Department of Civil and Environmental Engineering, Fall Semester, 2023

ENCE 201 Final Exam, Open Notes and Open Book

Name : Austin

E-mail (print neatly!): austin@umd.edu.

Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are five questions. Partial credit will be given for partially correct answers, so please show all your working.

Please see the **class web page for instructions on how to submit your exam paper**.

Question	Points	Score
1	20	
2	20	
3	10	
4	10	
5	10	
Total	70	

Question 1: 20 points.

This question covers solution of matrix equations using Gauss Elimination. Consider the matrix equations Ax = b, where:

$$\begin{bmatrix} 0 & 7 & 3 \\ 3 & 0 & 7 \\ 7 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 17 \end{bmatrix}. \tag{1}$$

[1a] (4 pts). Compute the det(A)?

$$\det(A) = 0 \det\begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix} - 7 \det\begin{bmatrix} 3 & 7 \\ 7 & 0 \end{bmatrix} + 3 \det\begin{bmatrix} 3 & 0 \\ 7 & 3 \end{bmatrix}$$

$$= 0(0-21) - 7(0-49) + 3(9-0)$$

$$= 370.$$

[1b] (4 pts). Based on your solution to part `1a`, what can you say about: (1) the rank of the system of equations, and (2) the number of solutions to the matrix equations?

1. $\det A \neq 0 \Rightarrow \text{rank}(A) = 3.$ = no rows in A.
2. A^{-1} exists $\Rightarrow Ax = b \Rightarrow x = A^{-1}b$.
3. Number of sol'ns to $Ax = b$ is 1.

[1c] (4 pts). In terms of "complexity of matrix structure" and "row operations," what are the goals of the method of Gauss Elimination? Why does the method work?

Design sequence of elementary row ops to simplify matrix structure, without changing the sol'n.

[1d] (8 pts). Use the method of Gauss Elimination with pivoting to compute the solution to equation 1. This is a hand calculation, so show all of your working.

Augmented Matrix $(A:b)$

$$\left[\begin{array}{ccc|c} 0 & 7 & 3 & 10 \\ 3 & 0 & 7 & 13 \\ 7 & 3 & 0 & 17 \end{array}\right] \xrightarrow{\text{Swap } R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 7 & 3 & 0 & 17 \\ 3 & 0 & 7 & 13 \\ 0 & 7 & 3 & 10 \end{array}\right] \xrightarrow[R_2 \to R_2 - 3R_1]{R_1/7}$$

$$\left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & -9/7 & 7 & 40/7 \\ 0 & 7 & 3 & 10 \end{array}\right] \xrightarrow[R_2/7]{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & 1 & 3/7 & 10/7 \\ 0 & -9/7 & 7 & 40/7 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + \frac{9}{7}R_2} \left[\begin{array}{ccc|c} 1 & 3/7 & 0 & 17/7 \\ 0 & 1 & 3/7 & 10/7 \\ 0 & 0 & 370/49 & 370/49 \end{array}\right]$$

Backsubstitution: $\frac{370}{49} x_3 = \frac{370}{49} \Rightarrow x_3 = 1.0$

$x_2 + 3/7\, x_3 = \frac{10}{7} \Rightarrow x_2 = 1.0$

$x_1 + 3/7\, x_2 = \frac{17}{7} \Rightarrow x_1 = 2.0$.

Check Soln:

$$\begin{bmatrix} 0 & 7 & 3 \\ 3 & 0 & 7 \\ 7 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 17 \end{bmatrix} \checkmark$$

Question 2: 20 points

[2a] (5 pts). Derive the Newton-Raphson formula:

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right] \tag{2}$$

Hint : Start off by writing down a Taylors series expansion for $f(x+h)$. State all of your assumptions.

$$f(x+h) = f(x) + h f'(x) + \underbrace{\frac{h^2}{2!} f''(x) \ \cdots}_{\text{higher order terms. (HOT)}} = 0$$

Ignoring HOT: $f(x_n+h) = f(x_n) + h f'(x_n) = 0$

$$\Rightarrow x_{n+1} = x_n + h = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right) \quad \text{— } A.$$

[2b] (5 pts). Consider the function

$$y(x) = \sin^2(x). \tag{3}$$

Show that the Newton-Raphson update formula can be written as:

$$x_{n+1} = x_n - \frac{1}{2}\tan(x_n). \tag{4}$$

Be sure to show all of your working.

$$y(x) = \sin^2(x) \Rightarrow \frac{dy}{dx} = 2\sin(x)\cos(x) = \sin 2x.$$

$$\Rightarrow x_{n+1} = x_n - \frac{\sin^2(x_n)}{2\sin(x_n)\cos(x_n)}$$

$$= x_n - \frac{1}{2}\tan(x_n). \quad \text{—— } B.$$

[2c] (5 pts). If $\dot{g}(x)$ and $\ddot{g}(x)$ are the first and second derivatives of $g(x)$, respectively, and

$$f(x) = \left[\frac{g(x)}{\dot{g}(x)}\right] \tag{5}$$

show that the modified Newton-Raphson formula is given by:

$$x_{n+1} = x_n - \left[\frac{g(x_n)\dot{g}(x_n)}{\dot{g}(x_n)\dot{g}(x_n) - g(x_n)\ddot{g}(x_n)}\right] \tag{6}$$

$$f(x) = \left[\frac{g(x)}{\dot{g}(x)}\right] \Rightarrow f'(x) = \frac{g'(x)^2 - g(x)g''(x)}{g'(x)^2} \quad - \text{C}$$

Plugging C into B & rearranging terms gives:

$$x_{n+1} = x_n - \left[\frac{g(x_n)\cdot\dot{g}(x_n)}{\dot{g}(x_n)\cdot\dot{g}(x_n) - g(x_n)\cdot\ddot{g}(x_n)}\right]$$

[2d] (5 pts). Use a starting value $x_o = \pi/4$ and the modified Newton Raphson Formula to find an improved estimate of the root of the polynomial:

$$y(x) = \sin^2(x). \tag{7}$$

Note: Do no more than 1 iteration !!.

$y(x) = \sin^2(x) \longrightarrow \dot{y}(x) = \sin(2x),\ \ddot{y}(x) = 2\cos(2x).$

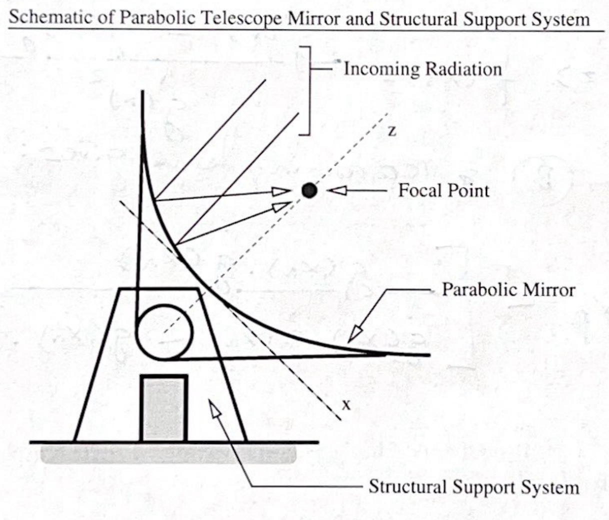
Let $x_0 = \pi/4$. $y(\pi/4) = \frac{1}{2}$, $\dot{y}(\pi/4) = 1.0$, $\ddot{y}(\frac{\pi}{4}) = 0$.

$$x_1 = x_0 - \left[\frac{\frac{1}{2}\cdot 1}{1^2 - \frac{1}{2}\cdot 0}\right] = \frac{\pi}{4} - 1 = -0.214.$$

Question 3: 10 points.

This question covers least squares analysis. Carefully read the problem statement and then answer the question that follows.

Problem Statement. Most structural engineering systems have metrics of performance (e.g., maximum allowable deformation) that are relatively insensitive to small variations in material strength, external loads, temperature, and details of construction. An important exception is structural systems required to support large mirrors in telescopes.



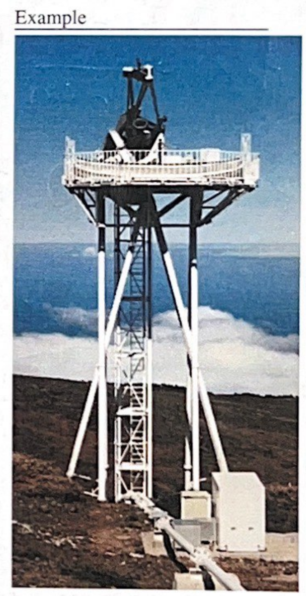


Figure 1. Schematic of Telescope and Structural Support System

Figure 1 shows that in order for the mirror to reflect incoming radiation on the focal point, the mirror surface must be precisely parabolic, i.e.,

$$z(x, y) = k\left[x^2 + y^2\right] \tag{8}$$

where k is a suitable design constant. But unfortunately, even a small variation in the structural support can cause the mirror surface to distort, thereby rendering the telescope inoperable. Instead of following equation 8 the as-built surface might be:

$$z(x, y) = ax + by + cx^2 + dy^2 \tag{9}$$

The terms $ax + by$ account for shear-like distortions in the mirror surface (ideally, a=b=0). The terms $cx^2 + dy^2$ account for mirror distortions in the axial/radial directions (ideally, c=d=k).

Suppose that an experiment is conducted to measure the profile of an as-built mirror surface, resulting in the data format:

```
----------------------------------
z (mm)        x (mm)         y (mm)
==================================

... details of data omitted ...

==================================
```

Least squares procedures can be used to detect the presence of distortions, which in turn, can guide adjustments to the support system.

Question. (10 pts.) Starting from first principles of least squares analysis, show that the least squares fit of equation 9 to the data set is given by the matrix equation:

$$\begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \cdot y_i & \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i \cdot y_i^2 \\ \sum_{i=1}^{N} x_i \cdot y_i & \sum_{i=1}^{N} y_i^2 & \sum_{i=1}^{N} x_i^2 \cdot y_i & \sum_{i=1}^{N} y_i^3 \\ \sum_{i=1}^{N} x_i^3 & \sum_{i=1}^{N} x_i^2 \cdot y_i & \sum_{i=1}^{N} x_i^4 & \sum_{i=1}^{N} x_i^2 \cdot y_i^2 \\ \sum_{i=1}^{N} x_i \cdot y_i^2 & \sum_{i=1}^{N} y_i^3 & \sum_{i=1}^{N} x_i^2 \cdot y_i^2 & \sum_{i=1}^{N} y_i^4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i \cdot z_i \\ \sum_{i=1}^{N} y_i \cdot z_i \\ \sum_{i=1}^{N} x_i^2 \cdot z_i \\ \sum_{i=1}^{N} y_i^2 \cdot z_i \end{bmatrix} \quad (10)$$

where N is the total number of data points. Be sure to show all of your working.

Let $S(a, b, c, d) = \sum_{i=1}^{N} (z_i - f(x_i, y_i))^2$

where $f(x_i, y_i) = (a x_i + b y_i + c x_i^2 + d y_i^2)$.

Line of best fit corresponds to:

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = \frac{\partial S}{\partial c} = \frac{\partial S}{\partial d} = 0.$$

$$\frac{\partial S}{\partial a} = \frac{\partial}{\partial a} \sum_{i=1}^{N} (z_i - a x_i - b y_i - c x_i^2 - d y_i^2)^2 = 0$$

$$\Rightarrow 2 \sum_{i=1}^{N} (z_i - a x_i - b y_i - c x_i^2 - d y_i^2) \cdot x_i = 0.$$

$$\Rightarrow a \sum_{i=1}^{N} x_i^2 + b \sum_{i=1}^{N} x_i y_i + c \sum_{i=1}^{N} x_i^3 + d \sum_{i=1}^{N} x_i y_i^2 = 0.$$

We have 3 similar expressions for $\frac{\partial S}{\partial b} = 0$, $\frac{\partial S}{\partial c} = 0$ & $\frac{\partial S}{\partial d} = 0$. Putting equations in matrix form gives:

$$\begin{bmatrix} \sum_1 x_i^2 & \sum_1 x_i y_i & \sum_1 x_i^3 & \sum_1 x_i y_i^2 \\ \sum_1 x_i y_i & \sum_1 y_i^2 & \sum_1 x_i^2 y_i & \sum_1 y_i^3 \\ \sum_1 x_i^3 & \sum_1 x_i^2 y_i & \sum_{i=1}^{N} x_i^4 & \sum_1 x_i^2 y_i^2 \\ \sum_1 x_i y_i^2 & \sum_1 y_i^3 & \sum_1 x_i^2 y_i^2 & \sum_{i=1}^{N} y_i^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i z_i \\ \sum_{i=1}^{N} y_i z_i \\ \sum_{i=1}^{N} x_i^2 z_i \\ \sum_{i=1}^{N} y_i^2 z_i \end{bmatrix}$$

Question 4: 10 points. This question covers numerical integration.

Theoretical considerations indicate that

$$\int_0^{\pi} \sin^2(x)dx = \left[\frac{x}{2} - \frac{\sin(x)\cos(x)}{2}\right]_0^{\pi} = \frac{\pi}{2}. \tag{11}$$

The values for $f(x) = \sin^2(x)$ can be summarized as follows:

x	0.0	pi/4	pi/2	3pi/4	pi
f(x)	0.0	0.5	1.0	0.5	0.0

[4a] (5 pts.) Suppose that equation 11 is evaluated using the Trapezoid Rule and only interval of integration (i.e., h = π and b-a = π). What is the maximum error that will occur with this numerical approximation? Is the actual error within this bound?

$T_1 = \frac{h}{2}\left[f(0) + f(\pi)\right] = \frac{\pi}{2}\left[0 + 0\right] = 0.$

Actual error $= \pi/2 \doteq 1.57.$

Estimate of numerical error $\leq \dfrac{|f''(\xi)|\, h^2 (b-a)}{12}.$

$h = \pi, \quad b - a = \pi, \quad 0 \leq \xi \leq \pi.$

$f(x) = \sin^2(x) \Rightarrow f''(x) = 2\cos(2x). \Rightarrow |f''(\xi)|_{max} = 2$

Max error $\leq \dfrac{2\pi^3}{12} = \dfrac{\pi^3}{6} = 5.16.$

Actual error < Max Error $\rightarrow$ It works ✓.

[4b] (5 pts.) Use Simpson's Rule with h = $\pi/4$ to estimate equation 11. Show all of your working.

Simpson's Rule with $h = \pi/4$.

$$S = \frac{h}{3}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right] +$$

$$\frac{h}{3}\left[f(\pi/2) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right]$$

$$= \frac{\pi}{12}\left[0 + 4 \cdot \frac{1}{2} + 1\right] + \frac{\pi}{12}\left[1 + 4 \cdot \frac{1}{2} + 0\right]$$

$$= \frac{\pi}{12}\left[3 + 3\right]$$

$$= \frac{\pi}{2} \longleftarrow \text{Exact!}$$

Question 5: 10 points

This question covers numerical integration. Consider the integration problem:

$$\int_{0.0}^{0.8} \left[\frac{sin(x)}{x}\right] dx \tag{12}$$

The values for $f(x) = sin(x)/x$ can be summarized as follows:

x	0.0	0.2	0.4	0.6	0.8
f(x)	1.0	0.99334	0.97355	0.94107	0.89670

[5a] (5 pts.) Use the method of Romberg integration to obtain an $O(h^4)$ accurate estimate of equation 12.

$$T_1 (h = 0.8) = \frac{0.8}{2}\left[f(0) + f(0.8)\right] = 0.75868 \; \{O(h^2)\}$$

$$T_2 (h = 0.4) = \frac{0.4}{2}\left[f(0) + f(0.4)\right] + \frac{0.4}{2}\left[f(0.4) + f(0.8)\right]$$

$$= 0.76876 \leftarrow O(h^2) \text{ error.}$$

<u>One step of Romberg Integration:</u>

$$R_{21} = \left[\frac{4T_2 - T_1}{(4-1)}\right] = 0.77212 \leftarrow O(h^4) \text{ error}$$

[5b] (5 pts.) Evaluate equation 12 using 2-pt Gauss Quadrature. Be sure to show all steps in your working.

$$I = \int_{0.0}^{0.8} \left[\frac{\sin(x)}{x} \right] dx.$$

Map domain $[0, 0.8] \longrightarrow [-1, 1]$.

Let $x = \frac{0.8}{2}[1+u] \Rightarrow dx = \frac{0.8}{2} du$.

$$\Rightarrow I = \int_0^{0.8} \frac{\sin(x)}{x} dx = \frac{.8}{2} \int_{-1}^{1} \frac{\sin(0.4(1+u))}{0.4(1+u)} du$$

Two-Point Quadrature: $\omega_0 = \omega_1 = 1$

$u_0 = \frac{-1}{\sqrt{3}}, \quad u_1 = \frac{1}{\sqrt{3}}$.

$$I = \frac{0.8}{2} \left[\underbrace{\frac{\sin 0.4(1 - 1/\sqrt{3})}{0.4(1 - 1/\sqrt{3})}}_{f_1 = 0.9952} + \underbrace{\frac{\sin 0.4(1 + 1/\sqrt{3})}{0.4(1 + 1/\sqrt{3})}}_{f_2 = 0.9349} \right]$$

$= 0.4 [0.9952 + 0.9349]$

$= 0.77204$.