Reconstruction from Magnitudes of Redundant Representations
Radu Balan, Department of Mathematics, CSCAMM, ISR and NWC

Problem Formulation

Typical signal processing pipeline:

Features:
- Relative low complexity O(Nlog(N))
- On-line version is possible

The Analysis/Synthesis Components:

Example: Short-Time Fourier Transform
\[ c_{k,f} = \langle x(t), g_{k,f}(t) \rangle = \sum_{t \in \mathbb{Z}} x(t)g(t-kb) e^{-2\pi j ft} \]

Problem: Given the Short-Time Fourier Amplitudes (STFA):
\[ d_{k,f} = \left| \langle x(t), g_{k,f}(t) \rangle \right|^2 = tr\left\{ K_{x,k} \cdot \overline{K_{y,k,f}} \right\}_{HS} \]
we want an efficient reconstruction algorithm:
- Reduced computational complexity
- On-line ("on-the-fly") processing

Solution/Approach

First observation [BBCE'09]:
\[ d_{k,f} = \left| \langle x(t), g_{k,f}(t) \rangle \right|^2 = tr\left\{ K_{x,k} \cdot \overline{K_{y,k,f}} \right\}_{HS} \]
\[ K_{x,y}(y) = \langle y, x \rangle , \quad K_{g,k,f}(y) = \langle y, g_{k,f} \rangle \]

Assume \( \{K_{g,k,f}\} \) form a frame for its span, \( E \). Then the projection \( P_E \) can be written as:
\[ P_E = \sum_{k,f} \left\langle \overline{Q_{k,f}}, K_{x,k} \right\rangle_{HS} Q_{k,f} \]

Frame operator
\[ X \mapsto S(X) = \sum_{k,f} \left\langle X, K_{x,k} \right\rangle_{HS} K_{x,k} \]
\[ Q_{k,f} = S^{-1}(K_{x,k}) \]

Second Observation: Since
\[ 1 \mapsto M^T g \]
\[ G_{k,f} = M^T \Theta^* g \]
where \( M : h \mapsto M h(t) , \quad T : h \mapsto Th(t) = h(t-b) \)

it follows:
\[ K_{x,k} = N' \Theta^* N \quad Q_{k,f} = N' \Theta^* Q_{0,0} \]

where \( N : X \mapsto AX = MXM^T , \quad \Theta : X \mapsto \Theta X = TXT^T \)

Explicitly
\[ \left\langle Q_{k,f}, \chi \right\rangle = e^{2\pi j f(t_1-t_2)/F} \left\langle Q_{0,0}, \chi_{-k,k} \right\rangle \]

Solution/Approach

Set:
\[ H(\omega, m) = \sum_{k} e^{-\frac{2\pi j k\omega}{F}} \left| g_{k,0} \right|^2 \]

Main Result:

Theorem Assume \( \{g_{k,0}\}_{k\in\mathbb{Z}} \) is a frame for \( F(\mathbb{Z}) \).

1. is a frame for its span in \( HS(F(\mathbb{Z})) \) iff for each \( m \in \mathbb{Z} \), \( H(m,\omega) \) either vanishes identically in \( \omega \), or it is never zero;
2. is a Riesz basis for its span in \( HS(F(\mathbb{Z})) \) iff for each \( m \in \mathbb{Z} \) and \( \omega \), \( H(m,\omega) \) is never zero.

Final step: Signal reconstruction from \( Q_{k,f} \) by solving:
\[ \min \left\{ \left| x - (K_{x,k}) \right| + w_1 |x_{-1} - (K_{x,k})| + \cdots + w_L |x_{-L} - (K_{x,k})| \right\} \]
for \( x \), assuming we already estimated \( x \) for \( s < t \).

Reconstruction scheme:

Example:

\[ W^T(z^-) = \sum_{m} \left\langle Q_{k,f}, \chi \right\rangle z^- \]

Example:

Set:
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