Consensus problem under stochastic communication topologies

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PROBLEM FORMULATION

Definition. A Discrete Markovian Random Graph is a stochastic process taking values in a set of graphs $\mathcal{G}=\{G_1,\ldots,G_n\}$ which evolves in time according to a underlying finite-state discrete time Markov chain $M(k)$.

Consider a group of $n$ agents which exchange locally information according to a communication protocol. The communication topology is assumed modeled by a Markovian Random Graph $G_{M(k)}$. The evolution of the agents is governed by

$$X(k+1) = F_{M(k)}X(k), \quad X(0) = X_0$$

where $X(k)$ represents the $n$-dimensional state vector of the agents and, $F_{M(k)}$ is a stochastic matrix determined by the communication protocol and which corresponds to the current topology determined by $G_{M(k)}$.

Definition (Almost sure consensus). Vector $X(k)$ converges almost surely to consensus if it asymptotically reaches a vector $\alpha \mathbf{1}$ in the almost sure sense. If the state vector converges to $\alpha \mathbf{1}$ we say that $X(k)$ converges almost surely to average consensus.

Problem. Given a discrete Markovian random graph $G_{M(k)}$ together with the state updating rule $F_{M(k)}$ we derive necessary and sufficient conditions such that the state vector $X(k)$ converges almost surely to consensus for any initial state $X_0$.

Example of local communication protocols (local state updating rules):

1. $A_1: F_t=I-L_t$
2. $A_2: F_t=(I+D)^{-1}(I-A)$

where $L_t$, $D_t$, and $A_t$ are the Laplacian, degree and adjacency matrices respectively of graph $G_t$.

MOTIVATION

- Presence of communication topologies that varies randomly in time due to link failures, packet drops, appearance or disappearance of nodes, obstacles that interfere with the communication, etc.
- Topology switching may depend on the previous state of the communication topology (systems with memory)
- Applications in Flocking Theory
  - Rendezvous in Space
  - Distributed Sensor Fusion in Sensor Networks
  - Distributed Formation Control

RESULTS

Theorem (almost sure convergence to consensus). Under protocol A1 or A2, the state vector $X(k)$ converges almost surely to consensus for any initial state $X_0$ if and only if each of the sets of graphs corresponding to the positive recurrent closed sets of the Markov chain admits a spanning tree. In particular, if protocol A1 is used and all graphs in $G$ are either undirected or directed but balanced, then the state vector converges almost surely to average consensus.

Bound on the rate of convergence to consensus in the mean square sense:

$$E\left[\|X(k)\|_2^2\right] = c\|X_0\|_2^2$$

where $c$ is the second largest eigenvalue in absolute value of the matrix $\Lambda$

$$\Lambda = \begin{bmatrix}
    p_{11} & p_{21} & \cdots & p_{n1} \\
n_{12} & p_{22} & \cdots & p_{n2} \\
n_{13} & n_{23} & \cdots & p_{n3} \\
\vdots & \vdots & \ddots & \vdots \\
    n_{1n} & n_{2n} & \cdots & p_{nn}
\end{bmatrix}$$

where $p_{ij} = \text{Prob}(M(k)=j|M(0)=i)$ for large values of $k$ and $c_i$ are vectors of positive entries which sum up to one.

APPROACH

Show for the state vector that $E[X(k)X(k)^T] \rightarrow \beta \mathbf{1} \mathbf{1}^T$ for some scalar $\beta$, which implies that the second moment of the error vector $e(k)$ converges to zero exponentially, which in turn shows that $\|e(k)\|_2 \rightarrow 0$.

Key property:

$$\lim_{k \rightarrow \infty} \Lambda^k = \begin{bmatrix}
    p_{11}^k & p_{21}^k & \cdots & p_{n1}^k \\
p_{12}^k & p_{22}^k & \cdots & p_{n2}^k \\
p_{13}^k & p_{23}^k & \cdots & p_{n3}^k \\
\vdots & \vdots & \ddots & \vdots \\
p_{1n}^k & p_{2n}^k & \cdots & p_{nn}^k
\end{bmatrix}$$

References

