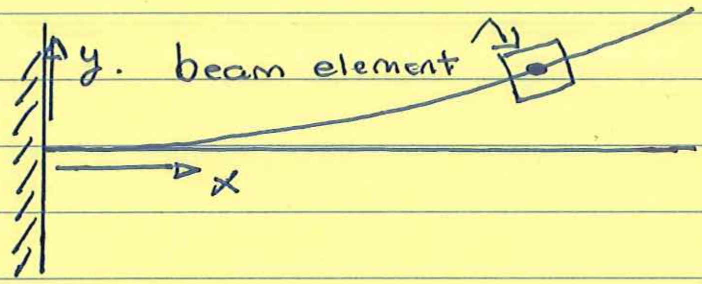
(A)

Elastic Curve for Beam Deflections.



deflection – $y(x)$

rotation – $\theta(x)$

Beam equations need to be compatible with geometry and internal forces (M, V, N).

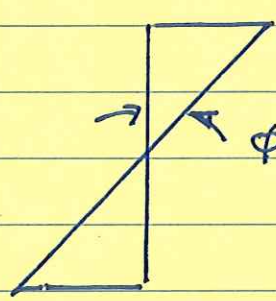
Procedure (beam element)

material level $\xrightarrow{\text{integrate}}$ section level $\xrightarrow{\text{integrate}}$ beam element level behavior

Material Level.

$$\sigma(z) = \left[\frac{E}{\rho}\right] z = E\epsilon$$

strain

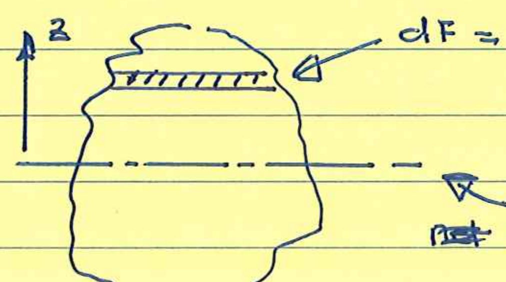


$\phi = \frac{1}{\rho}$ = curvature

radius of curvature

strain.

Section Level.



$dF = \sigma(z) . dA$

~~ne~~ neutral axis

(B)

$$M = \int_A dM = \frac{E}{\rho}\int z^2\, dA = \frac{EI}{\rho} = EI\phi$$

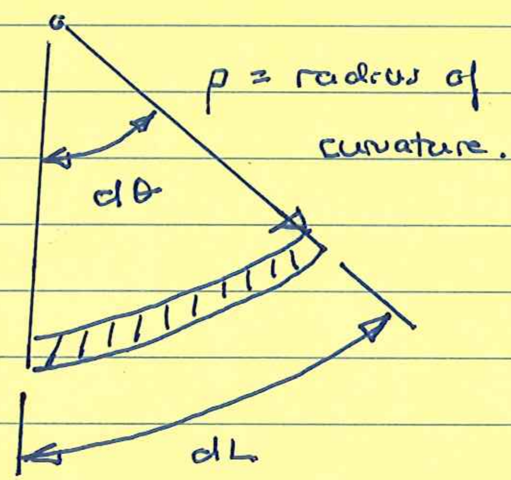
by definition; $I = \int_A z^2\, dA$

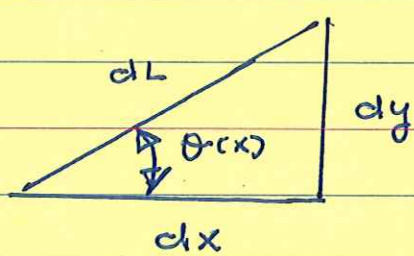
$$\Rightarrow \quad \phi = \left(\frac{M}{EI}\right) \quad \longrightarrow \quad (A)$$

Beam Geometry:

Need to relate ϕ & $\rho \longrightarrow y(x)$ and its derivatives.

Geometry of a beam element.





From geometry:

$$dL = \rho\, d\theta$$

$$\Rightarrow \quad \frac{1}{\rho} = \left(\frac{d\theta}{dL}\right) \quad — \quad (B)$$

Now relate dL to increments of dx & dy

$$dL^2 = dx^2 + dy^2$$

$$\Rightarrow \quad \frac{dL}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \quad — \quad (C)$$

Note; $\theta(x)$ may not be small. Need to relate $\theta(x)$ to $y(x)$ & its derivatives.

(C)

$$\tan(\theta(x)) = \left(\frac{dy}{dx}\right) \quad \text{—— (D)}$$

Recall chain rule:

$$\frac{d}{dx} f(\theta(x)) = \frac{df}{dx}(\theta(x)) \cdot \frac{d\theta}{dx}.$$

Differentiating (D) with respect to x.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\tan(\theta(x))\right]$$

$$= \sec^2(\theta(x)) \frac{d\theta}{dx}$$

$$= \left\{1 + \tan^2(\theta(x))\right] \frac{d\theta}{dx} \quad \text{—— (E)}$$

Plug (D) into (E)

$$\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right] \cdot \frac{d\theta}{dx} \quad \text{—— (F)}$$

$$\Rightarrow \frac{d\theta}{dx} = \left[\frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\right] \quad \text{—— (G)}$$

<u>Back to the Beam.</u> (see equation (B))

$1/(C)$

$$\phi = \frac{1}{\rho} = \frac{d\theta}{dL} = \frac{d\theta}{dL} \cdot \frac{dx}{dx} = \left(\frac{d\theta}{dx}\right)\left(\frac{dx}{dL}\right)$$

(G)

(D)

$$\phi = \frac{d\theta}{dx} \cdot \frac{dx}{dL} = \frac{d^2y/dx^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{M}{EI}.$$

For most Civil Engineering structures dy/dx is small.

$$\frac{dy}{dx} \rightarrow \text{small}, \quad \left(dy/dx\right)^2 \longrightarrow 0.$$

Thus, we can ignore denominator.

$$\phi = \frac{d^2y}{dx^2} = \frac{M(x)}{EI}.$$

$$\Rightarrow M(x) = EI\left(\frac{d^2y}{dx^2}\right) \quad \text{—— (H)}$$

<u>Relationship between Shear & Bending Moment.</u>

$$V(x) = \frac{dM}{dx} \quad \& \quad \frac{dV}{dx} = \underbrace{w(x)}_{\text{distributed loading}} \quad \text{—— (I)}$$

Combining (H) & (I)

$$\frac{dM}{dx} = \frac{d}{dx}\left[EI \frac{d^2y}{dx^2}\right].$$

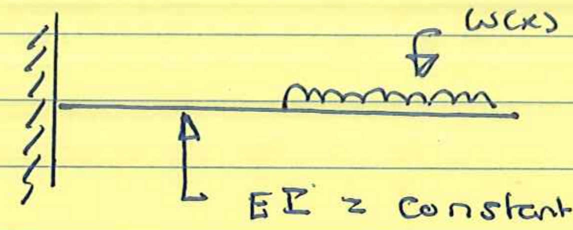
$$\Rightarrow \frac{d^2}{dx^2}\left[EI \frac{d^2y}{dx^2}\right] + w(x) = 0 \quad \text{—— (J)}$$

Equation (J) applies for small (dy/dx).

(E)

For our purposes:

1



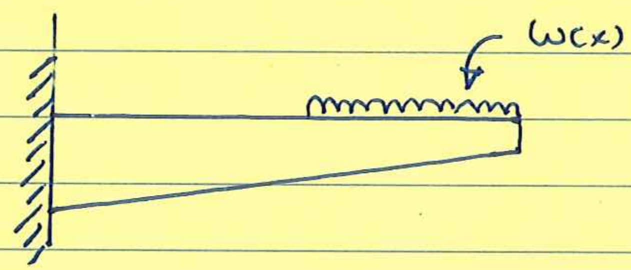
Equation (J) simplifies to

$$EI \frac{d^4y}{dx^4} + w(x) = 0$$

integrate 4 times

$y(x)$. – deflections.

2. Note: Not all beams will have
EI = constant.



Need to use equation (J) directly

integrate 4 times

$y(x)$ – deflections.

3. We will assume small displacements $y(x)$. In our derivation we said

$$\tan(\theta(x)) = \frac{dy}{dx}$$

For small displacements / rotations

$$\tan(\theta(x)) \doteq \theta(x)$$

$$\Rightarrow \quad \theta(x) = \left(\frac{dy}{dx}\right) = \text{rotation}.$$