1}^*} C_{i+1}(p^*(\lambda))\, d\lambda \right\}$$
$$+ U_{n+1}(p^*(\tau_n^*), \tau_n^*)\chi_{\tau_n^*<T}$$

where we use the notation

$$(3.62) \qquad K_i = \begin{cases} K_1 & \text{if } i \text{ is odd,} \\ K_2 & \text{if } i \text{ is even,} \end{cases}$$
$$C_i = \begin{cases} C_1 & \text{if } i \text{ is odd,} \\ C_2 & \text{if } i \text{ is even,} \end{cases}$$
$$U_i = \begin{cases} U_1 & \text{if } i \text{ is odd,} \\ U_2 & \text{if } i \text{ is even.} \end{cases}$$

However, observe that necessarily $\tau_n^* = T$ for n large enough (random). Otherwise we have $\tau_n^* < T$ for all n, on a set $\Omega_0 \subset \Omega$ of positive probability. But $\tau_n^* \uparrow \tau^* \leq T$ and

$$(3.63) \qquad (p^*(\tau_i^*), \mathbb{1}) \to (p^*(\tau^*), \mathbb{1})$$

where (since $(\pi, \mathbb{1}) = 1$)

$$(3.64) \qquad (p^*(\tau^*), \mathbb{1}) = 1 + \int_0^{\tau^*} p^* \delta^T \, dy$$

(see (2.66)) and

$$(3.65) \qquad (p^*(\tau^*), \mathbb{1}) = E\{\zeta(\tau^*) \mid \mathscr{F}_{\tau^*}^{y(\cdot, u^*)}\} > 0 \quad \text{a.s.}$$

where $\zeta(\cdot)$ is the process introduced by (2.8). Therefore on Ω_0, as $n \to \infty$

$$(3.66) \qquad \sum_{i=1}^{n} K_i(p^*(\tau_i^*)) \chi_{\tau_i^* < T} \to +\infty$$

and since Ω_0 has positive probability, as $n \to \infty$

$$(3.67) \qquad E\left\{ \sum_{i=1}^{n} K_i(p^*(\tau_i^*)) \chi_{\tau_i^* < T} \right\} \to \infty,$$

which contradicts (3.19).

We can thus assert that

$$(3.68) \qquad \chi_{\tau_n^* = T} \to 1 \quad \text{a.s.}$$

In particular, it follows that the sequence $\tau_1^*, \tau_2^*, \cdots$, defines an admissible schedule denoted by u^*. The corresponding state solution of (2.66) coincides with p^* and (3.61) implies

$$(3.69) \qquad U_1(\pi, 0) \geq J(u^*(\cdot)).$$

But by standard arguments, we check that

$$(3.70) \qquad U_1(\pi, 0) \leq J(u(\cdot)) \quad \forall u(\cdot) \in U_{\text{ad}}$$

and therefore $u^*(\cdot)$ is indeed optimal.

3.4. The main result. We want now to get rid of (3.11) and consider the original functional Ψ in (3.8). Let us consider the approximation (3.12) Ψ_n of Ψ. To Ψ_n corresponds a system of QVI:

$$(3.71) \quad
\begin{aligned}
&U_1^n, U_2^n \in C(0, T; \tilde{\mathscr{C}}_1), \\[4pt]
&U_1^n, U_2^n \geq 0, \\[4pt]
&U_1^n(\pi, T) = U_2^n(\pi, T) = \Psi_n(\pi), \\[4pt]
&U_1^n(\pi, t) \leq \Phi_1(s-t) U_1^n(\pi, s) + \int_t^s \Phi_1(\lambda - t) C_1(\pi) \, d\lambda, \\[4pt]
&U_2^n(\pi, t) \leq \Phi_2(s-t) U_2^n(\pi, s) + \int_t^s \Phi_2(\lambda - t) C_2(\pi) \, d\lambda \quad \forall s \geq t \\[4pt]
&U_1^n(\pi, t) \leq K_1(\pi) + U_2^n(\pi, t), \\[4pt]
&U_2^n(\pi, t) \leq K_2(\pi) + U_1^n(\pi, t).
\end{aligned}$$

From Theorem 3.2, we can assert that

$$(3.72) \qquad U_i^n(\pi, 0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J^n(u(\cdot)), \qquad i = 1, 2$$

where

$$(3.73) \qquad \begin{aligned} J^n(u(\cdot)) = E\Bigg\{ &\Psi^n(p(u(\cdot), T)) + \int_0^T (p(u(\cdot), t), C(u(t)))\, dt \\ &+ \sum_{i=1}^\infty \chi_{\tau_i < T}(p(u(\cdot), \tau_i), K(u_{i-1}, u_i)) \Bigg\}. \end{aligned}$$

Therefore we deduce that

$$(3.74) \qquad J^n(u(\cdot)) - J(u(\cdot)) = E\{\Psi_n(p(u(\cdot), T)) - \Psi(p(u(\cdot), T))\}$$

and from (3.12) we deduce

$$(3.75) \qquad \begin{aligned} |J^n(u(\cdot)) - J(u(\cdot))| \leq &\, E\left\{ \int \frac{p(u(\cdot), T)\|x\|^4}{n + \|x\|^2}\, dx \right\} \\ &+ E\Bigg\{ \left(\int p(u(\cdot), T)x \left(1 - \frac{1}{(1+\|x\|^2/n)^{1/2}}\right) dx \right)^T \\ &\qquad \cdot \left(\int p(u(\cdot), T)x \left(1 + \frac{1}{(1+\|x\|^2/n)^{1/2}}\right) dx \right) \\ &\qquad\qquad\qquad\qquad \cdot \frac{1}{\int p(u(\cdot), T)\, dx} \Bigg\}. \end{aligned}$$

But using (2.50) yields (see (2.1a))

$$\begin{aligned} &E\left\{ \int \frac{p(u(\cdot), t)\|x\|^4}{n + \|x\|^2}\, dx \right\} \\ &= E\Bigg\{ \int_0^t \int p(u(\cdot), s)(x) \Bigg\{ \frac{\partial a_{ij}}{\partial x_i} \frac{2\|x\|^2(2n + \|x\|^2)x_j}{(n + |x|^2)^2} \\ &\qquad\qquad + a_{ij}\left(\delta_{ij} \frac{2\|x\|^2(2n + \|x\|^2)}{(n + \|x\|^2)^2} + \frac{8x_i x_j n^2}{(n + \|x\|^2)^3} \right) \\ &\qquad\qquad - a_i \frac{2\|x\|^2(2n + \|x\|^2)x_i}{(n + \|x\|^2)^2} \Bigg\}\, ds + \int \frac{\pi(x)\|x\|^4}{n + \|x\|^2}\, dx \Bigg\} \end{aligned}$$

where we employ the summation convention over repeated indices. Hence after majorizing conveniently, we have

$$(3.76) \qquad \begin{aligned} &E\left\{ \int \frac{p(u(\cdot), t)(x)\|x\|^4}{n + \|x\|^2}\, dx \right\} \\ &\leq \int \frac{\pi(x)\|x\|^4}{n + \|x\|^2}\, dx + \Gamma \int_0^t E\left\{ \int \frac{p(u(\cdot), s)(x)\|x\|^4}{n + \|x\|^2}\, dx \right\} ds + \frac{\Gamma t}{n}. \end{aligned}$$

We shall use capital Greek letters, $\Gamma, \Delta, \cdots$, to indicate constants in the following estimates. Finally we deduce that

(3.77)
$$E\left\{\int \frac{p(u(\cdot), t)(x)\|x\|^4}{n+\|x\|^2} dx\right\} \leq \Gamma_t \left[\int \frac{\pi(x)\|x\|^4}{n+\|x\|^2} dx + \frac{1}{n}\right]$$

$$\leq \Gamma_t \left[\frac{1}{n}\int \pi(x)\|x\|^4 dx + \frac{1}{n}\right].$$

Next consider

$$\frac{p(u(\cdot), t)}{(p(u(\cdot), t), 1)} = \sigma(u(\cdot), t),$$

which is the normalized conditional probability measure and satisfies Kushner's equation

(3.78) $\quad d(\sigma(t)(\varphi)) = \sigma(t)(L\varphi)\, dt + (\sigma(t)(\tilde{h}\varphi) - \sigma(t)(\varphi)\sigma(t)(\tilde{h})) \cdot (dz - \sigma(t)(\tilde{h})\, dt).$

If we apply (3.78) with $\varphi = \|x\|^2 = \chi^2$, we obtain

(3.79)
$$dE\{\sigma(t)(\chi^2)\} = E\{\sigma(t)(L\chi^2) - \sigma(t)(\tilde{h})[\sigma(t)(\tilde{h}\chi^2) - \sigma(t)(\chi^2)\sigma(\tilde{h})]\}\, dt$$

$$\leq \Delta_0(1 + E\{\sigma(t)(\chi^2)\})\, dt.$$

Finally,

$$E\{\sigma(t)(\chi^2)\} \leq \Delta_t \int \pi(x)\|x\|^2\, dx.$$

But the second term in (3.75) is

$$E\left\{\sigma(T)\left(\chi\left(1+\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)^T (p(T))\left(\chi\left(1-\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right\}$$

$$\leq \left[E\left\{\left\|\sigma(T)\left(\chi\left(1+\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right\|^2\right\}\right]^{1/2}$$

$$\cdot \left[E\left\{\left\|p(T)\left(\chi\left(1-\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right\|^2\right\}\right]^{1/2}$$

(3.81)
$$\leq \Delta^1 (E\{\sigma(T)(\chi^2)\})^{1/2}\left(E\left\{\left\|p(T)\left(\chi\left(1-\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right\|^2\right\}\right)^{1/2}$$

$$\leq \Delta^2\left[E\left\{\left\|p(T)\left(\chi\left(1-\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right\|^2\right\}\right]^{1/2}$$

$$= \Delta^3\left[E\left\{\sum_i \left(p(T)\left(\chi_i\left(1-\frac{1}{(1+\chi^2/n)^{1/2}}\right)\right)\right)^2\right\}\right]^{1/2}$$

$$\leq \Delta^3\left[E\left\{p(T)(\chi^2)p(T)\left(\frac{\chi^2}{n+\chi^2}\right)\right\}\right]^{1/2}.$$

We easily check that

$$E\{(p(T)(\chi^2))^2\} \leq \Delta^4 + \left(\int \pi(x)\|x\|^2\, dx\right)^2 \leq \Delta^5,$$

$$dE\left\{\left|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right|^2\right\} \leq 2E\left\{p(t)\left(L\frac{\chi^2}{n+\chi^2}\right)p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right\}\, dt$$

$$+ \Delta^5 E\left\{\left|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right|^2\right\}\, dt.$$

But

(3.82)
$$L\frac{\chi^2}{n+\chi^2} \leqq \frac{\Delta^6}{\sqrt{n}};$$

hence

(3.83)
$$dE\left\{\left|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right|^2\right\} \leqq \left[\Delta^5 E\left\{\left|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right|^2\right\} + \frac{\Delta^7}{n}\right] dt,$$

which implies

(3.84)
$$E\left\{\left|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)\right|\right\} \leqq \Theta_t\left[\frac{1}{n} + \left(\int \frac{\pi(x)\|x\|^2\, dx}{n+\|x\|^2}\right)^2\right]$$
$$\leqq \frac{\Theta_t}{n}\left(1 + \int \pi(x)\|x\|^4\, dx\right).$$

Therefore, continuing from (3.81), the second term in (3.75) is majorized by $\Gamma_0/n^{1/4}$. Collecting results (from (3.75), (3.77), (3.81), (3.84)), we can assert that

(3.85)
$$|J^n(u(\cdot)) - J(u(\cdot))| \leqq \frac{\Delta}{n^{1/4}}$$

provided the initial distribution of $p(0)$, i.e., π satisfies

(3.86)
$$\int \pi(x)\|x\|^4\, dx < \infty.$$

The estimate in (3.85) is uniform with respect to n. Therefore

(3.87)
$$\left|U_i^n(\pi, 0) - \inf_{u(0)=i, p(0)=\pi} J(u(\cdot))\right| \leqq \frac{\Delta}{n^{1/4}}.$$

In fact we can replace zero by any $t \in [0, T]$ and consider the function

(3.88)
$$U_i(\pi, t) = \inf_{u(t)=i, p(t)=\pi} J_t(u(\cdot))$$

where $J_t(u(\cdot))$ corresponds to a problem analogous to (2.50), (2.61) starting in t instead of zero. Therefore we have

(3.89)
$$|U_i^n(\pi, t) - U_i(\pi, t)| \leqq \frac{\Delta}{n^{1/4}}.$$

However we must be careful of the fact that the constant in (3.89) depends on a bound on $\int \pi(x)\|x\|^4\, dx$. More precisely, we have proved that

(3.90)
$$U_i^n(\pi, t) - U_i(\pi, t)| \leqq \frac{\Delta'}{n^{1/4}}\left(1 + \int \pi(x)\|x\|^4\, dx\right)$$

where Δ' here does not depend on π (assuming that π is a probability). It follows that

(3.91)
$$U_i^n(\pi, t) \to U_i(\pi, t) \quad \text{in } C(0, T; \mathscr{C}_1).$$

Taking the limit in (3.71), we obtain that U_1, U_2 is a solution of (3.10), and moreover

(3.92)
$$U_i(\pi, 0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J(u(\cdot)).$$

However, by a probabilistic argument already used in § 3.3, any solution of (3.10) is smaller than the right-hand side of (3.92). This completes the proof of Theorem 3.1, and also provides the same statement as in Theorem 3.2, without assumption (3.11) and for our original Ψ given by (3.8).

REFERENCES

[1] M. HAZEWINKEL AND J. C. WILLEMS, EDS., *Stochastic Systems: The Mathematics of Filtering and Identification and Applications*, Proc. NATO Advanced Study Institute, Les Arcs, France, Reidel, Dordrecht, The Netherlands, 1981.

[2] W. H. FLEMING AND L. G. GOROSTIZA, EDS., *Advances in Filtering and Optimal Stochastic Control*, Proc. IFIP-WG 7/1 Working Conference, Cocoyoc, Mexico, 1982, Lecture Notes in Control and Information Sci. 42, Springer-Verlag, Berlin, New York, 1982.

[3] M. METIVIER AND E. PARDOUX, EDS., *Stochastic Differential Systems: Filtering and Control*, Proc. IFIP-WG 7/1 Working Conference, Marseille-Luminy, France, 1984, Lecture Notes in Control and Information Sci. 69, Springer-Verlag, Berlin, New York, 1985.

[4] W. H. FLEMING AND E. PARDOUX, *Optimal control for partially observed diffusions*, SIAM J. Control Optim., 20 (1982), pp. 261–285.

[5] J. M. BISMUT, *Partially observed diffusions and their control*, SIAM J. Control Optim., 20 (1982), pp. 302–309.

[6] J. S. BARAS, *Optimal sensor scheduling in multiple sensor platforms*, in preparation.

[7] J. D. KATTAR, *A solution of the multi-weapon, multi-target assignment problem*, WP-26597, The MITRE Co., Bedford, MA, February 1986.

[8] G. KALLIANPUR, *Stochastic Filtering Theory*, Springer-Verlag, Berlin, New York, 1980.

[9] A. BENSOUSSAN AND J. L. LIONS, *Contrôle impulsionnel et inéquations quasi-variationelles*, Dunod, Paris, 1982.

[10] A. BENSOUSSAN, *Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions*, Stochastics, 9 (1983), pp. 169–222.

[11] E. PARDOUX, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics, 9 (1983).

[12] J. L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*, Vols. 1 and 2, Dunod, Paris, 1968; English translation, Springer-Verlag, Berlin, New York, 1972.

[13] J. S. BARAS AND A. BENSOUSSAN, *Optimal sensor scheduling in nonlinear filtering of diffusion processes II: computational methods*, in preparation.

[14] R. S. LIPSTER AND A. N. SHIRYAYEV, *Statistics of Random Processes* I: *General Theory*, Springer-Verlag, Berlin, New York, 1977.