Opinion Dynamics over Signed Social Networks

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Social Networks
Opinions

iPhone, Blackberry, or Samsung? Republican or Democrat? Sell or buy AAPL? The rate of economic growth this year? Social cost of carbon?

French 1956; Benerjee 1992; Galam 1996
Dynamics of Opinions

\[ x_i(k + 1) = f_i \left( x_i(k); x_j(k), j \in \mathcal{N}_i \right) \]
De Groot Social Interactions

\[ x(k + 1) = P x(k) \]

- \( P \) is a stochastic matrix
De Groot Social Interactions

Before:

\[ x_i(k + 1) = x_i(k) + \alpha(x_j(k) - x_i(k)) \]

After:

\[ 0 < \alpha < 1 \]
De Groot Social Interactions

\[ x(k + 1) = Px(k) \]

An agreement is achieved if \( P \) is ergodic in the sense that

\[
\lim_{k \to \infty} x_i(k) = v^\top x(0)
\]

Trust and cooperation lead to social consensus!
Wisdom of Crowds (with Trust)

Golub and Jackson 2010
Disagreement Models

— Memory of initial values
  
  Friedkin and Johnsen (1999)

— Bounded confidence
  
  Krause (1997); Hegselmann-Krause (2002); Blondel et al. (2011); Li et al. (2013)

— Stubborn agents
  
  Acemoglu et al. (2013)

— Homophily
  
  Dandekar et al. (2013)
This Talk

A model and theory for opinion dynamics over social networks with friendly and adversarial interpersonal relations coexisting.
Friends and Adversaries
Strongly balanced if the node set can be divided into two disjoint subsets such that negative links can only exist between them;

Weakly balanced if such a partition contains maybe more than two subsets.

Heider (1947), Harary (1953), Cartwright and Harary (1956), Davis (1963)
Structural Balance
When do nodes interact?
Underlying World

\[ G = G^+ \cup G^- \]

- Fixed
- Undirected
- Deterministic
- Connected

\[ N_i = N_i^+ \cup N_i^- \]
Gossip Model
Gossip Model

Independent with other time and node states, at time $k$,
(i) A node $i$ is drawn with probability $1/N$;
(ii) Node $i$ selects one of its neighbor $j$ with probability $1/|\mathcal{N}_i|$.
How a pair of nodes interacts with each other when they meet?
A pair \((i,j)\) is randomly selected. The two selected nodes update.

\[
x_i(k + 1) = x_i(k) + \alpha (x_j(k) - x_i(k)) \quad 0 < \alpha < 1
\]

\[
x_i(k + 1) = x_i(k) + \beta (x_i(k) - x_j(k)) \quad \beta > 0
\]

Altafini 2013
Positive and Negative Interactions

Before:

\[ x_i(k + 1) = x_i(k) + \alpha(x_j(k) - x_i(k)) \]

After:

\[ x_i(k + 1) = x_i(k) - \beta(x_j(k) - x_i(k)) \]
Mean/Mean-Square Evolution
Relative-State-Flipping Model

\[ \mathbf{x}(k + 1) = W(k) \mathbf{x}(k) \]

\[ \mathbb{E}\{W(k)\} = I - \alpha L_{\text{pst}}^\dagger + \beta L_{\text{neg}}^\dagger. \]

- It’s an eigenvalue perturbation problem!
Phase Transition

**Theorem.** Suppose $G^+$ is connected and $G^-$ is non-empty. Then there exists $\beta_*$ such that

(i) $\lim_{k \to \infty} \mathbb{E}\{x_i(k)\} = \sum_{i=1}^{N} x_i(0)/N$ for all $i = 1, \ldots, N$ if $\beta < \beta_*$;

(ii) $\lim_{k \to \infty} \max_{i,j} ||\mathbb{E}\{x_i(k)\} - \mathbb{E}\{x_j(k)\}|| = \infty$ if $\beta > \beta_*$.

- It’s possible to prove that the expectation of the state transition matrix is *eventually positive*. 
Let $G$ be the complete graph. Let $G^-$ be the Erdos-Renyi random graph with link appearance probability $p$.

(i) If $p < \frac{\alpha}{\alpha + \beta}$, then

\[ P\left(\text{Consensus in expectation}\right) \to 1 \]

as the number of nodes $N$ tends to infinity;

(ii) If $p > \frac{\alpha}{\alpha + \beta}$, then

\[ P\left(\text{Divergence in expectation}\right) \to 1 \]

as the number of nodes $N$ tends to infinity.
Sample Path Behavior
Live-or-Die Lemma

Introduce

\[
\mathcal{C}_x^0 = \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\}, \quad \mathcal{D}_x^0 = \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\}
\]
\[
\mathcal{C}_x^* = \left\{ \liminf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\}, \quad \mathcal{D}_x^* = \left\{ \liminf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\}
\]

Lemma.
Suppose \( G^+ \) is connected. Then (i) \( \mathbb{P}(\mathcal{C}_x^0) + \mathbb{P}(\mathcal{D}_x^0) = 1 \); (ii) \( \mathbb{P}(\mathcal{C}_x^*) + \mathbb{P}(\mathcal{D}_x^*) = 1 \).
As a consequence, almost surely, one of the following events happens:

\[
\left\{ \lim_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \right\};
\]
\[
\left\{ \lim_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\};
\]
\[
\left\{ \liminf_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0; \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\}.
\]
Zero-One Law

\[ C \doteq \left\{ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = 0 \text{ for all } x^0 \in \mathbb{R}^n \right\}, \]

\[ D \doteq \left\{ \exists (\text{deterministic}) \ x^0 \in \mathbb{R}^n, \ s.t. \ \limsup_{k \to \infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty \right\} \]

**Theorem.** Both \( C \) and \( D \) are trivial events (i.e., each of them occurs with probability equal to either 1 or 0) and \( \mathbb{P}(C) + \mathbb{P}(D) = 1 \).
No-Survivor Theorem

**Theorem.** There always holds

\[
\mathbb{P}\left( \liminf_{k \to \infty} \left| x_i(k) - x_j(k) \right| = \infty \mid \liminf_{k \to \infty} \max_{i,j} \left| x_i(k) - x_j(k) \right| = \infty \right) = 1
\]

for all \( i \neq j \).
Phase Transition

Theorem.

(i) Suppose $G^+$ is connected. Then there is a $\beta_* > 0$ such that

$$\mathbb{P}\left( \lim_{k \to \infty} x_i(k) = \frac{\sum_{i=1}^{N} x_i(0)}{N} \right) = 1$$

for all $i$ if $\beta < \beta_*$. 

(ii) There is $\beta^* > 0$ such that

$$\mathbb{P}\left( \liminf_{k \to \infty} \max_{i,j} \| x_i(k) - x_j(k) \| = \infty \right) = 1$$

for all $i$ if $\beta > \beta^*$. 
Bounded State Model
Bounded States

• Let $A > 0$ be a constant and define $\mathcal{P}_A(\cdot)$ by $\mathcal{P}_A(z) = -A$, $z < -A$, $\mathcal{P}_A(z) = z$, $z \in [-A, A]$, and $\mathcal{P}_A(z) = A$, $z > A$.

• Define the function $\theta : E \to \mathbb{R}$ so that $\theta(\{i, j\}) = \alpha$ if $\{i, j\} \in E^+$ and $\theta(\{i, j\}) = -\beta$ if $\{i, j\} \in E^-$.

Consider the following node interaction under relative-state flipping rule:

$$x_s(t + 1) = \mathcal{P}_A((1 - \theta)x_s(t) + \theta x_{-s}(t)), \ s \in \{i, j\}.$$
Clustering of Opinions

**Theorem.** Let $\alpha \in (0, 1/2)$. Assume that $G$ is a weakly structurally balanced complete graph under the partition $V = V_1 \cup V_2 \cdots \cup V_m$ with $m \geq 2$. Let $\alpha \in (0, 1/2)$. When $\beta$ is sufficiently large, almost sure boundary clustering is achieved in the sense that for almost all initial value $x(0)$ w.r.t. Lebesgue measure, there are $m$ random variables, $l_1(x(0)), \ldots, l_m(x(0))$, each of which taking values in $\{-A, A\}$, such that:

$$P\left(\lim_{t \to \infty} x_i(t) = l_j(x(0)), i \in V_j, j = 1, \ldots, m\right) = 1.$$
Separation Events

- The power of minority groups.
Numerical Example
Theorem. Let \( \alpha \in (0, 1/2) \). Assume that \( G \) is a complete graph and the positive graph \( G^+ \) is connected. When \( \beta \) is sufficiently large, for almost all initial value \( x(0) \) w.r.t. Lebesgue measure, there holds for all \( i \in V \) that

\[
\mathbb{P} \left( \lim_{t \to \infty} \inf x_i(t) = -A, \lim_{t \to \infty} \sup x_i(t) = A \right) = 1.
\]
Numerical Example
Related Publications

Shi et al. 2013 *IEEE Journal on Selected Areas in Communications*;
Shi et al. 2015, 2016 *IEEE Transactions on Control of Network Systems*;
Thank you!