Constraint Reduced Algorithms for MPC-based Rotorcraft Control

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Abstract

Rotorcraft, in addition to being open-loop unstable, can have a complicated flight envelope, making them notoriously hard to control. Typically, rotorcraft control currently is based on a mixer, whose aim is to decouple the pitch, roll and yaw commands, and on single-axis controllers, usually of the proportional-integral type, with scheduled gains. Constraint avoidance is then added"after the fact" to the basic controllers.

Model-predictive control (MPC) is significantly more powerful. It has been used extensively in process control. Its application to aircraft control is challenging due to its high computational complexity (it involves that solution of a constrained optimization problem at every time step) in the face of the much smaller time constants in the systems dynamics.

In this project, we explore the feasibility of MPC in rotorcraft control, based on advances in computer hardware and software, in control theory, and in numerical optimization. In particular, the project features recent work by this research group on a "constraint reduction" approach to solving optimization problem with a large number of constraints.

Model Predictive Control

An MPC controller for rotorcraft control would proceed as follows. At time step 0, solve the quadratic program (QP):

$$\min_{u,x} \sum_{t=0}^{M-1}(u_t-u_s)^T R(u_t-u_s)+\sum_{t=1}^{P}(x_t-x_s)^T Q(x_t-x_s) \quad (1)$$

s.t.

$$u_{min} \le u_i \le u_{max} \quad \text{for } i=1,\cdots,M \quad (2)$$
$$x_{min} \le x_i \le x_{max} \quad \text{for } i=1,\cdots,P \quad (3)$$
$$\delta u_{min} \le u_{t+1}-u_t \le \delta u_{max} \quad \text{for } t=0,\cdots,M-1 \quad (4)$$
$$x_{t+1}=A_s x_t + B_s u_t \quad (5)$$

where $A_s \in \mathbb{R}^{m\times m}$ and $B_s \in \mathbb{R}^{m\times n}$ are constant matrix; $x_t \in \mathbb{R}^{m\times 1}, u_t \in \mathbb{R}^{n\times 1}$ denote the state and the input control respectively at time step t; the vector $x=\{x_1,\cdots,x_P\}$ and $u=\{u_1,\cdots,u_M\}$. After obtaining an optimal sequence of M controls, apply only the first control value to the dynamic system. Time then moves forward one step and the same QP is considered again using the new state of the system as initial state. Thus one continuously uses the current control based on the current state and accounts for the constraints on u over a control horizon of length M and the constraints on states x over a prediction horizon of length P. Thus at each time step, we solve QP (1)-(5) with varying initial states. Deriving the dynamic equation (5) for x in terms of u and substituting in (1)-(5) yields the QP in inequality form [GDS05]:

$$\min_{u} f^T u + \tfrac{1}{2}u^T H u \quad (6)$$
$$\text{s.t.} \quad Au \le b \quad (7)$$

with $A \in \mathbb{R}^{n\times m}$ and other matrices with appropriate dimensions by context.

Affine-Scaling Interior Point Methods

The KKT conditions for Problem (6)-(7) are:

$$Hx - A^T v + f = 0 \quad (8)$$
$$Ax - b - s = 0 \quad (9)$$
$$Sv = 0 \quad (10)$$
$$s \ge 0, v \ge 0 \quad (11)$$

where $s \in \mathbb{R}^n$ are the slack variables for the inequalities and $v \in \mathbb{R}^m$ is the Lagrange multiplier. Affine scaling interior point methods (IPMs) solve the QP by applying Newton's iteration (with respect to variables x, v and s) on equations (8)-(10) while selecting step size such that $s > 0$ and $x > 0$ are maintained thoughout. Block Gaussian elimination on the Newton system yields the "normal" system:

$$(H + A^T S^{-1} X A)\Delta x = -(Hx+f) \quad (12)$$
$$\Delta s = A\Delta x \quad (13)$$
$$\Delta v = -v - S^{-1}V\Delta s \quad (14)$$

with $S=\mathrm{diag}\{s\}, X=\mathrm{diag}\{x\}, V=\mathrm{diag}\{v\}$. The main computational task is to form the matrix $H + A^T S^{-1} X A$, at a cost of $O(m^2 n)$.

Constraint-Reduced Affine Scaling IPM

Suppose that the dimensions of control vector u_t and state vector x_t are N_u and N_x respectively. Then the QP (6)-(7) will have $n = 4MN_u + 2PN_x$ constraints while the number of variables is $m = MN_u$. Usually $P > M$ and $N_x > N_u$ so that $n \gg m$. The constraint-reduced affine scaling interior point method implemented by algorithm *CRCQP*, proposed and analyzed in [JOT08], is especially tailored to such situations; see also [TAW06] and [WNTO07]. *CRCQP* works with the constraints that are likely to be active (almost active) at each iteration while ignoring those that are not. We include the indices of those almost active constraints in index set Q. At each iteration, we solve a Newton system corresponding to the reduced problem:

$$\min_{u} f^T u + \tfrac{1}{2}u^T H u \quad (15)$$
$$\text{s.t.} \quad A_Q u \le b_Q \quad (16)$$

(Throughout, subscript Q indicates that only the constraints in index set Q are included.) Note that since x is updated at every iteration, Q also varies at every iteration. The normal system (12) reduces to:

$$(H + A_Q^T S_Q^{-1} X_Q A_Q)\Delta x = -(Hx+f) \quad (17)$$

Note that the cost of forming $A_Q^T S_Q^{-1} X_Q A_Q$ becomes $m^2|Q|$ where $m \le |Q| \le n$ is the number of elements in the set Q. When $n \gg m$, $m^2|Q|$ can be much less than $m^2 n$.

Results

- Our test problem is given by real world data provided by Sikorsky with $m = 30$ variables, $n = 520$ constraints.
- Figure 1 shows the total time and total number of iterations for the solutions of all 1000 QPs (one at each time step) vs. the size $|Q|$ of the reduced constraint set.

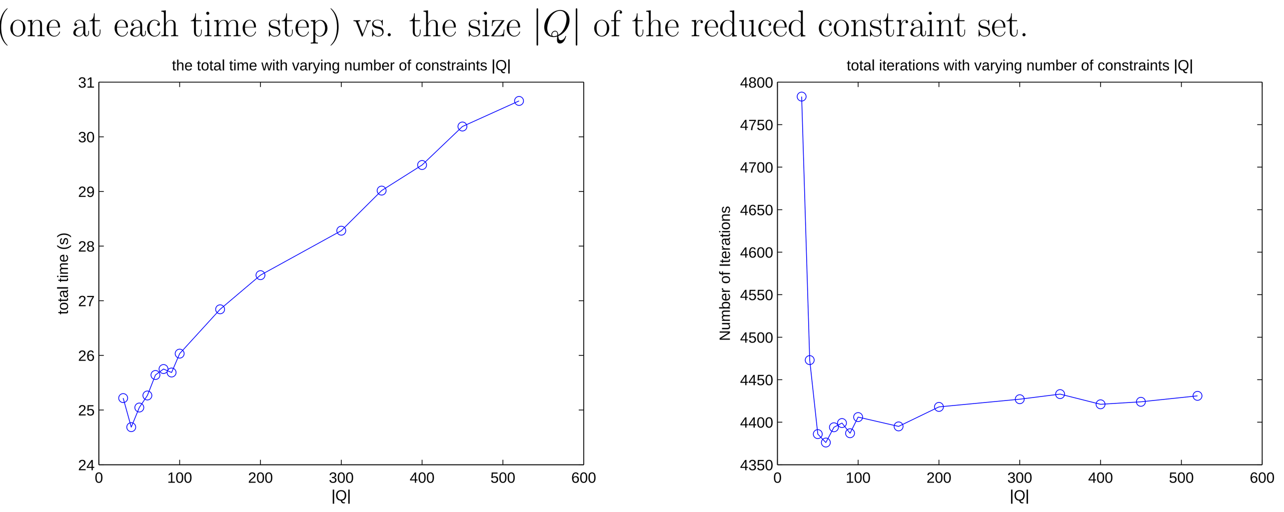

Figure 1: The total time and number of iterations vs. number of kept constraints.

The computation time is determined by both the number of iterations and the time per iteration. For each iteration, small Q leads to small computation cost given other same parameters. Note in Figure 1, the number of iteration does not change much for Q as small as about 10% of the number of constraints (right-hand plot). Accordingly, constraint reduction yields a dramatic speed-up (left-hand plot).

- Among the QPs solved at all 1000 time steps, we note that the one solved at time $t = 0.14s$ turns out to be the most time consuming, hence is the "bottleneck". Figure 2 shows the total time and number of iterations at time 0.14.

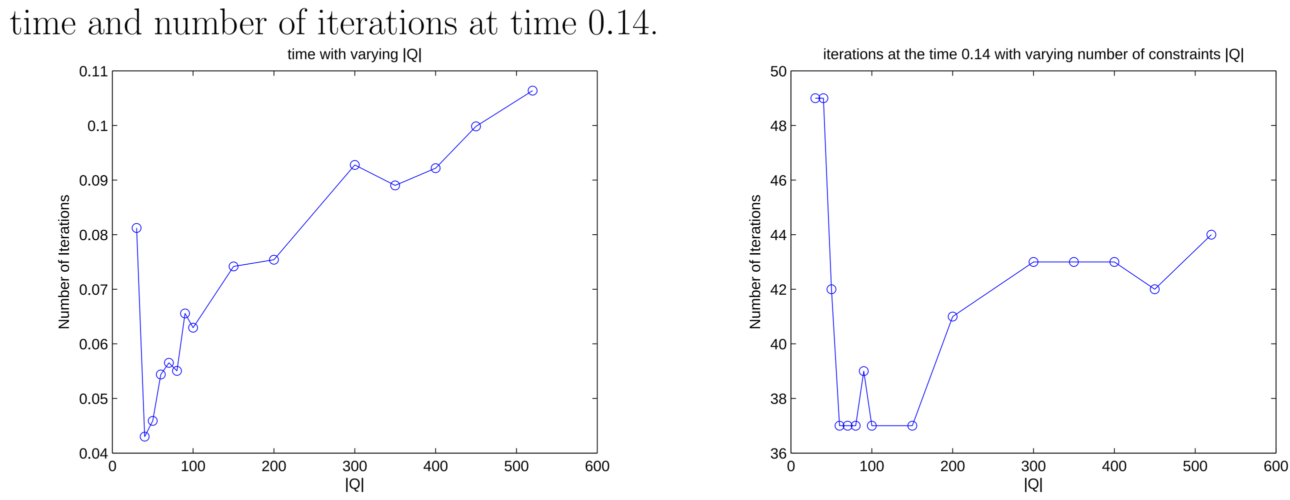

Figure 2: The time and iteration with varying kept constraints at t=0.14s.

- By the picture of total time in Figure 2, we can tell that $|Q| = 40$, i.e., about 8% of the constraints, is a good choice for the constraints reduction scheme at time $t = 0.14$. In the table below, we compare the total time for three algorithms: Matlab commands *qpdantz* and *quadprog*, and *CRCQP* with $|Q| = 40$. For *quadprog* and *CRCQP*, we use the same maximum iteration 100 and tolerance *tol* $= 10^{-4}$. All other parameters are default in the three algorithms. The strategy to choose Q is based on the slack variables of the primal problem which is the default in *CRCQP*. To see clearly how constraints reduction (CR) scheme saves time, we also list out the total time for *CRCQP* with $|Q| = 520$ (no constraint reduction).

Sample Time	qpdantz	quadprog	CR($\lvert Q\rvert = 40$)	CR($\lvert Q\rvert = 520$)
0.01	0.1803	0.0604	0.0430	0.1064

Table 1: Time for QP at time $t = 0.14$ using different methods. All the units are seconds.

Future Work

C Version . Translate Matlab code to C code for implementation within a flight simulator.

Feasible Initial Controls. Develop a scheme to fast generate initial feasible controls since algorithm *CRCQP* needs a strictly feasible initial point to start.

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