The capacity region of a fading CDMA channel under additive white noise is where the signature sequences are fixed, users have perfect CSI, and they allocate their powers as a function of the CSI subject to average power constraints.

The capacity region of a fading CDMA channel is not necessarily strictly convex, the flat portion on the boundary corresponds to the sum capacity maximizing power control policy. The problem of finding the power control policy that corresponds to the rate pair \((R_1', R_2')\) on the boundary is equivalent to maximizing \(\mu_1 R_1 + \mu_2 R_2\) subject to the average power constraints, for some \(\mu_1, \mu_2\).

Any rate pair \((R_1', R_2')\) on the curved portion of the capacity boundary is a corner of one of the pentagons.

For a given set of priorities \(\mu_1 > \ldots > \mu_T > 0\), define \(S = \{s_1, \ldots, s_T\}\), \(E = \{i, \ldots, K\}\) and \(D(h) - diag(p_i(h)I_2, \ldots, p_K(h)I_2)\). The optimization problem is:

\[
\max \frac{1}{2} E \left[ \sum_{i=1}^{K} \mu_i - \mu_{ij} \right] \left[ \log \left( \frac{1}{\rho_i} + \sigma^2 R(h) + \sum_{j \neq i} \rho_j p_j(h) s_j^T s_i \right) \right] \quad \text{s.t.} \quad \frac{1}{2} E \left[ \sum_{i=1}^{K} \mu_i - \mu_{ij} \right] \left[ \log \left( \frac{1}{\rho_i} + \sigma^2 R(h) + \sum_{j \neq i} \rho_j p_j(h) s_j^T s_i \right) \right] \quad \forall h, k = 1, \ldots, K
\]

Objective function is concave in \(p(h)\), strictly concave in \(\mu(h)\), and constraints are convex: solution satisfies the extended KKT conditions.

Theorem: The capacity region of a fading CDMA channel is not strictly convex, provided \(E_i \neq 0\), \(i \neq j\) and \(0 < s_i^T s_j < 1\).

Each of the pentagons correspond to a valid power allocation policy.

If channel state information (CSI) \(h\) is available at the transmitters and the receiver of a fading MAC, the transmitters can allocate transmit powers (resources) and adjust their coding strategies, and the receiver can adjust its decoding strategy.

Our goal is to obtain the capacity region, and the power control policies that achieve arbitrary points on the boundary of the capacity region.

The capacity region is as in [Hanly-Tse]. Strict convexity of the boundary is observed (red curve).

In this example, assume \(c.<0.5\) for discrete fading states \(h^1, h^2, \ldots, h^K\).

As we gradually increase water level, using (1), we solve for corresponding powers.

We stop when all available power is used, and the current water level gives 1/\(\lambda_e\). As we iterate over users, \(\lambda_e\) converges to \(\lambda_e\).

The power control policy that achieves the sum capacity of a fading CDMA system is simultaneous waterfilling, and can be obtained by a one-user-at-a-time iterative waterfilling algorithm [Kaya-Ulukus].

The capacity region of the fading MAC is the union of rate regions achievable by all valid power control policies, and the power control policies needed to achieve each point on the boundary can be obtained by using a greedy algorithm [Hanly-Tse].

The power control policy that achieves the sum capacity of a fading MAC system is simultaneous waterfilling, and can be obtained by a one-user-at-a-time iterative waterfilling algorithm [Kaya-Ulukus].

The capacity region of fading vector MACs is not known. Here, we provide the capacity region of fading CDMA and corresponding power control policies.

Capacity Region of Fading CDMA

Focus on the power update of a single user.

Generalized waterfilling to solve the KKT conditions at all \(h\).

Define the base levels, obtained by letting \(p(h) = 0\) in KKT conditions, as

\[
b_1(h) = \left( \sum_{i=1}^{K} \frac{1}{\rho_i} \right)^{-1} \quad \text{and} \quad b_2(h) = \left( \sum_{i=1}^{K} \frac{1}{\rho_i} \right)^{-1}
\]

Sort \(b_1(h)\) in increasing order, start pouring power at the channel state which yields the minimum \(b_2(h)\), say \(h^*\).

Next, pick another state \(h^+\) s.t. \(b_2(h^+) < b_2(h^*)\) User \(k\) starts transmitting at \(h^+\) iff

- it has already poured some powers \(q_i(h)\) to all states \(h\) s.t. \(b_2(h) < b_2(h^+)\).
- it still has some power to allocate; and
- the already allocated powers \(q_i(h)\) satisfy

\[
\sum_{i=1}^{K} \frac{1}{\rho_i} (q_i(h) + q_i(h^+)) = b_2(h^+) = b_2(h^+) (1)
\]

The value of \(q_i(h)\) at each step is not given by the water level at each state as in sum capacity case, but can be found by solving a 4th order polynomial equation.

By construction, when the average power constraint is met with equality, letting \(p_i(h) = q_i(h)\), this allocation satisfies the KKT conditions and is optimal.

When sequences are identical, the problem reduces to scalar MAC, and capacity region is as in [Hanly-Tse]. Strict convexity of the boundary is observed (red curve).

When sequences are orthogonal, the capacity region is a rectangle (blue curve).

When sequences are arbitrarily correlated, there is a flat portion on the boundary, supporting the analytical results on non-strict convexity of the capacity region of power controlled fading CDMA (e.g., green curve).