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Introduction and Motivation

Distributed algorithms
- Group of agents with simple/complex abilities
- Agents sense their local neighborhood
- Communicate with neighbors and process the information
- Perform a local action
- Emergence of a global behavior.

Effectiveness of these algorithms depends on:
- The speed of convergence
- Robustness to agent/connection failures
- Energy/communication efficiency

Graph theoretic abstraction of network
- Group topology affects group performance critically
- Graphs as structural abstractions of neighborhoods/connectivity
- Agents’ knowledge of connectivity effects their dynamics
- Structural properties of graphs characterized by relevant matrices

Important graph-related matrices
- Graph Laplacian: \( L = D - A \)
- Natural Random walk matrix \( P = D^{-1}A \)
- Spectrum provides important structural information

Objective

**Design problem:** Find graph topologies with favorable tradeoff between performance improvement (benefit) of collaborative behaviors vs. costs of collaboration

**Performance measures**
- The speed of convergence of many distributed algorithms is determined by Second Largest Eigenvalue (SLEM) of \( P \)
- The number of spanning trees of a graph is a measure of robustness to losses in many applications

**Problem 1** Characterization of Small World networks as efficient topologies (Asymptotic)

**Problem 2** Optimal performance enhancement by adding few links

Problem statement and Analysis

**Problem 1** Characterization of Small World graphs

- **\( \Phi \)-model:** Adding small number of new edges into a regular lattice \( G_0 = C(n,k) \), \( n \)-number of nodes, \( 2k \)-number of initial neighbors of each node

Capture performance measure of \( G \), as property of \( F(G) \)

Perturb zero elements of \( F_0 \) by \( \epsilon = Kn^{-1} \), \( \rightarrow F_0' \)

Start from base structure \( G_0, F_0 \)

Analyze \( \Delta(F_i) \) for large \( n \) as \( \alpha \) varies

Interpret the result as structural perturbation

Results for spectral gap gain

<table>
<thead>
<tr>
<th>Initial S.G.</th>
<th>SW onset</th>
<th>SW</th>
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<tbody>
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<td>( O(n^3) )</td>
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<td>( O(n) )</td>
<td>( x = \frac{1}{n} )</td>
<td>( x = \frac{1}{\sqrt{n}} )</td>
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**Problem 2:** Robust network design

Given a base graph topology, add \( k \) edges that result in maximum possible number of spanning trees

Matrix-tree theorem

\[
\tau(G) = \frac{1}{n} \prod_{i=1}^{n} \lambda_i(L) = \frac{1}{n} \det \left( L + \frac{1}{n} \right)
\]

Optimization problem

Let \( f_l \) denote the incidence vector for edge \( l \)

Maximize \( \tau \left( I_0 + \sum_{m=1}^{n} x_l f_l \right) \) or equivalently

\[
\log \det \left( I_0 + \frac{1}{n} J + \sum_{m=1}^{n} x_l f_l \right)
\]

Subject to:

\( 1^T x = k \)

We relax the problem to find heuristics for designing optimal topologies

Result:
- The optimal graph is determined as a compromise between symmetrizing the graph and minimizing a notion of distance, effective resistance distance.
- Effective resistance distance between two nodes \( i \) and \( j \): If we consider the graph as a resistive network with unit resistance on the edges, this is the effective resistance between \( i \) and \( j \) when a unit potential difference is applied between the two nodes
- The small world effect holds for spanning trees. Asymptotically, union of uniformly generated random spanning trees leads to construction of expander graphs.

References

[1] Baras and Hovareshti, Effects of topology in networked systems: stochastic methods and small worlds, CDC08.
[2] Baras and Hovareshti, Efficient and robust communication topologies for distributed decision making in networked systems, CDC09, Submitted