Capacity Region of Power Controlled Fading CDMA: Generalized Iterative Waterfilling

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Introduction

- Capacity of wireless systems subject to fading can be improved via resource allocation
- We consider a power controlled fading CDMA channel

$$\mathbf{r} = \sum_{i=1}^{K} \sqrt{p_i(\mathbf{h})h_i}\, b_i \mathbf{s}_i + \mathbf{n}$$

- If channel state information (CSI) **h** is available at the transmitters and the receiver of a fading MAC, the transmitters can allocate transmit powers (resources) and adjust their coding strategies, and the receiver can adjust its decoding strategy.
- Our goal is to obtain the capacity region, and the power control policies that achieve arbitrary points on the boundary of the capacity region.

Background

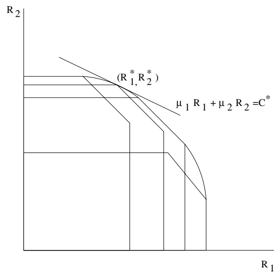
- Capacity achieving power allocation for a single user system is waterfilling over the inverse of the channel states, where more power is used at better channel states [Goldsmith-Varaiya].
- For a scalar MAC, sum of rates is maximized by a TDMA-like scheme where only the user with the best channel state transmits; optimal power allocation is again waterfilling [Knopp-Humblet].
- The capacity region of fading scalar MAC is the union of rate regions achievable by all valid power control policies, and the power control policies needed to achieve each point on the boundary can be obtained by using a greedy algorithm [Hanly-Tse].
- The power control policy that achieves the sum capacity of a fading CDMA system is simultaneous waterfilling, and can be obtained by a one-user-at-a-time iterative waterfilling algorithm [Kaya-Ulukus].
- The capacity region of fading vector MACs is not known. Here, we provide the capacity region of fading CDMA and corresponding power control policies.

Capacity Region of Fading CDMA

- **Theorem:** *The capacity region of a fading CDMA channel under additive white Gaussian noise, where the signature sequences are fixed, users have perfect CSI) and they allocate their powers as a function of the CSI subject to average power constraints $E_{\mathbf{h}}[p_i(\mathbf{h})] \le \bar{p}_i$, is given by,*

$$\bigcup_{\{\mathbf{p}(\mathbf{h}):E_{\mathbf{h}}[p_i(\mathbf{h})]\le\bar{p}_i,\ \forall i\}} \left\{ \mathbf{R} : \sum_{i\in\Gamma} R_i \le E_{\mathbf{h}}\left[\frac{1}{2}\log\left|\mathbf{I}_N + \sigma^{-2}\sum_{i\in\Gamma} h_i p_i(\mathbf{h})\mathbf{s}_i\mathbf{s}_i^T\right|\right]\right\}, \forall \Gamma \subset \{1,\ldots,K\}$$

- **Theorem:** *The capacity region of a power controlled fading CDMA channel is not strictly convex, provided $\exists i,j \in \{1,\ldots,K\}$ such that $i \ne j$ and $0 < \mathbf{s}_i^T\mathbf{s}_j < 1$.*



Boundary of the Capacity Region

- Each of the pentagons correspond to a valid power allocation policy.
- The capacity region is not necessarily strictly convex, the flat portion on the boundary corresponds to the sum capacity maximizing power control policy.
- The problem of finding the power control policy that corresponds to the rate pair (R_1^*, R_2^*) on the boundary is equivalent to maximizing $\mu_1 R_1 + \mu_2 R_2$ subject to the average power constraints, for some μ_1, μ_2.
- Any rate pair (R_1^*, R_2^*) on the curved portion of the boundary of the capacity region is a corner of one of the pentagons.

Optimum Power Allocation Policies

- For a given set of priorities $\mu_K > \ldots > \mu_1 > \mu_0 = 0$, define $\mathbf{S} = [\mathbf{s}_1 \ \ldots \ \mathbf{s}_K]$, $E_i = [i,\ldots,K]$ and $\mathbf{D}(\mathbf{h}) = \mathrm{diag}[p_1(\mathbf{h})h_1, \ldots p_K(\mathbf{h})h_K]$. The optimization problem is,

$$\max_{\mathbf{p}(\mathbf{h})} \ \frac{1}{2} E_{\mathbf{h}}\left[\sum_{i=1}^{K}(\mu_i - \mu_{i-1})\log\left|\mathbf{I}_N + \sigma^{-2}\mathbf{S}_{E_i}\mathbf{D}_{E_i}(\mathbf{h})\mathbf{S}_{E_i}^T\right|\right]$$
$$\text{s.t.} \quad E_{\mathbf{h}}[p_k(\mathbf{h})] \le \bar{p}_k, \quad k = 1,\ldots,K$$
$$p_k(\mathbf{h}) \ge 0, \ \forall \mathbf{h}, \quad k = 1,\ldots,K$$

- Objective function is concave in $\mathbf{p}(\mathbf{h})$, strictly concave in $p_i(\mathbf{h})$, and constraints are convex: solution satisfies the extended KKT conditions

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + p_k(\mathbf{h})} \le \lambda_k, \quad \forall \mathbf{h}, \quad k = 1,\ldots,K$$

- $a_{ki}(\mathbf{h})$ is the inverse of the SIR user k would obtain at output of an MMSE filter if it transmitted with unit power, and only users $i,\ldots,K$ were active in the system.
- By concavity-convexity properties, a **one-user-at-a-time algorithm**, which treats the powers of all but one users fixed, and uses the KKT condition for the user of interest to solve for its power, converges to global optimum.

Iterative Generalized Waterfilling

- Focus on the power update of a single user.
- Generalized waterfilling to solve the KKT conditions at all **h**.
- Define the base levels, obtained by letting $p_k(\mathbf{h}) = 0$ in KKT conditions, as

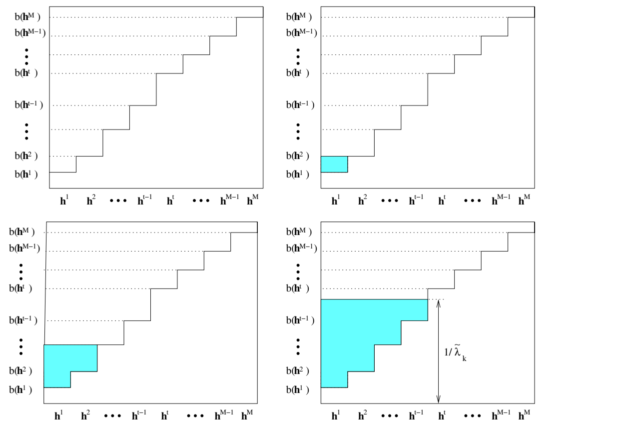
$$b_k(\mathbf{h}) = \left(\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h})}\right)^{-1}$$

- Sort $b_k(\mathbf{h})$ in increasing order, start pouring power at the channel state which yields the minimum $b_k(\mathbf{h})$, say $\mathbf{h}'$.
- Next, pick another state $\mathbf{h}''$ s.t $b_k(\mathbf{h}') < b_k(\mathbf{h}'')$. User k starts transmitting at $\mathbf{h}''$ iff
  - it has already poured some powers $q_k(\mathbf{h})$ to all states $\mathbf{h}$ s.t. $b_k(\mathbf{h}) < b_k(\mathbf{h}'')$,
  - it still has some power to allocate, and
  - the already allocated powers $q_k(\mathbf{h})$ satisfy

$$\sum_{i=1}^{k} \frac{\mu_i - \mu_{i-1}}{a_{ki}(\mathbf{h}) + q_k(\mathbf{h})} = b_k^{-1}(\mathbf{h}''), \quad \forall \mathbf{h} : b_k(\mathbf{h}) \le b_k(\mathbf{h}'') \quad (1)$$

- The value of $q_k(\mathbf{h})$ at each step is not given by the water level at each state as in sum capacity case, but can be found by solving a kth order polynomial equation.
- By construction, when the average power constraint is met with equality, letting $p_k(\mathbf{h}) = q_k(\mathbf{h})$, this allocation satisfies the KKT conditions and is optimal.

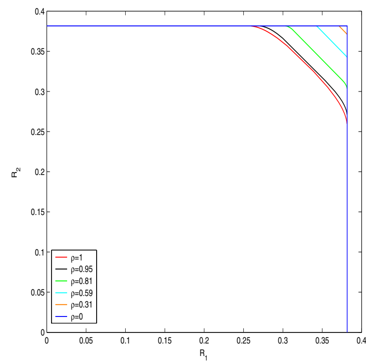
Generalized Waterfilling: Illustration



- In this example, assume w.l.o.g for discrete fading states $\mathbf{h}^i$, $b_k(\mathbf{h}^1) < \ldots < b_k(\mathbf{h}^M)$.
- As we gradually increase water level, using (1), we solve for corresponding powers.
- We stop when all available power is used, and the current water level gives $1/\tilde{\lambda}_k$. As we iterate over users, $\tilde{\lambda}_k$ converges to λ_k.

Simulations

- We perform the generalized iterative waterfilling algorithm for K=2, N=2.
- Fixing a set of signature sequences, compute the optimal power allocations and corresponding rate pairs for a wide range of μ_i values, to generate capacity region.
- We obtain capacity regions for several correlation values ρ among the sequences.



- When sequences are identical, the problem reduces to scalar MAC, and capacity region is as in [Hanly-Tse]. Strict convexity of the boundary is observed (red curve).
- When sequences are orthogonal, the capacity region is a rectangle (blue curve).
- When sequences are arbitrarily correlated, there is a flat portion on the boundary, supporting the analytical results on non-strict convexity of the capacity region of power controlled fading CDMA (e.g., green curve).