



PROBLEM FORMULATION

Definition. A *Discrete Markovian Random Graph* is a stochastic process taking values in a set of graphs $\mathcal{G}=\{G_1, \dots, G_s\}$ which evolves in time according to a underlying finite-state discrete time Markov chain $M(k)$.

Consider a group of n agents which exchange locally information according to a communication protocol. The communication topology is assumed modeled by a *Markovian Random Graph* $G_{M(k)}$. The evolution of the agents is governed by

$$X(k+1) = F_{M(k)} X(k), \quad X(0) = X_0$$

where $X(k)$ represents the n -dimensional state vector of the agents and, $F_{M(k)}$ is a stochastic matrix determined by the communication protocol and which corresponds to the current topology determined by $G_{M(k)}$.

Definition (Almost sure consensus). Vector $X(k)$ converges almost surely to consensus if it asymptotically reaches a vector $\alpha \mathbf{1}$ in the almost sure sense. If the state vector converges to $av(X_0) \mathbf{1}$ we say that $X(k)$ converges almost surely to *average consensus*.

Problem. Given a discrete Markovian random graph $G_{M(k)}$ together with the state updating rule $F_{M(k)}$, we derive necessary and sufficient conditions such that the state vector $X(k)$ converges almost surely to consensus for any initial state X_0 .

Example of local communication protocols (local state updating rules):

$$A1) F_i = I - \epsilon L_i$$

$$A2) F_i = (I + D_i)^{-1} (I + A_i)$$

where L_i , D_i and A_i are the Laplacian, degree and adjacency matrices respectively of graph G_i .

MOTIVATION

- Presence of communication topologies that varies randomly in time due to link failures, packet drops, appearance or disappearance of nodes, obstacles that interfere with the communication, etc.
- Topology switching may depend on the previous state of the communication topology (systems with memory)
- Applications in
 - Flocking Theory
 - Rendezvous in Space
 - Distributed Sensor Fusion in Sensor Networks
 - Distributed Formation Control

RESULTS

Theorem (almost sure convergence to consensus). Under protocol A1 or A2, the state vector $X(k)$ converges almost surely to consensus for any initial state X_0 if and only if each of the sets of graphs corresponding to the positive recurrent closed sets of the Markov chain admits a spanning tree. In particular if protocol A1 is used and all graphs in \mathcal{G} are either undirected or directed but *balanced*, then the state vector converges almost surely to *average consensus*.

Bound on the *rate of convergence* to consensus in the mean square sense:

$$E[\|\epsilon(k)\|^2] \leq \alpha |\lambda_2|^k \|\epsilon(0)\|^2$$

$$\epsilon(k) = X(k) - \arg \min_{z \in \text{span}\{\mathbf{1}\}} \|X(k) - z\|$$

where
the matrix

$$\Lambda = \begin{bmatrix} p_{11} F_1 \otimes F_1 & p_{21} F_2 \otimes F_2 & \dots & p_{s1} F_s \otimes F_s \\ p_{21} F_1 \otimes F_1 & p_{22} F_2 \otimes F_2 & \dots & p_{s2} F_s \otimes F_s \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} F_1 \otimes F_1 & p_{s2} F_2 \otimes F_2 & \dots & p_{ss} F_s \otimes F_s \end{bmatrix}$$

and λ_2 is the second largest eigenvalue in absolute value of

APPROACH

Show for the state vector that

$$E[X(k)X(k)^T] \longrightarrow \beta \mathbf{1}\mathbf{1}^T$$

for some scalar β , which implies that the second moment of the error vector

$$\epsilon(k) = \left(I - \frac{\mathbf{1}\mathbf{1}^T}{\mathbf{1}^T \mathbf{1}} \right) X(k)$$

converges to zero exponentially, which in turns shows that

$$\|\epsilon(k)\|^2 \xrightarrow{a.s.} 0$$

Key property:

$$\lim_{k \rightarrow \infty} \Lambda^k = \begin{bmatrix} p_{11}^{\infty} \mathbf{1} c_{11}^T & p_{21}^{\infty} \mathbf{1} c_{12}^T & \dots & p_{s1}^{\infty} \mathbf{1} c_{1s}^T \\ p_{12}^{\infty} \mathbf{1} c_{21}^T & p_{22}^{\infty} \mathbf{1} c_{22}^T & \dots & p_{s2}^{\infty} \mathbf{1} c_{2s}^T \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1}^{\infty} \mathbf{1} c_{s1}^T & p_{s2}^{\infty} \mathbf{1} c_{s2}^T & \dots & p_{ss}^{\infty} \mathbf{1} c_{ss}^T \end{bmatrix}^k$$

where $p_{ij}^{\infty} = \text{Prob}(M(k) = j | M(0) = i)$

for large values of k and c_{ij} are vectors of positive entries which sum up to one.

References

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- M. Porfiri and D.J. Stilwell, "Consensus seeking over Random Directed Weighted Graphs", *IEEE Trans. Autom. Control*, vol. 52, no. 9, Sept. 2007.
- Tahbaz Salehi and A. Jadbabaie, "Necessary and Sufficient Conditions for Consensus over random networks", *IEEE Transactions on Automatic Control*, accepted, August 2007.