Large Scale Curvature of Networks:
And Implications for Network Management and Security

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Understanding Large-Scale Networks

- Hard to visualize due to scale
- Unclear what is essential and what is not for overall performance, reliability and security
- Much of the existing work on “complex networks” focuses on local parameters, degree distribution, clustering coefficients, etc.
- Need more fundamental ways to “summarize” critical network information
- A promising direction is to look at key geometric characteristics: dimension and curvature

Rocketfuel dataset 7018
10152 nodes, 28638 links, diameter 12
Dimension -- Degrees of Freedom
Dimension of a Lattice & Average Shortest Path Lengths

- How fast does a “ball” grow in a lattice?

<table>
<thead>
<tr>
<th>Circumference of Configuration (dimension D, degree d)</th>
<th>1-hop away</th>
<th>2-hops away</th>
<th>3-hops away</th>
<th>4-hops away</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square (D=2, d=4)</td>
<td>4=4*1₁</td>
<td>8=4*2₁</td>
<td>12=4*3₁</td>
<td>16=4*4₁</td>
</tr>
<tr>
<td>Hexagon (D=2, d=3)</td>
<td>3=3*1₁</td>
<td>6=3*2₁</td>
<td>9=3*3₁</td>
<td>12=3*4₁</td>
</tr>
<tr>
<td>Triangle (D=2, d=6)</td>
<td>6=6*1₁</td>
<td>12=6*2₁</td>
<td>18=6*3₁</td>
<td>24=6*4₁</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>General (D, d)</td>
<td>d*1^{(D-1)}</td>
<td>d*2^{(D-1)}</td>
<td>d*3^{(D-1)}</td>
<td>d*4^{(D-1)}</td>
</tr>
</tbody>
</table>

In a Euclidean grid in dimension D: (a) Volume within h hops scales like h^D & (b) average length of a shortest path <h> ≈ (D/D+1)(DN/d)^{1/D} ≈ O(N^{1/D})
Dimension

Dimension of a Network & Its Average Shortest Path Length

Measure the number of neighbors of a node $X$ $h$ hops away. How does this number scale with $h$? If roughly like $h^\Delta$ then we say $\Delta$ is the dimension of the graph in the neighborhood of $X$.

<table>
<thead>
<tr>
<th>Node</th>
<th>1-hop away</th>
<th>2-hops away</th>
<th>3-hops away</th>
<th>4-hops away</th>
</tr>
</thead>
<tbody>
<tr>
<td>J (NSF)</td>
<td>3</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E (NSF)</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>B (NSF)</td>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A (lattice)</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>B (lattice)</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

$v(r) \sim r^D \Rightarrow \log(v(r)) \sim D \log(r)$

$D_{\text{small-graph}} \approx 1.7$
Look at scaling of the average shortest path length $<h>$

- In 2-dim grid, $<h> \sim \sqrt{N}$ (or $\sim N^{1/D}$ in D-dimensional grid)

- Look at “Rocketfuel” data, [Washington University researchers’ detailed connectivity data from various ISPs 2002-2003]

- $<h>$ does not scale like $\sqrt{N}$ or $N^{1/D}$ but are more like $\log(N)$ -- “Small World” like

**=> RF networks do not appear to be grid-like (or flat) nor do they exhibit characteristics of finite dimensions**
Curvature -- Deviation from the Flat
Curvature

Basic Geometry: Vertex Curvature of Polyhedra

- A vertex has “angle defect” when
  \[ k(v) = 2\pi - \sum_{f \text{ face } f} \alpha_f > 0 \]
  -- positive curvature or “spherical”

- A vertex has “angle excess” when
  \[ k(v) = 2\pi - \sum_{f \text{ face } f} \alpha_f < 0 \]
  -- negative curvature or “hyperbolic”

- By Descartes’ theorem for polyhedra
  \[ \sum_{v \in P} k(v) = 2\pi \chi(P) \]
  where \( \chi(P) \) is the Euler characteristic of the polyhedron (typically equal to \( V - E + F = 2 \))
Curvature

Combinatorial Vertex Curvature for Planar Graphs

One could imitate the previous definition to define a \textit{combinatorial angular defect/excess} at vertices of a planar graph (net of $2\pi$). E.g.,

$$k(v) = 2\pi - \left(\frac{1}{2} \cdot \frac{2}{4} \cdot 2\pi + \frac{\pi}{3} + \frac{1}{2} \cdot \frac{3}{5} \cdot 2\pi + \frac{\pi}{3} + \frac{\pi}{3}\right)$$

In effect, assume each face is a regular \textit{n-gon}, compute the facial angles, add up and subtract from $2\pi$ \cite{Higuchi01}

$$k(v) = 2\pi - \sum_{f \in G} \frac{1}{2} \cdot \frac{2\pi}{f} (f-2) = 2\pi \left(1 - \frac{d(v)}{2} \right) + \sum_{v \in G} \frac{1}{f}$$

\textbf{Gauss-Bonnet theorem (extension of Descartes’)} then states

$$\sum_{v \in G} k(v) = 2\pi \chi(G) = 2\pi (2 - 0)$$

\begin{itemize}
  \item \(G\) is a \textit{planar} graph.
  \item \(\chi(G)\) is the \textit{Euler characteristic} of \(G\).
  \item \(k(v)\) is the \textit{combinatorial angular defect} at vertex \(v\).
\end{itemize}
What can be said about non-planar graphs? Use the fact that all finite graphs are \textit{locally} planar.

\textbf{[Ringel-Youngs ‘68]} (‘‘All graphs with \(N \geq 3\) nodes are locally 2-dimensional.’’) For \(N \geq 3\), any \(G=(N,L)\) can be embedded in \(T_g\), a torus with \(g\) holes, where

\[
g \leq \left\lfloor \frac{(N-3)(N-4)}{12} \right\rfloor
\]

The minimal \(g\) is called the \textit{genus} of the graph \(G\).
Curvature
Non-Planar Graphs: Combinatorial Curvature

But there is more that we need:

[Edmonds-Heffter? see Mohar-Thomassen and others]. The above embedding can always be done “strongly”, i.e., where the resulting embedding on $T^g$ has faces that are 2-cells (equivalent to disks).

Now with well-defined faces, the previous definition of vertex curvature can be reused:

$$k(v) = 2\pi(1 - \frac{d(v)}{2} + \sum_{f \in \mathcal{F}} \frac{1}{f})$$

Gauss-Bonnet then states:

$$\sum_{v \in G} k(v) = 2\pi \chi(G) = 2\pi(2 - 2g)$$

$\kappa(A) = 1 - \frac{4}{2} + 4 \cdot \frac{1}{3} = \frac{1}{3}$

$\kappa(B, C, D, E) = -\frac{1}{12}$

B, C, D and E are vertices of the one octagonal face.
Curvature
Summary (So Far)

[Exercise] What is the genus of $K_7$? Identify all faces of the strong minimal embedding of $K_7$ on $T'$. Compute the curvature at each vertex. Verify that $\chi(K_7)=2-2g$.

The Euler Characteristic of a graph is an intrinsic invariant that determines its total (combinatorial) curvature*. We say a graph is

- “flat” when $\chi(G)=0$
- “spherical” when $\chi(G)>0$
- “hyperbolic” when $\chi(G)<0$

Note. It is not easy to compute $\chi(G)$ for large scale networks!

* There is also a similar concept of “discrete curvature” for graphs that uses actual edge lengths and angles. It results in the same $\chi(G)$.
Dimension and Curvature

So far:

Managed to define (relatively) satisfactory notions of dimension and curvature for networks but

- **dimension** does not appear to be finite
- **curvature** does not appear to be computable!

Need something better to work with!

Possible alternative: Consider metric structure of networks
Other Locally 2-Dimensional Models: The Poincaré Disk $H^2$

Consider the unit disk $\{x \in \mathbb{R}^2; |x| < 1\}$ with length metric given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

the hyperbolic metric.

Advantages

- In the small scale it is 2-dimensional, but has much slower scaling of geodesics (shortest paths) than $\sqrt{N}$
- Has meaningful small-scale and large-scale curvatures

Relationship to graphs? The Poincaré disk comes with numerous natural “scaffoldings” or “tilings”.

A few geodesics
Scaffoldings of $H^2$: Hyperbolic Regular Graphs

Consider $X_{p,q}$ tilings (isometries) of $H^2$, that at each vertex consist of $q$ regular $p$-gons for integers $p$ & $q$ with $(p-2)(q-2)>4$ (flat with equality)

Examples:

Note. Since networks of interest to us are typically finite, we’ll consider truncations of $X_{p,q}$, the part within a (large enough) radius $r$ from the center. Call this $TX_{p,q}$. 
Some Key Properties of $X_{p,q}$

1. Negative local curvature. The local combinatorial curvature at each node of $X_{p,q}$ is negative
   \[ \kappa_v = 2\pi \left\{ 1 - \frac{q}{2} + \frac{q}{p} \right\} = 2\pi \left\{ \frac{4 - (p-2)(q-2)}{2p} \right\} < 0 \]
   or $\kappa_v = 2\pi q \left( \frac{1}{p} + \frac{1}{q} - \frac{1}{2} \right) < 0$

2. Exponential growth. Number of nodes within a ball of radius $r$ is proportional to $\lambda^r$ for some $\lambda \equiv \lambda(p,q) > 1$ (e.g., for $X_{3,7}$, $\lambda = \phi$, the golden ratio) or equivalently

2’. Logarithmic scaling of geodesics. For (a finite truncation of) $X_{p,q}$ with $N$ nodes, the average geodesic (shortest path length) scales like $O(\log(N))$
Curvature in the Large: Geodesic Metric Spaces

- Computation of total curvature of non-flat networks with varying nodal degrees via $\sum_{v \in G} \kappa_v$ does not appear to be possible/easy nor does it provide information about the large-scale properties of networks.

- A more direct definition of (negative) curvature in the large is the thin-triangle condition for a geodesic metric space (or a CAT($\kappa$) space):

[M. Gromov’s Thin Triangle Condition for a hyperbolic geodesic metric space] There is a (minimal) value $\delta \geq 0$ such that for any three nodes of the graph connected to each other by geodesics, each geodesic is within the $\delta$-neighborhood of the union of the other two.

Example. For $H^2$, $\delta = \ln(\sqrt{2} + 1)$. [Sketch. Largest inscribed circle must be in largest area triangle, $\text{Area}_H(ABC) = \pi(-\alpha + \beta + \chi)$, maximized to $\pi$ when $\alpha, \beta, \chi = 0$ or when $A, B, C$ are on the boundary.]
What Can We Say About Communication Networks?

Communication networks are (geodesic) metric spaces via reasonable link metrics (e.g., the hop metric).

Is there evidence for negative curvature in real networks?

We consider 10 Rocketfuel networks and some prototypically flat or curved famous synthetic networks to test this hypothesis.

<table>
<thead>
<tr>
<th>Network ID</th>
<th>Network Name</th>
<th>Number of nodes</th>
<th>Number of links</th>
<th>Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>1221</td>
<td>Telstra (Aust.)</td>
<td>2998</td>
<td>3806</td>
<td>12</td>
</tr>
<tr>
<td>1239</td>
<td>Sprintlink (US)</td>
<td>8341</td>
<td>14025</td>
<td>13</td>
</tr>
<tr>
<td>1755</td>
<td>EBONE (US)</td>
<td>605</td>
<td>1035</td>
<td>13</td>
</tr>
<tr>
<td>2914</td>
<td>Verio (US)</td>
<td>3045</td>
<td>12291</td>
<td>13</td>
</tr>
<tr>
<td>3257</td>
<td>Tiscali (EU)</td>
<td>855</td>
<td>1173</td>
<td>14</td>
</tr>
<tr>
<td>3356</td>
<td>Level 3 (US)</td>
<td>3447</td>
<td>9390</td>
<td>11</td>
</tr>
<tr>
<td>3967</td>
<td>Exodus (US)</td>
<td>895</td>
<td>2070</td>
<td>13</td>
</tr>
<tr>
<td>4755</td>
<td>VSNL (India)</td>
<td>121</td>
<td>228</td>
<td>6</td>
</tr>
<tr>
<td>6461</td>
<td>AboveNet (US)</td>
<td>2720</td>
<td>3824</td>
<td>12</td>
</tr>
<tr>
<td>7018</td>
<td>AT&amp;T (US)</td>
<td>10152</td>
<td>14319</td>
<td>12</td>
</tr>
</tbody>
</table>

| Hyperbolic 3-7 grid | X₃,₇, synthetic | 4264          | 7511            | 14       |
| Barabasi-Albert   | (B-A), synthetic | 10000         | 19997           | 9        |
| Watts-Strogatz    | (W-S), synthetic | 80x80         | 13289           | 20       |
| Triangular lattice | synthetic        | 469           | 1260            | 24       |
| Square lattice    | synthetic        | 80x80         | 12640           | 158      |
| Erdos-Renyi       | (E-R), synthetic | 7902          | 20132           | 30       |

In RF data, a node is a unique IP address and a link is a (logical) connection between a pair of IP addresses enabled by routers, physical wires between ports, MPLS, etc.
Rocketfuel IP Networks

1221/Telstra
1239/Sprintlink
1755/Ebone
2914/Verio
3257/Tiscali
3356/Level3
3967/Exodus
4755/VSNL
6461/Abovenet
7018/AT&T
Some “Famous” Synthetic Networks: E-R, W-S, B-A

W-S: Grid with small (1-5%) additional random links

B-A: Start with some nodes and add nodes sequentially and at each iteration join new node to existing node $i$ with probability

$p = d(i)/\Sigma d(i)$

Then $P(k) \sim k^{-3}$

$p < 1/N$, GC $\sim O(\ln(N))$

$p = d/N$, GC $\sim O(N)$ for $d > 1$

$p \sim \ln(N)/N$, $G(N,p)$ a.s. connected
Experiments and Methodology

We ran experiments on all Rocketfuel networks plus a few prototypical flat/curved networks to test our key hypothesis:

1. Dimension. “Growth test” - Polynomial or exponential?
   - Consider the volume $V(r)$ as a function of radius $r$ for arbitrary centers
   
   [In flat graphs volume growth is typically polynomial in radius $r$]

2. Curvature. “Triangle test” - Are triangles are universally $\delta$-thin
   - Randomly selected 32M, 16M, 1.6M triangles for networks with more than 1K nodes and exhaustively for the remainder
   - For each triangle noted shortest side $L$ and computed the $\delta$
   - Counted number of such triangles, indexed by $\delta$ and $L$
   
   [In flat graphs $\delta$ grows without bound as the size of the smallest side increases]

We conduct “growth” and “triangle” tests
1. Growth Charts

Recall that:
Euclidean growth $V(r) \approx r^D$
then dimension is “D”
Exponential growth $V(r) \approx \theta^r$
then dimension is “infinity”

*Volume* (number of points within distance $r$) as a function of *radius* $r$
from a “center” of the graph. Flattening of curves for larger $r$ is due
to boundary effects / finite size of network.
2. Triangle Test - Rocketfuel 7018 & Triangular Grid

(a) Probability $P_L(\delta)$ for randomly chosen triangles whose shortest side is L to have a given $\delta$ for the network 7018 (AT&T network) which has 10152 nodes and 14319 bi-directional links and diameter 12. The quantities $\delta$ and L are restricted to integers, and the smooth plot is by interpolation.

(b) Similar to (a), for a (flat) triangular lattice with 469 nodes and 1260 links. (The smaller number of nodes is sufficient for comparing with (a) since the range for L is large due to the absence of the small world effect.)
The average δ as a function of L, E[δ](L), for the 10 IP-layer networks studied here, and for the Barabasi-Albert model with k = 2 and N = 10000 (11th curve) and the hyperbolic grid X3,7 (12th curve). On the other hand, a Watts- Strogatz type model on a square lattice with N = 6400, open boundary conditions and 5% extra random connections (13th curve) and two flat grids (the triangular lattice with diameter 29 and the square lattice with diameter 154) are also shown.
Where to go from here?

- OK, these ten RF datasets and some “well-known” large-scale networks exhibit
  - Exponential growth / logarithmic scaling of shortest paths
  - Negative curvature in the large

So what?

Turns out negatively curved networks exhibit specific features that affect their critical properties -- Existence of a “core”:
  - $O(N^2)$ scaling of “load” (1 unit between all node pairs)
  - Non-random points of critical failures
  - Non-random points of security
The Downside of Hyperbolicity: Quadratic Scaling of Load ("Betweenness Centrality" and Existence of "Core")

Plot of the maximum load $L_c(N)$ -- maximal number of geodesics intersecting at a node -- for each network in the Rocketfuel database as a function of the number of nodes $N$ in the network. Also shown are the maximum load for the hyperbolic grid $X_{3,7}$, the Barabasi-Albert model with $k = 2$, the Watts-Strogatz model and a triangular lattice, for various $N$. The dashed lines have slopes of 2.0 and 1.5, corresponding to the hyperbolic and Euclidean cases respectively.
Metric Properties of RF and Other networks

- So far we worked with the unit-cost (hop) metric
- Can things change significantly through changes in the metric?
- Yes, and no! Look at toy networks again:

- Metrics can change things but evidently not by that much! (Have some rigorous proofs that show by how much)
Downside of Metric Changes: Long Paths (w.r.t. the hop metric)

Even if we can eliminate $O(N^2)$ scaling of load via metric changes, we’re liable to pay a (big) price:

[Bridson-Haefliger] Let $X$ be a $\delta$-hyperbolic geodesic metric space. Let $C$ be a path in $X$ with end points $p$ and $q$. Let $[p,q]$ be the geodesic path. Then for every $x$ on $[p,q]$,

\[
d(x, C) \leq \delta \| \log_2 l(C) \| + 1
\]

where $l(C)$ is the length of $C$.

Open Question. Can paths with small deviations from geodesics decrease “load” by much? [Unlikely in the mathematical sense but perhaps yes in practice.]
Key Claims:
Network Curvature -> Congestion, Reliability and Security

Numerical studies show that congestion is a property of the large-scale geometry of the networks - large-scale curvature -- and does not necessarily occur at vertices of high degree but rather at the points of high cross-section (the “core”)

At the “core” -- intersection of largest number of shortest paths - load scales as quadratic as function of network size

Shortest path routings

- (Upside) Are very effective, as diameter is small compared to N, e.g., TTL of ~20 good enough for all of the Internet!
- (Downside) Lead to
  - congestion
  - non-random failure can be severe
  - certain nodes are exhibit more significant security-wise

\[
X_{3,7}
\]

Nodal loads need not be related to nodal degrees
A Taxonomy for Large-Scale Networks Based on the Curvature Plot

Taxonomy of networks based on large-scale curvature characteristics of networks as indicated by measurements and curvature plots

Can we formally prove any of this?
Taxonomy Verification. Easy Cases

Small World Networks

- Erdős-Rényi model
- Watts-Strogatz model
- Hyperbolic grids
- Bethe lattice
- Chains

δ-Hyperbolic Networks

- PLDD Networks
- "Hairy" chains
- Euclidean grids

"Hairy" Euclidean grids
"Hairy" Watts-Strogatz model

Power-law trees
Analytical Verification of CP Results

1) $E_{p,q}$ (regular Euclidean grid, degrees=q, all faces p-gons, $(p-2)(q-2)=4$)

1. Local curvature. The local combinatorial curvature at each interior node of $X_{p,q}$ is negative

$$\kappa_v = 2\pi \left\{1 - \frac{q}{2} + \frac{q}{p}\right\} = 2\pi \left\{\frac{4-(p-2)(q-2)}{2p}\right\} = 0$$

1’. Large-scale curvature. For $E^2$, $\delta = \infty$. [Radius of the largest inscribed circle increases without bound as the perimeter of triangle / size of smallest side increases.]

2. Polynomial growth. Number of nodes within a ball of radius $r$ is proportional to $r^2$

2’. Scaling of geodesics. For (a finite truncation of) $E_{p,q}$ with $N$ nodes, the average geodesic (shortest path length) scales like $O(\sqrt{N})$
Analytical Verification of CP Results

2. \( X_{p,q} \) (regular grid degrees=\( q \), all faces \( p \)-gons, \( (p-2)(q-2)>4 \))

1. **Local curvature.** The local combinatorial curvature at each interior node of \( X_{p,q} \) is negative

\[
\kappa_v = 2\pi \left( 1 - \frac{q}{2} \right) = 2\pi \left( \frac{4 - (p-2)(q-2)}{2p} \right) < 0
\]

or

\[
\kappa_v = 2\pi q \left( \frac{1}{p} + \frac{1}{q} - \frac{1}{2} \right) < 0
\]

1’. **Large-scale curvature.** For \( H^2 \), \( \delta = \ln(\sqrt{2}+1) \).

[Largest inscribed circle must be in largest area triangle, \( \text{Area}_{H}(\text{ABC}) = \pi \cdot (\alpha+\beta+\gamma) \), maximized to \( \pi \) when \( \alpha, \beta, \gamma=0 \) or when \( A, B, C \) are on the boundary.]

2. **Exponential growth.** Number of nodes within a ball of radius \( r \) is proportional to \( \lambda^r \) for some \( \lambda \equiv \lambda(p,q) > 1 \) (e.g., for \( X_{3,7} \), \( \lambda = \phi \), the golden ratio) or equivalently \( \Rightarrow \) “D” = \( \infty \)

2’. **Logarithmic scaling of geodesics.** For (a finite truncation of) \( X_{p,q} \) with \( N \) nodes, the average geodesic (shortest path length) scales like \( O(\log(N)) \)
Analytical Verification of CP Results

3) *E-R random graph (the sparse regime)* $G(N,p=d/N)$

1. **Negative local curvature.**
   1. Clearly $E[q] = d$
   2. What are $g$ (typical genus) and $p$ (typical size of faces) of $G(N,p=d/N)$ as $N \to \infty$?

1’. **Large-scale curvature. Curvature Plot:**
   Does $\delta \to \infty$ with $L$?

2. **Exponential growth & Logarithmic scaling of geodesics.** It is well-known that diameter of $G(N,p=d/N)$ *scales with log(N)*. So *E-R trivially is Small World*
The Case of $G(N, p=d/N)$ -- 1

Some basic facts about $G(N, d/N)$ with $d$ fixed and $> 1$.

- Giant component size $O(N)$
- Average degree is $d$ with Poisson distribution as $N \to \infty$
- Locally tree-like -- thus likely hyperbolic but CP says differently!

![Diagram of $G(1000; p=2/1000)$]
The Case of $G(N,p=d/N)$ -- 2

**Theorem [NST].** Given any $\delta \geq 0$, for large enough $n$ the probability of a $\delta$-wide triangle in $G(n, d/n)$ has a strictly positive lower bound independent of $n$.

**Sketch of Proof.** 1) Estimate prob. of a given $6\delta$-long loop with a single connection to the GC, 2) Derive lower bound on prob. of a $6\delta$-long loop, 3) Show limit as $n \to \infty$ is positive. Let $\Delta = 6\delta$ and pick any $\Delta$ points out of $n$.

1) $q = \Pr\{(n_1, n_2, \ldots, n_{\Delta})\} = \frac{\Delta!}{2} (n - \Delta) \cdot p^{\Delta+1} (1 - p) \frac{n(n-1)}{2} - \frac{(n-\Delta)(n-\Delta-1)}{2} - (\Delta+1)$

\[= \frac{\Delta!}{2} (n - \Delta) \cdot p^{\Delta+1} (1 - p) \frac{\Delta(2n-3)}{2} - \frac{\Delta^2 + 2}{2}\]
The Case of $G(N,p=d/N)$ -- 3

**Sketch of Proof.** 2) Derive lower bound on prob. of a $6\delta$-long loop

2) $\Pr\{(n_1, n_2, \ldots, n_\Delta) \text{ and some other } \Delta\text{-wide loop}\} \leq \left(\frac{n-\Delta}{\Delta}\right) q^2$

Thus

$\Pr\{(n_1, n_2, \ldots, n_\Delta) \text{ and no other } \Delta\text{-wide loop }\} \geq q - \left(\frac{n-\Delta}{\Delta}\right) q^2$

& therefore

$\Pr\{\text{some 1-connected } \Delta\text{-wide loop}\} \geq \left(\frac{n}{\Delta}\right) (q - \left(\frac{n-\Delta}{\Delta}\right) q^2) > \left(\frac{n}{\Delta}\right) q - \left(\frac{n}{\Delta}\right) q^2$

3) After some asymptotics and algebra, get

$$1 > 2e^{-d\Delta} d^{\Delta+1} > \left(\frac{n}{\Delta}\right) q$$

which establishes that

$$\lim_{n \to \infty} \Pr\{1\text{-connected } \Delta\text{-wide loop}\} > 0.//$$
The actual density of $\delta$-wide triangles is much higher than the lower bound just proved. As the curvature plot shows that a significant fraction of triangles in $G(n,p=d/n)$ are $\delta$-wide.
SOME CHALLENGES: Impact of Curvature on CDNs, Cloud, etc.

• Analysis of larger datasets
  ▪ Communication data
  ▪ Biological data
  ▪ Social network data

• Scaling of algorithms for detection of hyperbolicity in much larger graphs (of ~10^9 nodes)

• How does “negative curvature in the large” affect performance, reliability and security?
  ▪ Speed of information/virus spread \rightarrow spectral properties of large graphs
  ▪ Impact of correlated failures \rightarrow Core versus non-core

• How does the O(N^2) scaling of load change as a function of alternative load profiles, e.g., for localization in CDNs?

• How O(N^2) affect reliability and security? Does a core add or diminish robustness / security?

• How to leverage hyperbolicity for data centers / cloud / virtualization? Are there fundamental designs?

• How to leverage hyperbolicity for caching and CDNs? DHTs?
Some Recent References


