(A)

Moment-Area Method.

Quick Review: General equation for elastic behavior of a beam.

$$\frac{d}{dx^2}\left[EI \frac{d^2 y}{dx^2} \right] + w(x) = 0 \quad \text{— (A)}$$

$$\frac{d^2 y}{dx^2} = \phi(x) = \frac{M(x)}{EI} \quad \text{— (B)}$$

We can integrate equations (A) & (B) to get $y(x)$ (displacements) & dy/dx (Slopes).

Good news: The method works — $y(x)$ describes desplacement at all points x.

Bad news: Math can be tedious! Time consuming!

Motivation: Often, whole elastic curve $y(x)$ is not required. We only want to know displacements at specific points.

Moment-Area Method.

The moment-area method is based on two theorems.

Theorem 1: The change in angle $\theta_A - \theta_B$ between two points on an elastic curve is equal to the area under M/EI diagram.

(B)

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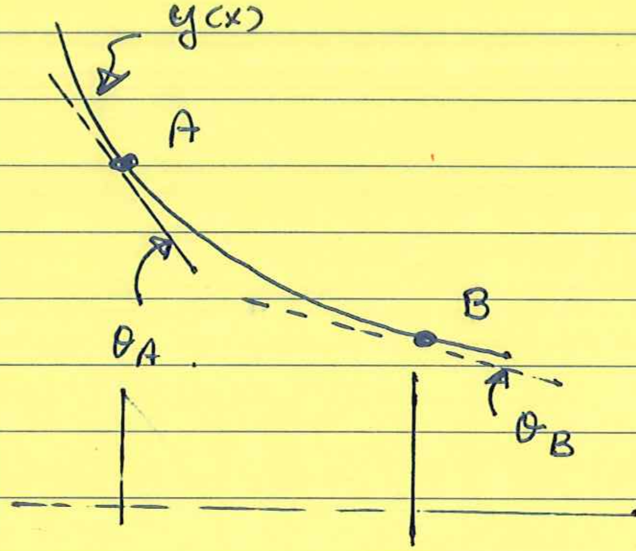


Simple Proof.

$$\frac{d^2y}{dx^2} = \phi(x) = \frac{M(x)}{EI} \quad \text{-(C)}$$

curvature of elastic curve.

Integrating (C) gives.

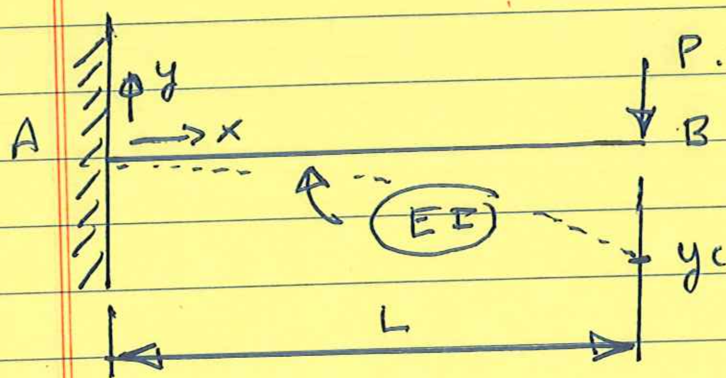
$$\int_A^B \frac{d^2y}{dx^2}\, dx = \int_A^B \frac{M(x)}{EI}\, dx.$$

$$\Rightarrow \underbrace{\left[\frac{dy}{dx}\right]_A^B = \theta_B - \theta_A}_{\text{change in slope / rotation.}} = \underbrace{\int_A^B \frac{M(x)}{EI}\, dx}_{\text{area under } M/EI \text{ diagram}}$$

Theorem II: The distance of A to the tangent at B measured perpendicular (⊥) to the original beam axis equals the first moment of area between A & B evaluated about A.

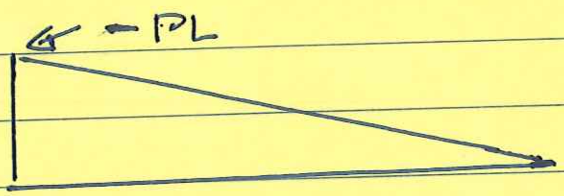
(c) Let's do a few examples to see how the theorems work in practice.

Example 1: Cantilever beam.



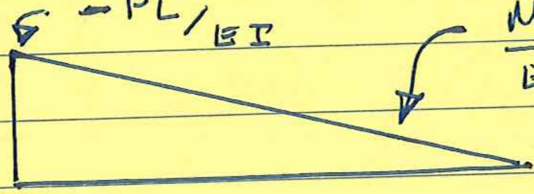
$y(L), \left.\frac{dy}{dx}\right|_{x=L} = \theta(L).$

BMD



drawing bending moment on tension side of the beam.

$\frac{M}{EI}$



$$\frac{M(x)}{EI} = \frac{-P(L-x)}{EI}$$

From Theorem I. $\theta_B - \theta_A = \int_A^B \frac{M(x)}{EI}\,dx.$

$$= \int_0^L \frac{-P(L-x)}{EI}\,dx$$

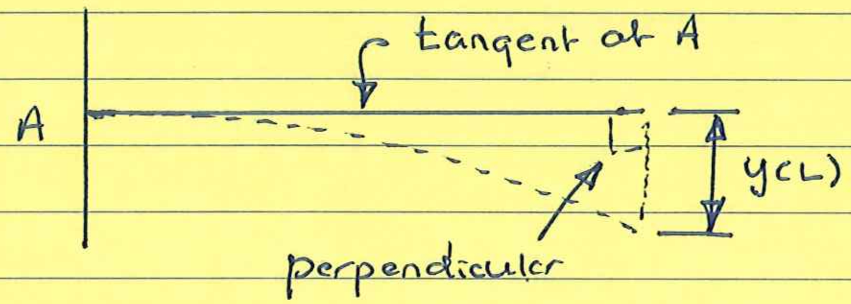
$$= \frac{-PL^2}{2EI}$$

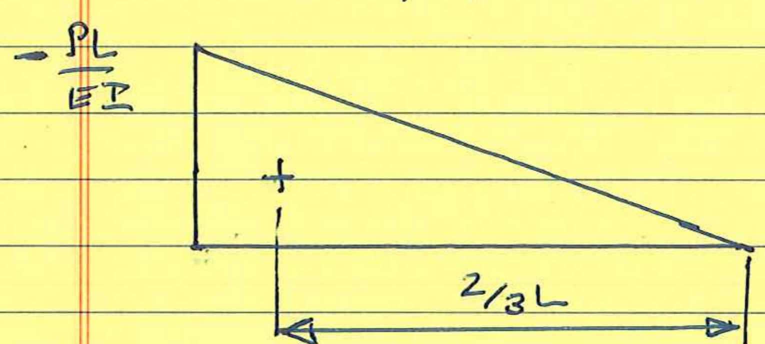
Note $\theta_A = 0$ (cantilever is ~~ffetty~~ fully fixed at A).

(D)

$$\Rightarrow \quad \theta_B = \frac{-PL^2}{2EI}$$ ← clockwise rotation of B.

From Theorem II.





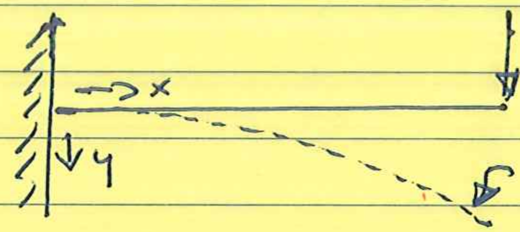
$$y(L) = \int_0^L (L-x)\left[\frac{-P(L-x)}{EI}\right] dx.$$

$$= \left(\frac{2}{3}L\right)\left(\frac{-PL^2}{2EI}\right).$$

$$y(L) = \frac{-PL^3}{3EI}$$ ← Tip deflection of cantilever.

E

Example II. (From Midterm II, 2017).



$$y(x) = \frac{P(x^2)(3L-x)}{6EI} \quad \text{— (D)}$$

From our studies of elastic curve.

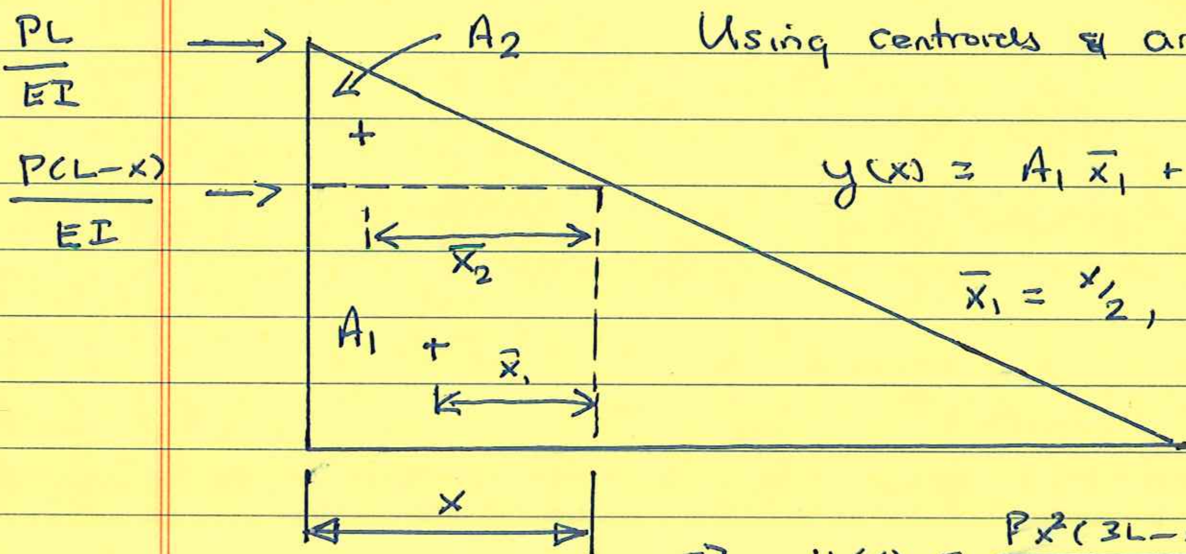
integrate | apply boundary conditions.

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = \frac{P(L-x)}{EI}$$

Now lets use Moment-area to derive equation (D)

Note: This is tricky - we need to compute the first moment of area of M/EI between O and a generic point x, evaluated about x.

The M/EI diagram is as follows.



Using centroids & areas

$$y(x) = A_1\bar{x}_1 + A_2\bar{x}_2$$

$$\bar{x}_1 = x/2, \quad \bar{x}_2 = 2/3\,x,$$

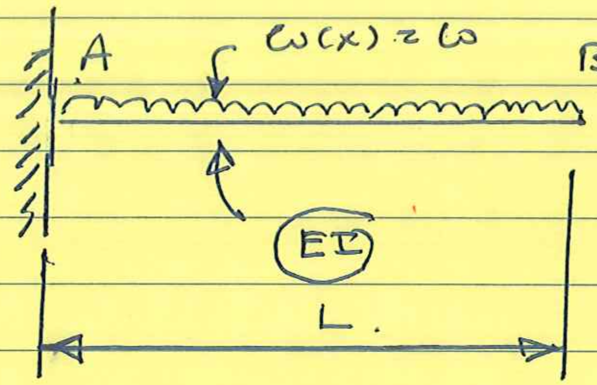
$$A_1 = \frac{P(L-x)x}{EI}$$

$$A_2 = Px^2/2EI$$

$$\Rightarrow \quad y(x) = \frac{Px^2(3L-x)}{6EI}$$

(F)

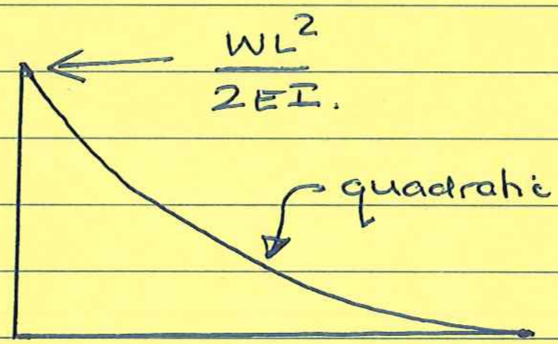
Example 3: Cantilever carrying uniform load.



What are θ_B and Δ_B.

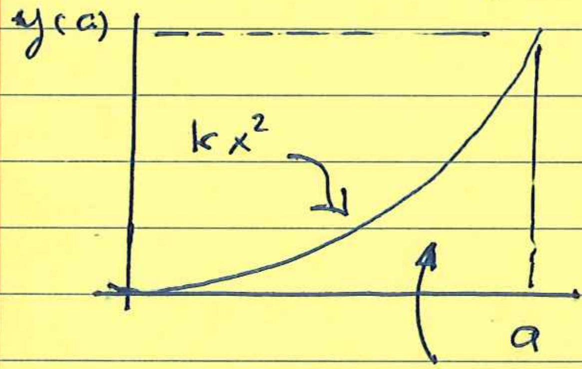
θ_B = area under M/EI diagram

M/EI



Δ_B = first moment of area of M/EI evaluated about B.

Aside: Consider geometry of a curve $y = kx^2$.

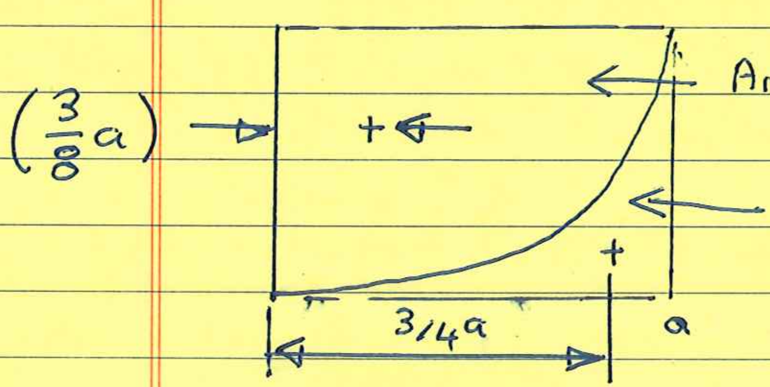


Area under curve A_1

$$A_1 = \int_0^a kx^2 \, dx = \frac{ka^3}{3}.$$

Area under curve = $\frac{1}{3}$ of area of rectangle

Centroids



(G)

Back to the cantilever

$$\theta_B = \frac{1}{3}\underbrace{\frac{wL^2}{2EI}\cdot L}_{\text{area of rectangle}} = \frac{wL^3}{6EI}.$$

$$\Delta_B = \left(\begin{array}{c}\text{area under}\\ \text{quadratic}\end{array}\right) \times \left(\begin{array}{c}\text{distance to}\\ \text{centroid}\end{array}\right)$$

$$= \left(\frac{wL^3}{6EI}\right)\times\left(\frac{3}{4}L\right) = \frac{wL^4}{8EI}.$$

Summary.