(A)

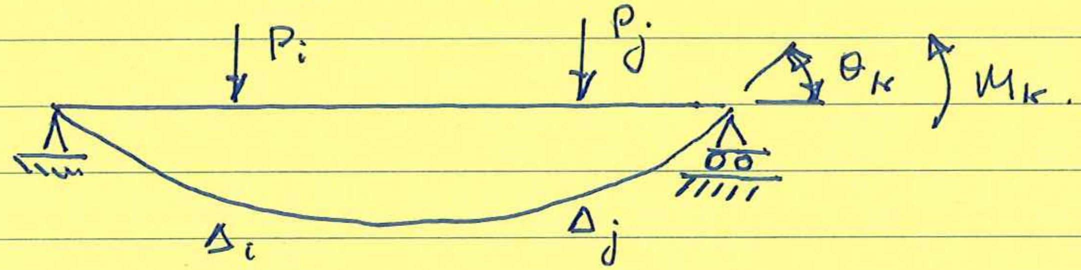
# Flexibility Method.

**Goal:** Compute flexibility coefficients & flexibity matrices.

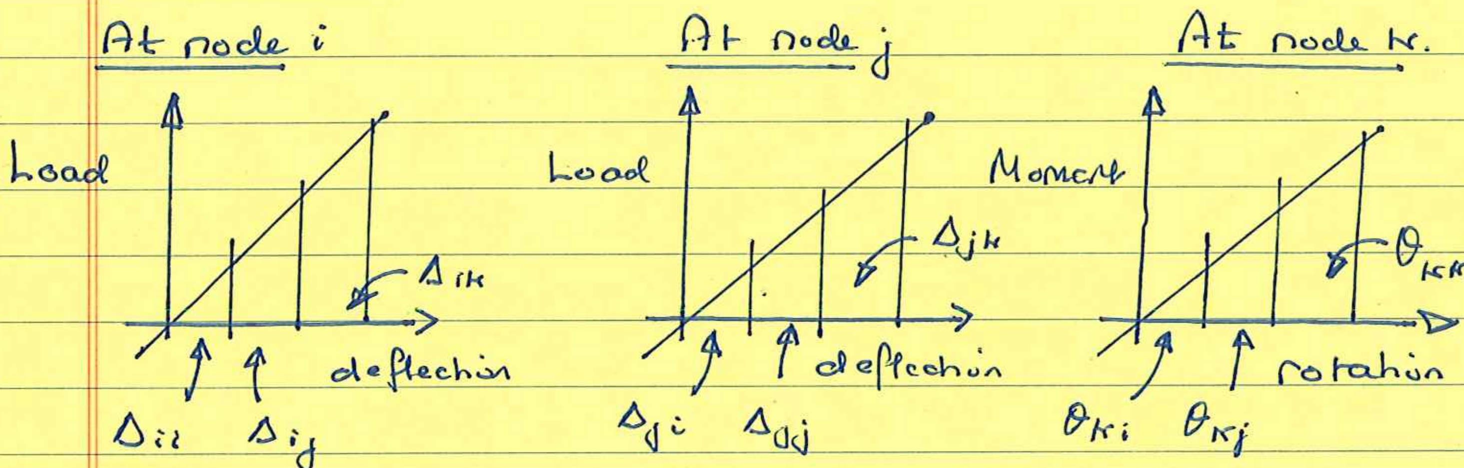
**Note:** In a linear elastic structure:

(1) direct force $\propto$ deflection
(2) bending moment $\propto$ rotation.
(3) principal of superposition applies.

Consider the following test structure:




Load $P_i$, $P_j$ & $M_k$ are applied at points $i, j$ & $k$.




Apply loads in the following order: $P_i$, $P_j$ & $M_k$.

(B)

We have:

$\Delta_{ii}$ = deflection at node $i$ due to load at $i$

$\Delta_{ij}$ = " " " " " " " " $j$

$\theta_{ki}$ = rotation at $k$ due to an applied load at $i$.

Principle of Superposition holds:

$$\left.\begin{aligned}\Delta_i &= \Delta_{ii} + \Delta_{ij} + \Delta_{ik}\\ \Delta_j &= \Delta_{ji} + \Delta_{jj} + \Delta_{jk}\\ \theta_k &= \theta_{ki} + \theta_{kj} + \theta_{kk}.\end{aligned}\right\} \quad — \text{(A)}$$

Now let.

$\Delta_{ij} = f_{ij}.P_j$ where $f_{ij}$ = deflection at node $i$ due to a unit load at $j$

& $\theta_{ki} = f_{ki}\,P_i$ where $f_{ki}$ = rotation at $k$ due to unit load at $i$.

We can write equations (A) in matrix form:

$$\begin{bmatrix}\Delta_i\\ \Delta_j\\ \theta_k\end{bmatrix} = \begin{bmatrix} f_{ii} & f_{ij} & f_{ik}\\ f_{ji} & f_{jj} & f_{jk}\\ f_{ki} & f_{kj} & f_{kk}\end{bmatrix} \cdot \begin{bmatrix}P_i\\ P_j\\ M_k\end{bmatrix} \quad — \text{(B)}$$

OR $\{\Delta\} = \{F\}[P\}$.

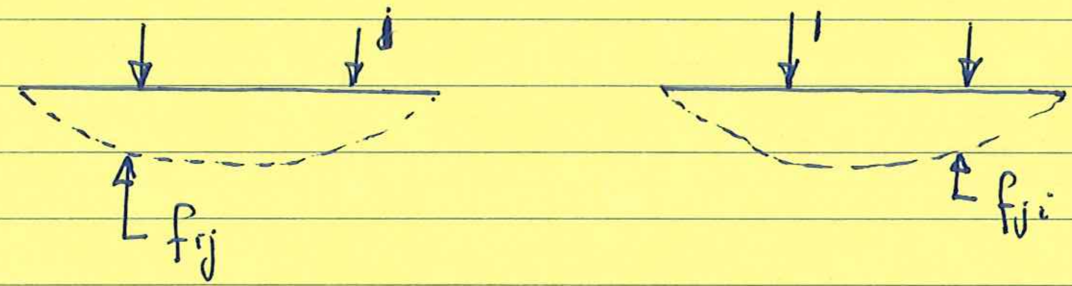
$\Delta$ = displacement column vector

(C)

$\{f\}$ = flexibility matrix
$\{P\}$ = force matrix.

## Maxwell's Reciprocal Theorem (1870).

- In a linear elastic structure, the deflection at node $i$ due to an applied load a $j$ (unit load) is equal to the deflection at $j$ due to a unit load at $i$.




$\Rightarrow f_{ij} = f_{ji} \Rightarrow$ flexibility matrix is symmetric about leading diagonal.

$\Rightarrow$ Can use moment area, or virtual forces to comput flexibility coefficients in equations (B)

$$f_{ij} = \int_0^L M_i \left( \frac{M_j}{EI} \right) dx$$ } principal of virtual forces.

Where

$L$ = length of member $i$.

$M_i$ = bending moment expression due to unit force applied at direction of required deflection (rotation)

$M_j/EI$ = curvature expression due to unit force $j$.

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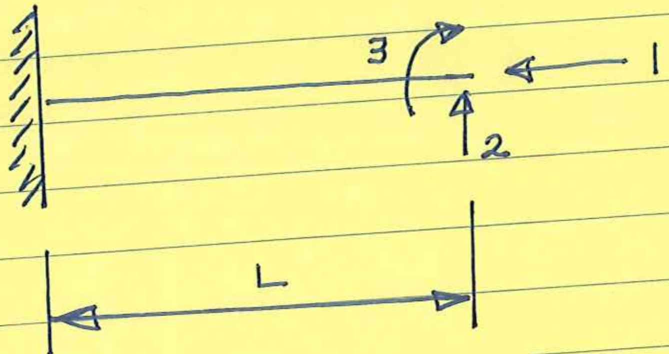
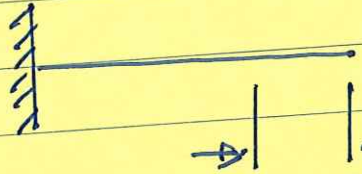
**Example 1.** Compute flexibility matrix for a uniform cantilever having 3 point directions.




The flexibility matrix is:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ M_3 \end{bmatrix}$$

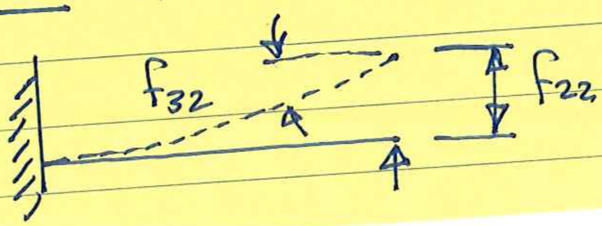
**Step 1:** Apply unit force in direction 1 to find $f_{11}, f_{21}, f_{31}$.




$$\Delta L = \frac{PL}{AE} = f_{11} P$$

$$\Rightarrow f_{11} = L/AE \,.\; f_{21} = f_{31} = 0$$

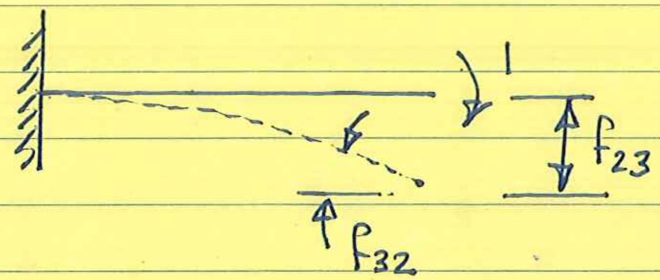
**Step 2:** Apply unit force in direction 2.




$f_{22} = \frac{L^3}{3EI}$, $f_{32} = \frac{-}{2}$

E

Step 3: Apply unit moment in direction 3.




$f_{13} = 0$

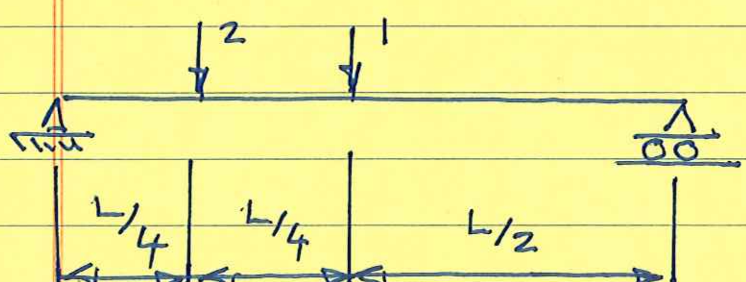
$f_{23} = \dfrac{-L^2}{2EI}$

$f_{33} = L/EI$.

Hence:

$$[f] = \begin{bmatrix} L/AE & 0 & 0 \\ 0 & L^3/3EI & -L^2/2EI \\ 0 & -L^2/2EI & L/EI \end{bmatrix}$$

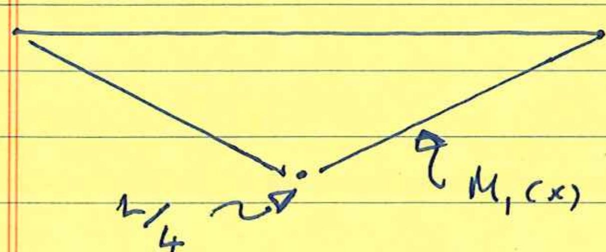
Example 2: Use energy methods to compute the (2×2) flexibility matrix.




ie, $[f] = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

Step 1: Apply unit force in direction 1.

BMD

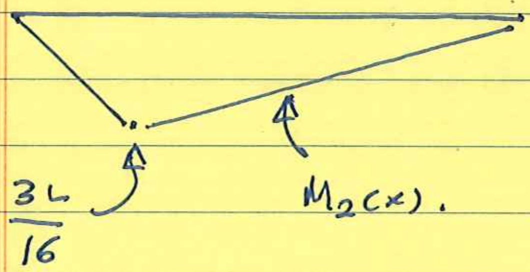



$$M_1(x) = \begin{cases} x/2 \,; & 0 \le x \le L/2 \\ \dfrac{L-x}{2} \,; & L/2 \le x \le L. \end{cases}$$

(F)

Step 2: Apply unit force in direction 2.

BMD.




$$M_2(x) = \begin{cases} 3x/4 \ ; & 0 \le x \le L/4 \\ \dfrac{L-x}{4} \ ; & L/4 \le x \le L. \end{cases}$$

Step 3: Compute flexibility matrix coefficients:

$$f_{11} = \int_0^L \frac{M_1(x)}{EI} \cdot M_1(x)\, dx = \frac{L^3}{48EI}.$$

$$f_{22} = \int_0^L \frac{M_2(x)}{EI} M_2(x)\, dx = \frac{3L^3}{256\,EI}.$$

$$f_{12} = f_{21} = \int_0^L \frac{M_1(x)}{EI} \cdot M_2(x)\, dx = \frac{11L^3}{768EI}$$

Maxwell's Reciprocal Theorem.

$$\Rightarrow [f] = \begin{bmatrix} L^3/48EI & 11L^3/256EI \\ 11L^3/256EI & \dfrac{3L^3}{256EI} \end{bmatrix}$$

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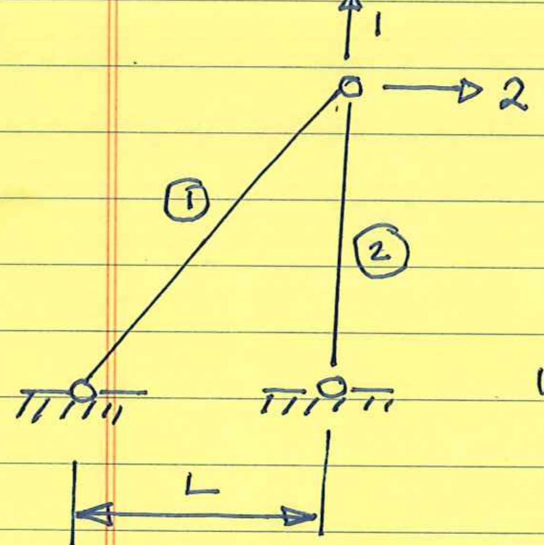
## Example 3: Flexibility Matrix for a truss structure.



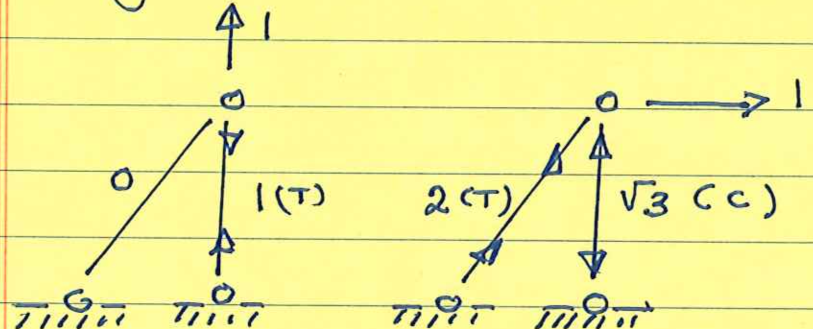

$AE$ = constant

$$f_{ij} = \sum_{i=1}^{N} \left\{ \frac{f_i f_j L_i}{AE} \right\}$$

Where $f_i$ = real force due to unit force applied in direction $\hat{i}$.

$N$ = no truss members.

Apply unit loads in directions 1 & 2.




Compute flexibility coefficients.

$$f_{11} = \sum \frac{f_1^2 L}{AE} = \frac{\sqrt{3} L}{AE}$$

$$f_{22} = \sum \frac{f_2^2 L}{AE} = \frac{13 L}{AE}$$

$$f_{12} = \sum \frac{f_1 f_2 L}{AE} = \frac{-3L}{AE}$$

| Member No | $L/AE$ | $f_1$ | $f_2$ |
|---|---|---|---|
| 1 | $2L/AE$ | 0 | 2 |
| 2 | $\sqrt{3}L/AE$ | 1 | $-\sqrt{3}$. |

$$\Rightarrow \{f\} = \begin{bmatrix} \frac{\sqrt{3}L}{AE} & -\frac{3L}{AE} \\ -3L/AE & \frac{13L}{AE} \end{bmatrix}$$