Analysis of Beam Structures

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Overview

1 Types of Beam Structure

2 Connection to Mechanics

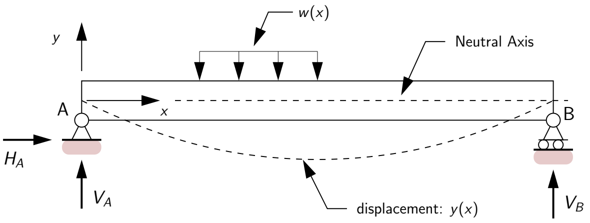
3 Relationship between Shear Force and Bending Moment
- Mathematical Preliminaries
- Derivation of Equations

4 Examples

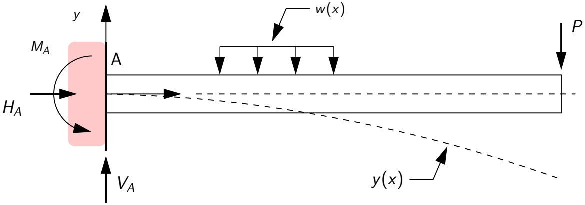
Types of Beam Structure

Types of Beam Structures

Simply Supported Beam:

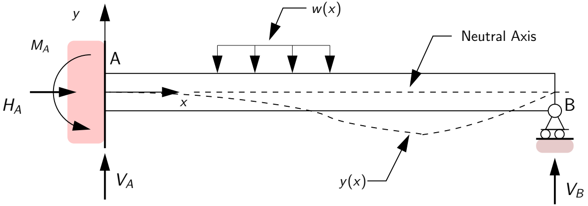


Cantilever:

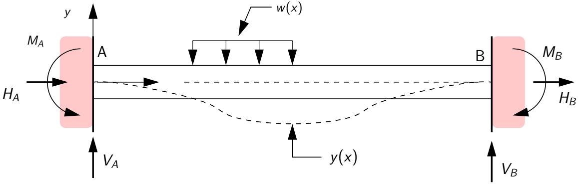


Types of Beam Structures

Supported Cantilever:



Fixed-Fixed Beam Structure:



Types of Beam Structures

Boundary Conditions

Simply Supported Beam

- $y(0) = y(L) = 0.$

Cantilever Beam

- $y(0) = 0, \frac{dy}{dx}|_{x=0} = 0$

Supported Cantilever Beam

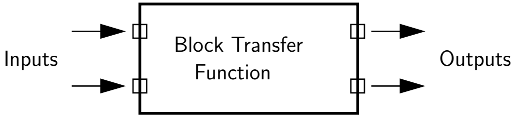
- $y(0) = y(L) = 0, \frac{dy}{dx}|_{x=0} = 0$

Fixed-Fixed Beam

- $y(0) = y(L) = 0, \frac{dy}{dx}|_{x=0} = \frac{dy}{dx}|_{x=L} = 0$

Basic Questions

Q1. What is the relationship between inputs and outputs?



Inputs

- Applied loads (P and w)
- Boundary conditions
- Beam geometry (L and I)
- Material Properties (E)

→ mechanics →

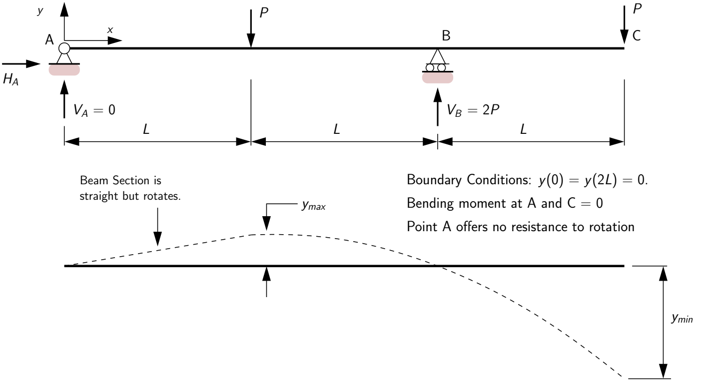
Outputs

- Shear force – $V(x)$
- Bending moment – $M(x)$
- Axial force – $N(x)$
- Displacement – $y(x)$
- Rotation – $\theta(x) = \left[\frac{dy}{dx}\right]$

Decisions will be based on estimates of outputs.

Basic Questions

Typical problem: Given input parameters, compute $y(x)$, find location and magnitude of y_{min} and y_{max}.



For simple problems, can rely on intuition. Otherwise, need math and mechanics.

Basic Questions

Q2. What is the relationship among the outputs? Are they dependent?

$$\begin{bmatrix} P \\ w(x) \end{bmatrix} \underset{\text{affected by}}{\overset{\text{affects}}{\rightleftarrows}} \begin{bmatrix} M(x) \\ V(x) \end{bmatrix} \underset{\text{affected by}}{\overset{\text{affects}}{\rightleftarrows}} \begin{bmatrix} y(x) \\ \theta(x) \end{bmatrix}$$

Applied loads + geometry. | Bending moments + shear forces. | Displacements + rotations.

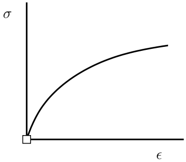
We will need to work with a chain of dependencies.

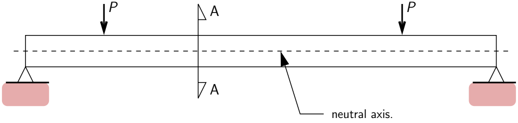
Q3. What is the relationship between $V(x)$ and $M(x)$? Are they independent? No!
We will see: $V(x) = \frac{dM(x)}{dx}$, but not always true!

Connection to Mechanics

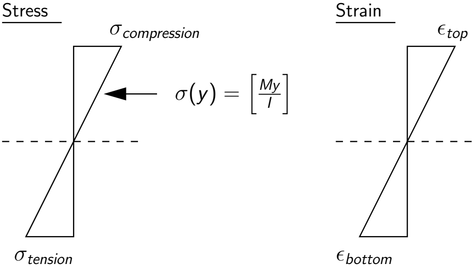
Connection to Mechanics

Problem Setup





Stress-Strain Relationships



Connection to Mechanics

For design purposes we need to make sure:

$$\sigma_{tension} < \sigma_{\text{max tension}} \tag{1}$$

and

$$\sigma_{compression} < \sigma_{\text{max compression}} \tag{2}$$

Also,

$$\epsilon_{\text{max compression}} \leq \epsilon(y) \leq \epsilon_{\text{max tension}} \tag{3}$$

These constraints limit the amount of load that a beam can carry.

Connection to Mechanics

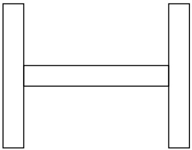
Section-Level Behavior

From a design standpoint we can reduce $\sigma(y)$ and $\epsilon(y)$ by increasing the moment of inertia in

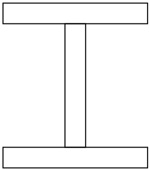
$$\sigma(y) = \left[\frac{My}{I} \right]. \tag{4}$$

To maximise I, maximize distance of material from neutral axis.

Poor Choice of Inertia



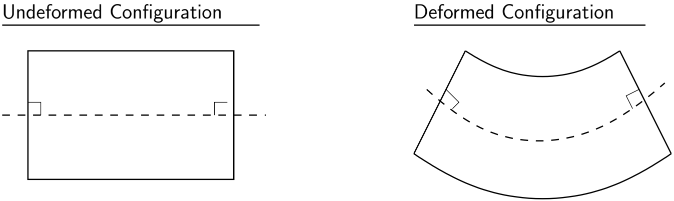
Good Choice of Inertia



Assumptions on Beam Displacements

Assumptions. We will assume beam length / depth $\gg 10$.

Therefore, displacements will be dominated by flexural bending.



Sections remain perpendicular to the deformed neutral axis.

This is not the case for shear deformations.

Relationship between Shear Force and Bending Moment

Relationship between Shear Force and Bending Moment

Basic Questions

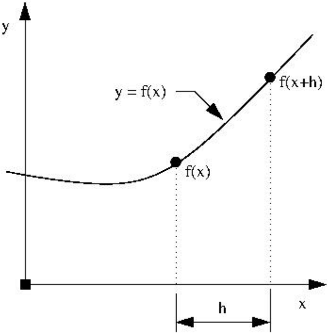
- Are V(x) and M(x) independent? No!
- Under what conditions does a dependency relationship exist?

Strategy

- Introduce relavant mathematics.
- Extract a thin section from a beam and examine its equilibrium.
- See where the mechanics takes us!

Mathematical Preliminaries

Taylor Series Expansion. Let $y = f(x)$ be a smooth differentiable function.



Given $f(x)$ and derivatives $f^{'}(a)$, $f^{''}(a)$, $f^{'''}(a)$, etc, the purpose of Taylor's series is to estimate $f(x+h)$ at some distance h from x.

Mathematical Preliminaries

The Taylor series is as follows:

$$f(x+h) = \sum_{k=0}^{\infty} \frac{f^k(x)}{k!} h^k = f(x) + f^{'}(x)h + \frac{f^{''}(x)}{2!} h^2 + \frac{f^{'''}(x)}{3!} h^3 + \cdots \quad (5)$$

For a Taylor series approximation containing $(n+1)$ terms

$$f(x+h) = \sum_{k=0}^{k=n} \frac{f^k(x)}{k!} h^n + O(h^{(n+1)}) \quad (6)$$

The big-O notation indicates how quickly the error will change as a function of h, e.g., $O(h^2) \rightarrow$ magnitude of error proportional to h squared.

Mathematical Preliminaries

Finite Difference Derivatives. Truncating equation 6 after two terms gives:

$$f(x+h) = f(x) + f^{'}(x)h + O(h^2). \tag{7}$$

A simple rearrangement of equation 7 gives:

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]. \tag{8}$$

Similarly, we require:

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x) - f(x-h)}{h} \right]. \tag{9}$$

In order for the derivative to exist, equations 8 and 9 need to be the same!

Mathematical Preliminaries

Simple Example. Let y $= x^2$.

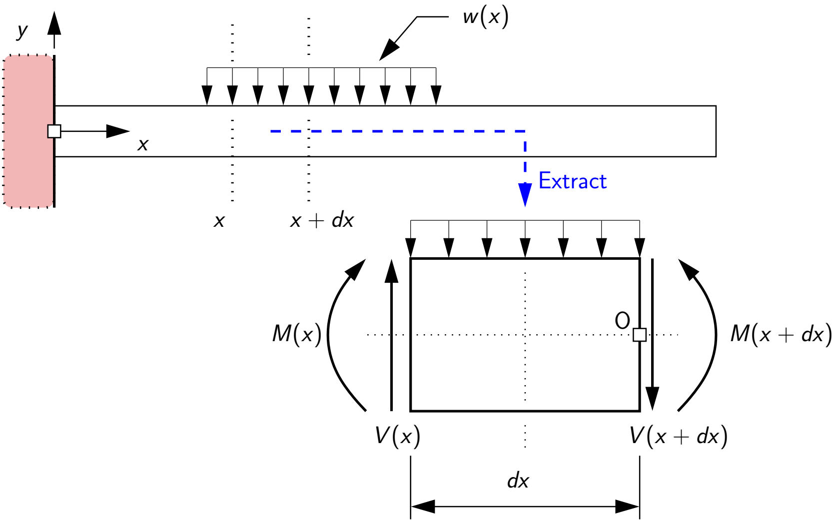
$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} \right] = \lim_{h \to 0} [2x + h] = 2x. \qquad (10)$$

Home Exercise. Use first principles to find dy/dx when:

$$y(x) = \left(x^2 - 4x + 3\right)^2 \qquad (11)$$

Counter Example. $y(x) = |x|$ is not differentiable at x = 0.

Test Problem for Derivation of Equations



Derivation of Equations

Part 1: Equilibrium in Vertical Direction:

$$\sum F_y = 0 \;\rightarrow\; V(x) - V(x+dx) - w(x)dx = 0 \tag{12}$$

From the Taylors series expansion:

$$V(x+dx) = V(x) + \frac{dV}{dx}dx + O(dx^2) \tag{13}$$

Plugging equation 13 into 12 and ignoring higher-order terms:

$$\sum F_y = 0 \;\rightarrow\; V(x) - \left[V(x) + \frac{dV}{dx}dx\right] - w(x)dx = 0 \tag{14}$$

Derivation of Equations

Hence,

$$\frac{dV}{dx} + w(x) = 0 \leftarrow \text{gradient of shear force equals -w(x).} \qquad (15)$$

Part 2: $\sum M_o = 0$ (anticlockwise +ve)

$$-V(x)dx - M(x) + M(x + dx) + w(x)dx \cdot \frac{dx}{2} = 0 \qquad (16)$$

Note:

- The term $w(x)dx$ is the vertical load acting on the element.
- The term $dx/2$ is the distance from O to the centroid of loading.

Derivation of Equations

From the Taylor Series expansion:

$$M(x + dx) = M(x) + \frac{dM}{dx}dx + O(dx^2) \tag{17}$$

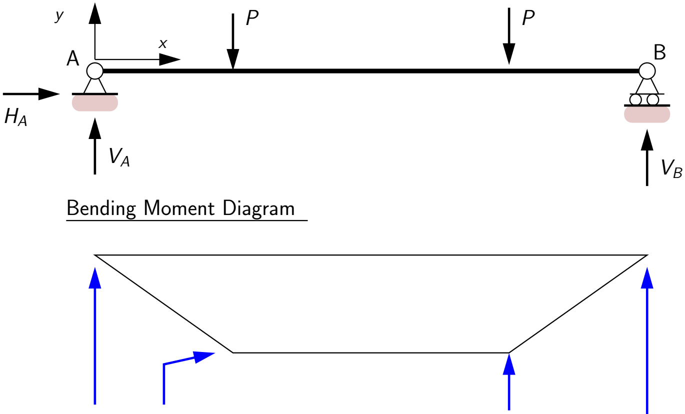
Plugging equation 17 into 16 and ignoring terms $O(dx^2)$ and higher:

$$V(x) = \frac{dM}{dx} \leftarrow \text{shear force} = \text{gradient of bending moment.} \tag{18}$$

Note. Equation 18 only applies when the derivatives of M(x) with respect to x exist.

Derivation of Equations

Illustrative Example



$V(x) = \frac{dM}{dx}$ does not apply at four points....

Shear Force and Bending Moment

Interpretation. Consider an interval $[a, b]$ on a beam:

$$dV = -w(x)dx \rightarrow \int_a^b dV = -\int_a^b w(x)dx = V(b) - V(a). \quad (19)$$

Key Point: Change in shear force between points a and b = total loading within interval.

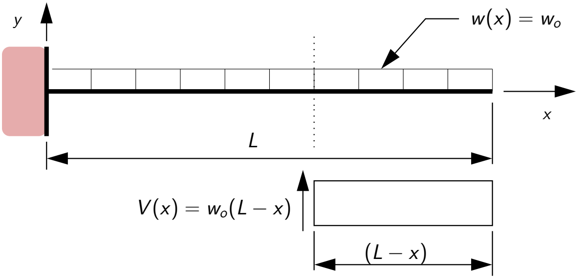
$$dM = V(x)dx \rightarrow \int_a^b dM = \int_a^b V(x)dx = M(b) - M(a). \quad (20)$$

Key Point: Change in moment between points a and b = area under the shear force diagram.

Examples

Shear Force and Bending Moment

Example 1.



Check Shear Loading ($a = 0,\ b = L$):

$$V(b) - V(a) = V(L) - V(o) = -wL = -\int_0^L w_o dx. \checkmark \qquad (21)$$

Shear Force and Bending Moment

Check Relationship between Shear and Bending Moment:

$$V(x) = \frac{dM(x)}{dx} = w_o(L - x). \tag{22}$$

For $a = 0$ and $b = L$ we expect:

$$\int_0^L V(x)dx = w_o \int_0^L () \, dx = M(L) - M(0). \tag{23}$$

For a general value x:

$$M(x) = w_o \int_x^L (L - s)ds = w_o Lx - \frac{1}{2} w_o x^2 + A. \tag{24}$$

Shear Force and Bending Moment

Apply Boundary Conditions:

$$M(L) = 0 \rightarrow A = -\frac{1}{2}wL^2. \qquad (25)$$

Hence,

$$M(x) = wLx - \frac{1}{2}wx^2 - \frac{1}{2}wL^2 = -\frac{1}{2}w(L-x)^2. \qquad (26)$$

Check Moment at Boundary Conditions:

- $M(L) = wL^2 - \frac{1}{2}2wL^2 = 0.$ ✓
- $M(0) = -\frac{1}{2}wL^2.$ ✓

Shear Force and Bending Moment

Physical Interpretation

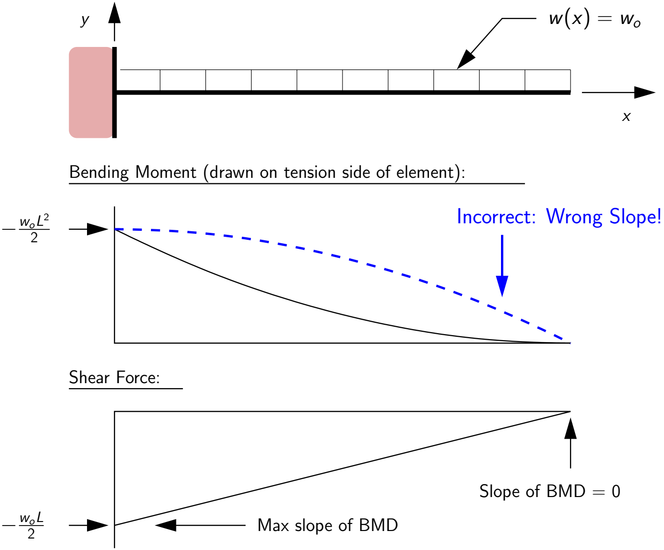
For the extracted element:

$$\sum F_y(x) = 0 \rightarrow V(x) = w_o\,(L - x)\,. \tag{27}$$

Similarly,

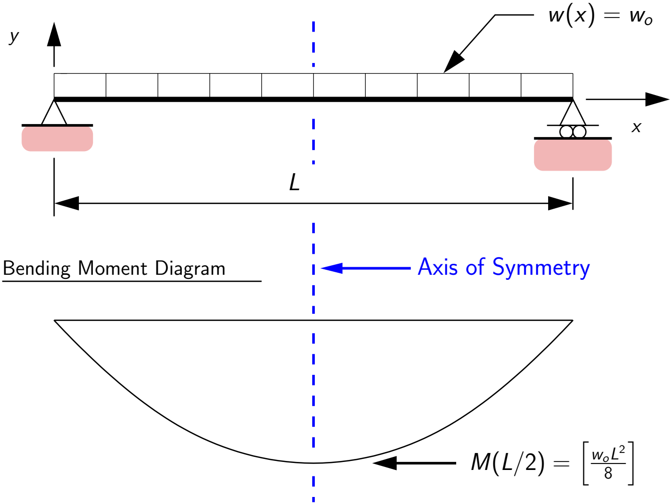
$$\sum M_z(x) = 0 \rightarrow M(x) = \underbrace{w_o\,(L - x)}_{\text{total load}} \cdot \underbrace{\frac{(L - x)}{2}}_{\text{centroid}} \tag{28}$$

Shear Force and Bending Moment Diagrams



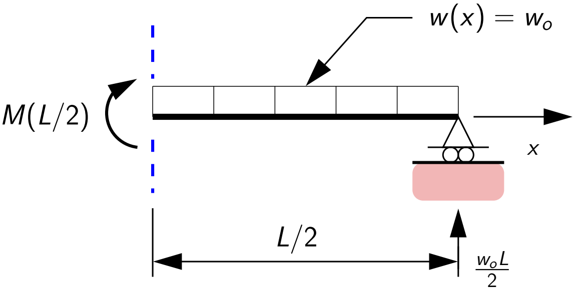
Shear Force and Bending Moment

Example 2.



Shear Force and Bending Moment

Bending Moment at x = L/2 (extract substructure):



Taking moments:

$$M(L/2) = \underbrace{\frac{w_o L}{2}}_{reaction} \frac{L}{2} - \underbrace{\frac{w_o L}{2}}_{loading} \underbrace{\frac{L}{4}}_{centroid} = \frac{w_o L^2}{8}. \quad (29)$$

Shear Force and Bending Moment

Equation for M(x)?

We have:

- Axis of symmetry at $x = L/2$.
- M(x) will have roots at $x = 0$ and $x = L$.

Hence, let $M(x) = Ax\left(x - L\right)$, then use midpoint moment to determine A:

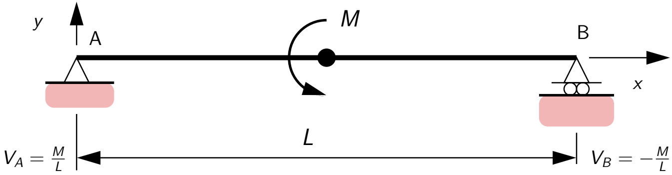
$$M(L/2) = A\frac{L}{2}\left(\frac{-L}{2}\right) - > A = -\frac{w_o}{2}. \tag{30}$$

Thus,

$$M(x) = \frac{w_o}{2}x\left(L - x\right). \tag{31}$$

Shear Force and Bending Moment

Example 3.



Bending Moment Diagram