An Explicit Optimal Scheme for Distributed Lossy Compression

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Abstract
We show that polar codes can be used to achieve the rate-distortion functions in the problem of hierarchical lossy compression also known as the successive refinement problem.

Introduction
Polar coding was introduced by Arikan in the seminal paper [Arıkan ’09]. Let \( n = 2^n; G_n = \left( \frac{1}{2} \right)^n \). Arikan showed that given a binary-input channel \( W \), there is a sequence of linear codes, whose generator matrices are appropriately chosen from the rows of \( G_n \) achieving the symmetric capacity of \( W \). Later it was proved that polar codes work equally well for source coding [Arıkan ’10].

Polar Coding Scheme [Honda and Yamamoto ’13]

\[ \mathcal{L}_X \mid Y \quad \mathcal{H}_X \mid Y \]

\{1, 2, \ldots, n\}

- \( X \sim P_X \)
- \( X \) and \( Y \) are independent
- \( \mathcal{L}_X \mid Y \) and \( \mathcal{H}_X \mid Y \) are uniformly random
- \( \lim_{n \to \infty} \text{H}(X | Y) = \text{H}(X) \)

Achieve optimal rates for both channel and source coding.

Polar Codes for Lossy Source Coding

Rate-Distortion:
- Source \( X \sim P_X \) over a finite alphabet \( \mathcal{X} \)
- Distortion function \( d : \mathcal{X} \times \{0, 1\} \to [0, \infty) \)
- Rate-distortion function \( \mathcal{R}(D) = \min_{P_{1|X}} \text{I}(X; T) \), where \( P_{1|X} \) is such that \( \text{I}(X; T) \leq D \).

Polar Coding Scheme [Honda and Yamamoto ’13]

- Objective: to approximate the distribution \( P_{1|X} \).

Succesive Refinement of Information

The source \( X \) is said to be successively refinable with distortions \( D_1 \) and \( D_2 \) if and only if there exists a conditional distribution \( P_{1|W} \) with \( \text{E}_{X,W}(X,T) \leq D_1 \), \( \text{E}_{X,W}(X,W) \leq D_2 \).

Theorem 1. (Koshelev ’80, Equitz and Cover ’91) Let \( X \) be a source and let \( T, W \) be two binary random variables. The source is successively refinable with distortions \( D_1 \) and \( D_2 \) if and only if there exists a conditional distribution \( P_{1|W} \) with

\[ \frac{1}{n} \sum_{i=1}^{n} I(X; T) \to R(D_1), \quad \frac{1}{n} \sum_{i=1}^{n} I(X, W) \to R(D_2), \]

and such that \( X, W, T \) satisfy the Markov condition

\[ X \to W \to T. \]

References