Analysis of Beam Structures

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ENCE 353, Fall Semester 2020

September 3, 2020

Overview

Part 3

Relationship between Shear Force and Bending Moment

Relationship between Shear Force and Bending Moment

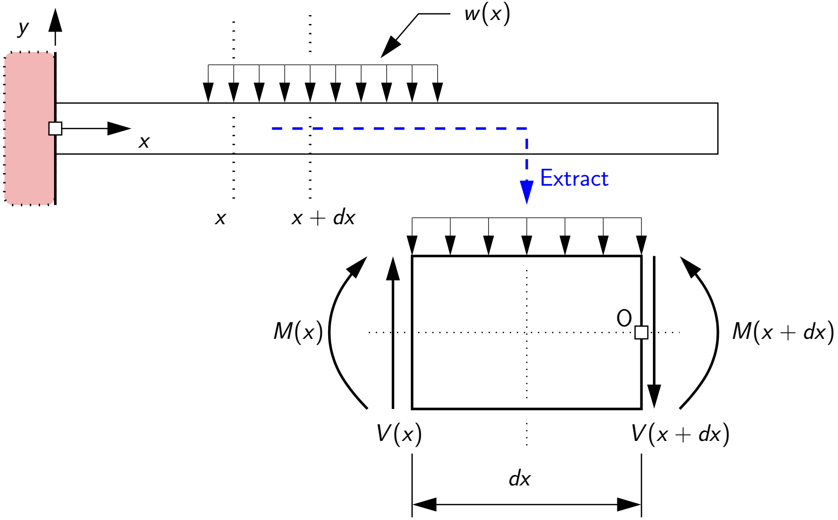
Basic Questions

- Are V(x) and M(x) independent? No!
- Under what conditions does a dependency relationship exist?

Strategy

- Introduce relavant mathematics.
- Extract a thin section from a beam and examine its equilibrium.
- See where the mechanics takes us!

Test Problem for Derivation of Equations



Derivation of Equations

Hence,

$$\frac{dV}{dx} + w(x) = 0 \leftarrow \text{gradient of shear force equals -w(x).} \qquad (15)$$

Part 2: $\sum M_o = 0$ (anticlockwise +ve)

$$-V(x)dx - M(x) + M(x + dx) + w(x)dx \cdot \frac{dx}{2} = 0 \qquad (16)$$

Note:

- The term $w(x)dx$ is the vertical load acting on the element.
- The term $dx/2$ is the distance from O to the centroid of loading.

Derivation of Equations

From the Taylor Series expansion:

$$M(x + dx) = M(x) + \frac{dM}{dx}dx + O(dx^2) \tag{17}$$

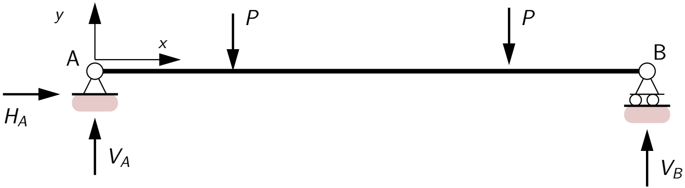
Plugging equation 17 into 16 and ignoring terms $O(dx^2)$ and higher:

$$V(x) = \frac{dM}{dx} \leftarrow \text{shear force} = \text{gradient of bending moment.} \tag{18}$$

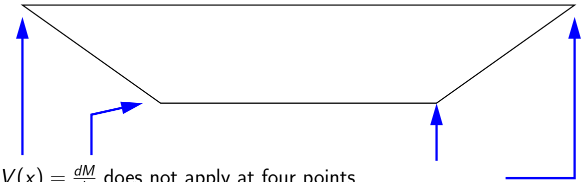
Note. Equation 18 only applies when the derivatives of M(x) with respect to x exist.

Derivation of Equations

Illustrative Example



Bending Moment Diagram



$V(x) = \frac{dM}{dx}$ does not apply at four points....

Shear Force and Bending Moment

Interpretation. Consider an interval $[a, b]$ on a beam:

$$dV = -w(x)dx \to \int_a^b dV = -\int_a^b w(x)dx = V(b) - V(a). \quad (19)$$

Key Point: Change in shear force between points a and b = total loading within interval.

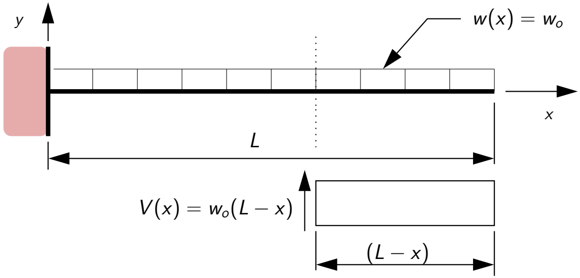
$$dM = V(x)dx \to \int_a^b dM = \int_a^b V(x)dx = M(b) - M(a). \quad (20)$$

Key Point: Change in moment between points a and b = area under the shear force diagram.

Examples

Shear Force and Bending Moment

Example 1.



Check Shear Loading ($a = 0,\ b = L$):

$$V(b) - V(a) = V(L) - V(o) = -wL = -\int_0^L w_o dx. \checkmark \qquad (21)$$

Shear Force and Bending Moment

Check Relationship between Shear and Bending Moment:

$$V(x) = \frac{dM(x)}{dx} = w_o(L - x). \qquad (22)$$

For $a = 0$ and $b = L$ we expect:

$$\int_0^L V(x)dx = w_o \int_0^L ()\, dx = M(L) - M(0). \qquad (23)$$

For a general value x:

$$M(x) = w_o \int_x^L (L - s)ds = w_o Lx - \frac{1}{2} w_o x^2 + A. \qquad (24)$$

Shear Force and Bending Moment

Apply Boundary Conditions:

$$M(L) = 0 \rightarrow A = -\frac{1}{2}wL^2. \tag{25}$$

Hence,

$$M(x) = wLx - \frac{1}{2}wx^2 - \frac{1}{2}wL^2 = -\frac{1}{2}w(L-x)^2. \tag{26}$$

Check Moment at Boundary Conditions:

- $M(L) = wL^2 - \frac{1}{2}2wL^2 = 0.$ ✓
- $M(0) = -\frac{1}{2}wL^2.$ ✓

Shear Force and Bending Moment

Physical Interpretation

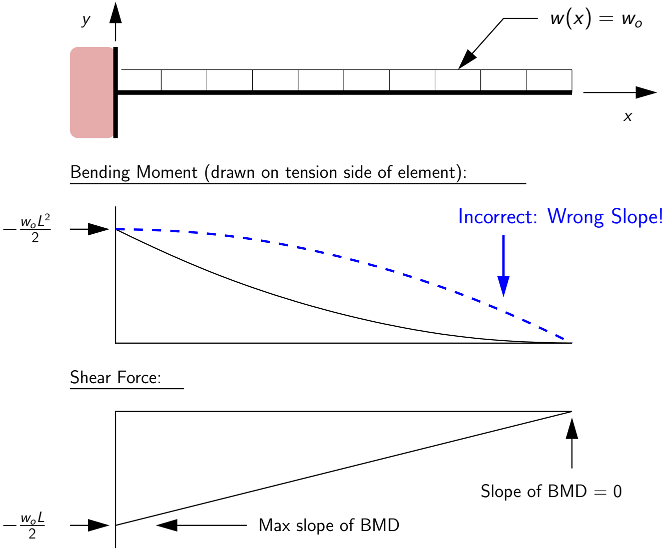
For the extracted element:

$$\sum F_y(x) = 0 \rightarrow V(x) = w_o\,(L - x)\,. \tag{27}$$

Similarly,

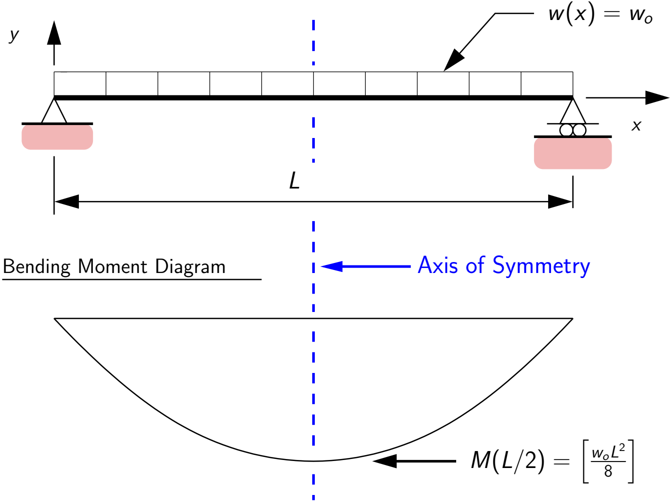
$$\sum M_z(x) = 0 \rightarrow M(x) = \underbrace{w_o\,(L - x)}_{\text{total load}} \cdot \underbrace{\frac{(L - x)}{2}}_{\text{centroid}} \tag{28}$$

Shear Force and Bending Moment Diagrams



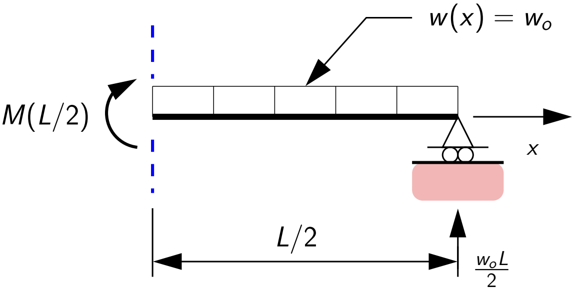
Shear Force and Bending Moment

Example 2.



Shear Force and Bending Moment

Bending Moment at x = L/2 (extract substructure):



Taking moments:

$$M(L/2) = \underbrace{\frac{w_o L}{2}}_{reaction} \frac{L}{2} - \underbrace{\frac{w_o L}{2}}_{loading} \underbrace{\frac{L}{4}}_{centroid} = \frac{w_o L^2}{8}. \quad (29)$$

Shear Force and Bending Moment

Equation for M(x)?

We have:

- Axis of symmetry at x = L/2.
- M(x) will have roots at x = 0 and x = L.

Hence, let $M(x) = Ax\,(x - L)$, then use midpoint moment to determine A:

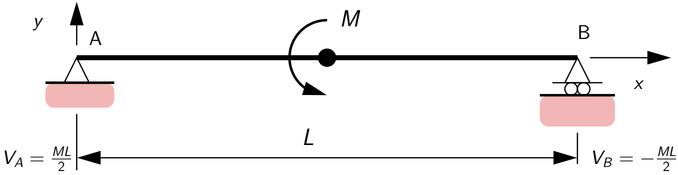
$$M(L/2) = A\frac{L}{2}\left(\frac{-L}{2}\right) - > A = -\frac{w_o}{2}. \tag{30}$$

Thus,

$$M(x) = \frac{w_o}{2}x\,(L - x)\,. \tag{31}$$

Shear Force and Bending Moment

Example 3.



Bending Moment Diagram