Analysis of Beam Structures

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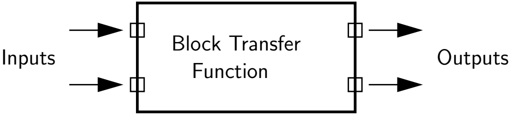
Overview

Types of Beam Structure

Basic Questions

Q1. What is the relationship between inputs and outputs?



Inputs

- Applied loads (P and w)
- Boundary conditions
- Beam geometry (L and I)
- Material Properties (E)

→ mechanics →

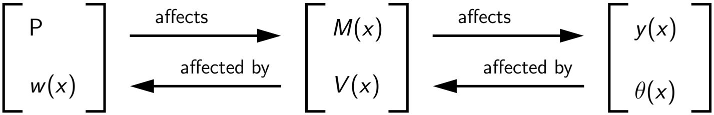
Outputs

- Shear force – $V(x)$
- Bending moment – $M(x)$
- Axial force – $N(x)$
- Displacement – $y(x)$
- Rotation – $\theta(x) = \left[\frac{dy}{dx}\right]$

Decisions will be based on estimates of outputs.

Basic Questions

Q2. What is the relationship among the outputs? Are they dependent?



Applied loads + geometry.

Bending moments + shear forces.

Displacements + rotations.

We will need to work with a chain of dependencies.

Q3. What is the relationship between $V(x)$ and $M(x)$? Are they independent? No!
We will see: $V(x) = \frac{dM(x)}{dx}$, but not always true!

Relationship between Shear Force and Bending Moment

Relationship between Shear Force and Bending Moment

Basic Questions

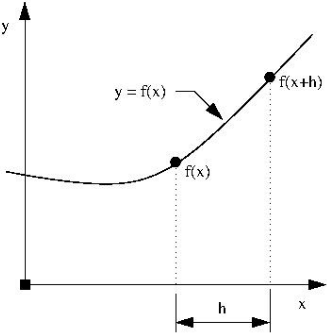
- Are V(x) and M(x) independent? No!
- Under what conditions does a dependency relationship exist?

Strategy

- Introduce relavant mathematics.
- Extract a thin section from a beam and examine its equilibrium.
- See where the mechanics takes us!

Mathematical Preliminaries

Taylor Series Expansion. Let $y = f(x)$ be a smooth differentiable function.



Given $f(x)$ and derivatives $f^{'}(a)$, $f^{''}(a)$, $f^{'''}(a)$, etc, the purpose of Taylor's series is to estimate $f(x+h)$ at some distance h from x.

Mathematical Preliminaries

The Taylor series is as follows:

$$f(x+h)=\sum_{k=0}^{\infty}\frac{f^{k}(x)}{k!}h^{k}=f(x)+f^{'}(x)h+\frac{f^{''}(x)}{2!}h^{2}+\frac{f^{'''}(x)}{3!}h^{3}+\cdots \tag{5}$$

For a Taylor series approximation containing $(n+1)$ terms

$$f(x+h)=\sum_{k=0}^{k=n}\frac{f^{k}(x)}{k!}h^{n}+O(h^{(n+1)}) \tag{6}$$

The big-O notation indicates how quickly the error will change as a function of h, e.g., $O(h^{2})\rightarrow$ nagnitude of error proportional to h squared.

Mathematical Preliminaries

Finite Difference Derivatives. Truncating equation 6 after two terms gives:

$$f(x+h) = f(x) + f'(x)h + O(h^2). \tag{7}$$

A simple rearrangement of equation 7 gives:

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]. \tag{8}$$

Similarly, we require:

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x) - f(x-h)}{h} \right]. \tag{9}$$

In order for the derivative to exist, equations 8 and 9 need to be the same!

Mathematical Preliminaries

Simple Example. Let y $= x^2$.

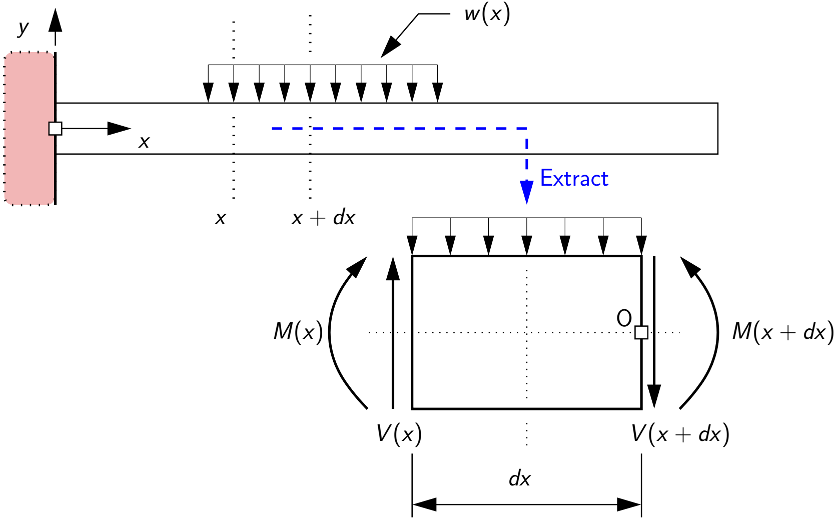
$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} \right] = \lim_{h \to 0} [2x + h] = 2x. \qquad (10)$$

Home Exercise. Use first principles to find dy/dx when:

$$y(x) = \left(x^2 - 4x + 3\right)^2 \qquad (11)$$

Counter Example. $y(x) = |x|$ is not differentiable at x = 0.

Test Problem for Derivation of Equations



Derivation of Equations

Part 1: Equilibrium in Vertical Direction:

$$\sum F_y = 0 \;\rightarrow\; V(x) - V(x+dx) - w(x)dx = 0 \tag{12}$$

From the Taylors series expansion:

$$V(x+dx) = V(x) + \frac{dV}{dx}dx + O(dx^2) \tag{13}$$

Plugging equation 13 into 12 and ignoring higher-order terms:

$$\sum F_y = 0 \;\rightarrow\; V(x) - \left[V(x) + \frac{dV}{dx}dx\right] - w(x)dx = 0 \tag{14}$$

Derivation of Equations

Hence,

$$\frac{dV}{dx} + w(x) = 0 \leftarrow \text{gradient of shear force equals -w(x).} \qquad (15)$$

Part 2: $\sum M_o = 0$ (anticlockwise +ve)

$$-V(x)dx - M(x) + M(x + dx) + w(x)dx \cdot \frac{dx}{2} = 0 \qquad (16)$$

Note:

- The term $w(x)dx$ is the vertical load acting on the element.
- The term $dx/2$ is the distance from O to the centroid of loading.

Derivation of Equations

From the Taylor Series expansion:

$$M(x+dx) = M(x) + \frac{dM}{dx}dx + O(dx^2) \tag{17}$$

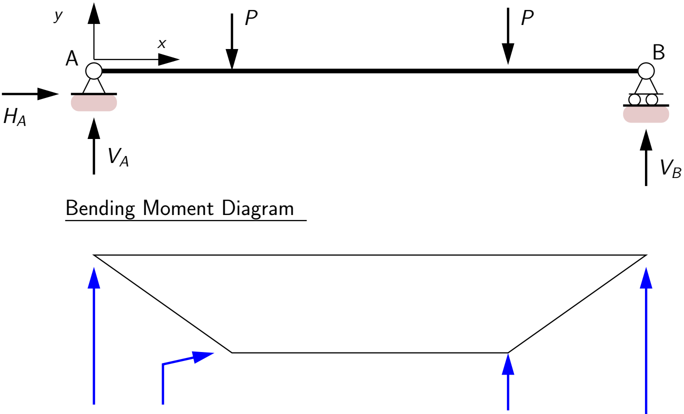
Plugging equation 17 into 16 and ignoring terms $O(dx^2)$ and higher:

$$V(x) = \frac{dM}{dx} \leftarrow \text{shear force} = \text{gradient of bending moment.} \tag{18}$$

Note. Equation 18 only applies when the derivatives of M(x) with respect to x exist.

Derivation of Equations

Illustrative Example



$V(x) = \frac{dM}{dx}$ does not apply at four points....

Shear Force and Bending Moment

Interpretation. Consider an interval $[a, b]$ on a beam:

$$dV = -w(x)dx \rightarrow \int_a^b dV = -\int_a^b w(x)dx = V(b) - V(a). \quad (19)$$

Key Point: Change in shear force between points a and b = total loading within interval.

$$dM = V(x)dx \rightarrow \int_a^b dM = \int_a^b V(x)dx = M(b) - M(a). \quad (20)$$

Key Point: Change in moment between points a and b = area under the shear force diagram.