Optimal Control for Reconstruction

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**Problem Definition** (Regularized Inversion)

Given a time series of observed positions in three dimensional space, our objective is to reconstruct a smooth trajectory from these data points by solving an optimization problem.

- **Penalty term** is introduced to assure smoothness of the reconstructed trajectory and to make the problem well posed. The objective function to be minimized is

\[
\sum_{i=0}^{N} \| r(t_i) - r_i \|^2 + \int_{t_0}^{t_N} \{ \text{Suitable Path Cost} \} dt
\]

- **Fit Error**

- **Penalty Term**

The relative significance of the fit error with respect to the penalty term is not known a priori.

- An optimal amount of regularization is picked using ordinary cross validation (OCV); signal-to-noise ratio in the data affects the corresponding value of \( \lambda \).

**Generative Model: I**

\[
\begin{align*}
\dot{r} &= v/x \\
\dot{x} &= v (y + vz) \\
\dot{y} &= -v (ux) \\
\dot{z} &= -v (ux)
\end{align*}
\]

**Penalty Term**

\[
\int_{t_0}^{t_N} (\dot{v}^2 + \dot{w}^2 + \dot{z}^2) dt
\]

**Generative Model: II**

\[
\begin{align*}
\dot{r} &= v \\
\dot{v} &= a \\
\dot{a} &= u
\end{align*}
\]

**Penalty Term**

\[
\int_{t_0}^{t_N} (r^T u) dt
\]

- **Approximation** by piecewise constant speed and curvature, transforms regularized inversion (for model I) into a non-convex numerical optimization problem \[2,3\]. (Matlab routine: \text{fminunc})

- **Sub-frame adaptable**, but the algorithm is time consuming.

**Dictionary between Generative Model I and II**

\[
\begin{align*}
\nu &= \| v \| \\
\kappa &= \| x \| \\
\tau &= \nu \cdot (a \times u) / |v \times a|^2 \\
\hat{x} &= \frac{1}{\nu}(a - (a \cdot x)x) \\

x(t) &= x(0) + \int_{0}^{t} \nu(\sigma)w(\sigma)x(\sigma) d\sigma \\
y(t) &= y(0) - \int_{0}^{t} \nu(\sigma)w(\sigma)x(\sigma) d\sigma \\
z(t) &= z(0) - \int_{0}^{t} \nu(\sigma)w(\sigma)x(\sigma) d\sigma
\end{align*}
\]

**Reconstruction via Jerk Path Integral Minimization**

- **State**:

\[
x \triangleq [ r \quad v \quad a ]^T
\]

- **Input**:

\[
u \triangleq u
\]

- **Output**:

\[
y \triangleq x
\]

- **Minimize**

\[
\int_{t_0}^{t_N} u^T u dt + \sum_{i=0}^{N} \| y(t_i) - r_i \|^2
\]

**subject to**

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)
\]

- \( x : [t_0, t_N] \rightarrow \mathbb{R}^9 \)

- \( u \in U : [t_0, t_N] \rightarrow U \subseteq \mathbb{R}^3 \)

- \( u \) — piecewise continuous

- **Optimal Control Input**:

\[
u^*(t) = -B^T p(t)
\]

- **Costate Variable**:

\[
\frac{d}{dt} \left[ \begin{array}{c} x(t) \\ p(t) \end{array} \right] = \left[ \begin{array}{cc} A & -B^T \\ 0 & -A^T \end{array} \right] \left[ \begin{array}{c} x(t) \\ p(t) \end{array} \right]
\]

\[
p(t_N) = p(t_0) = 0 \quad \text{Optimal Initial Condition}:
\]

**Salient Features of The Approach** \[1,3\]

- Path-independence lemmas and Riccati equation ensure global optimality of the solution.

- This linear-quadratic approach yields an analytical solution.

- Reconstructed positions can be expressed as linear combinations of raw data.

- The proposed algorithm is fast, with complexity of the order of sample size \( O(N) \).

- OCV is carried out using leaving-out-one strategy, wherein a trajectory is reconstructed by considering all but one data point and then prediction error is computed at the left-out point. An optimal \( \lambda \) minimizes the total prediction error.

**References**


**Reconstruction of Starling Flocks**

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