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Dynamics and Controllability of a Planar Rigid Body with a Thruster *

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Abstract

In this paper we discuss the dynamics of a planar rigid body with a thruster. We identify the position and orientation of the planar rigid body with an element of the Lie group $SE(2)$. The invariance of the kinetic energy Lagrangian and the forcing term (thrust) under the $SE(2)$ action of translations and rotations is used to obtain a set of Poisson-reduced equations on $T^*SE(2)/SE(2)$ which is isomorphic to $se(2)^*$, the dual of the Lie algebra of $SE(2)$. Depending on the control authority available, we distinguish two versions of the problem which we call the *"jet-puck"* problem and the *"hovercraft"* problem respectively. In the jet-puck problem the control enters linearly, and we show that, using only a scalar control we have controllability over $se(2)^*$. In the hovercraft problem the controls enter nonlinearly. In the equations of motion of the above cases the drift vectorfield is periodic (and hence trivially weakly positively Poisson stable). This, along with the fact that the Lie algebra rank condition is satisfied at all points on $se(2)^*$, is used to show controllability of the systems.

1 Introduction

In recent years a geometric approach to studying mechanical systems has led to a deeper understanding of their behavior, controllability aspects and in the design of feedback controls. Key to this has been the development in reduction theory (cf. [1]), that exploits the invariance of the dynamics to a group of transformations (the symmetry group). The existence of a symmetry group permits the dropping of the dynamics to a lower dimensional (reduced) space. Lagrangian reduction involves dropping the Euler-Lagrange equations to the quotient of the velocity phase space given by the symmetry group while Hamiltonian reduction involves projecting the Poisson bracket to the reduced (quotient) space which also inherits a Poisson structure. The complete dynamics can then be reconstructed from those of the reduced system. In particular if the transformation group is a solvable Lie group then the solutions can be reconstructed using quadratures. The geometric phase [2] associated with a path in the reduced system describes the motion of the complete (lifted) system. While earlier work did not include external forces and constraints (nonholonomic), more recently the geometry of systems with constraints and external forces have been developed in [3, 4, 5, 6]. This has provided a framework for control theoretic development of such systems.

The primary goal of this paper is to illustrate the concept of reduction, and its utility using an example of a planar rigid body with a thruster. We identify the position and orientation of the planar rigid body with an element of the Lie group $SE(2)$. The invariance of the kinetic energy Lagrangian and the forcing term (thrust) under the $SE(2)$ action of translations and rotations is used to obtain a set of Poisson-reduced equations on $T^*SE(2)/SE(2)$ which is isomorphic to $se(2)^*$, the dual of the Lie algebra of $SE(2)$. The reduced set of equations, thus obtained, can be shown to be controllable as they fall into the class of nonlinear systems that are accessible and whose drift term is Poisson stable. In addition we observe that the base-space equations appear similar to the symmetric controlled Euler equations with one control. The controllability of the base-space equations and their resemblance to the symmetric controlled Euler equations would not have otherwise been obvious if one simply examined the equations derived from a standard Lagrangian framework.

The work in this paper is partially motivated by its application to steering hovercraft. Though hovercraft are rarely able to compete economically with traditional models of transportation in most environments, the amphibious versatility of hovercraft have given them a role in specialized applications including search and

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rescue, emergency medical services, ice breaking and recreational activities [7]. Hovercraft have proven to be one of the most promising vehicles for future Arctic offshore transportation. One of the most important ice characteristics affecting mid-winter offshore hovercraft operations is the distribution of ice roughness features, either as isolated ridges or as extensive rubble fields. In this environment hovercraft require accurate route marking and positioning systems combined with suitable obstacle avoidance capabilities. An ideal hovercraft for mid-winter offshore arctic operations combines a deep cushion with thrust vectoring for control. The models and analysis of this paper are relevant to the design and (autonomous) operation of such vehicles.

The paper is organized as follows. In section 2 we present the kinematics and dynamics of a planar rigid body with a thruster. In section 3 the Poisson-reduced set of equations are derived. The reduction procedure is discussed in some detail. In section 4 we address issues related to controllability over $se(2)^*$ and $T^*SE(2)$. In section 5 we present our conclusions and some future work.

2 Kinematics and Dynamics

The configuration of the system is shown in Figure 1. Let $\{e_1^r, e_2^r, e_3^r\}$ be an inertial frame of reference fixed at $\mathbf{O}$ and $\{e_1^b, e_2^b, e_3^b\}$ be a body frame fixed on the rigid body $\mathcal{B}$ at its center of mass. Since the rigid body is restricted to move in the $e_1^r e_2^r$ plane, a typical material point q^b in the rigid body is then represented in the inertial frame as $q^r = Rq^b + r$ where R is an element of $SO(2)$, the special orthogonal group of 2×2 matrices and $r = (x, y)$ is a vector from $\mathbf{O}$ to the center of mass of $\mathcal{B}$. Hence at any instant, the configuration $X(t)$ of $\mathcal{B}$ can be uniquely identified by the pair (R, r) or equivalently as an element of $SE(2)$, the Special Euclidean group of 2×2 matrices. Recall

$$SE(2) \triangleq \{ \begin{pmatrix} R & r \\ 0 & 1 \end{pmatrix} \mid R \in SO(2), r \in \mathbb{R}^2 \}$$

Let us assume that the thruster is mounted at the point C defined by the vector d^b in body coordinates and d^r in the inertial frame of reference. The thrusters exert a force f^r in inertial coordinates such that the line of action of the force passes through C and makes an angle ϕ with the vector d^b. We now derive the equations of motion of a rigid body subject to a force f^r along a specified line of action.

The kinematics are defined by

$$\dot{g} = g\xi$$

where $g \in SE(2)$ and ξ is a curve in $se(2)$, the Lie

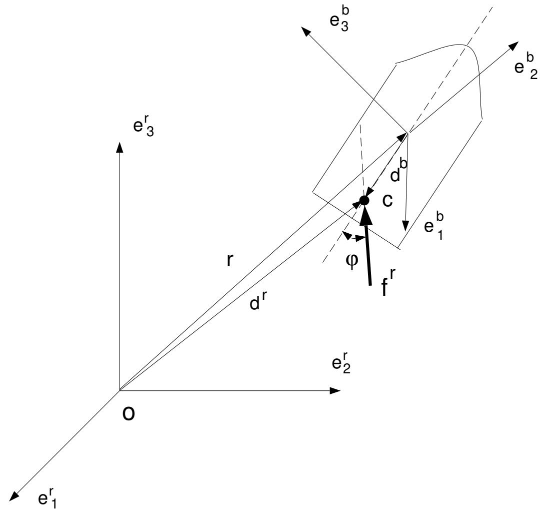

Figure 1: Planar rigid body with thruster

algebra of $SE(2)$. Equivalently they could be written as

$$\begin{aligned} \dot{R} &= R\widehat{\Omega} && \text{(1-a)} \\ \dot{r} &= p/m && \text{(1-b)} \end{aligned}$$

where

$$\widehat{\Omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Omega,$$

and $\Omega = \dot{\theta}$. Ω is the scalar body angular velocity about the center of mass, θ is the orientation of the body fixed frame w.r.t. to the inertial frame and p is the spatial linear momentum.

The kinetic energy of the rigid body relative to the inertial frame is

$$\begin{aligned} T &= \frac{1}{2} \int_{\mathcal{B}} \|\dot{q}^r\|^2 dm(q^b) \\ &= \frac{1}{2} < \Omega, I\Omega > + \frac{m}{2} \|\dot{r}\|^2 \end{aligned}$$

where m is the total mass and I is the moment of inertia of $\mathcal{B}$ in the body frame. We assume for now that the rigid body (which will later be approximated to a hovercraft) has sufficient lift and glide on the surface with no friction. Models with lift and friction will be studied later. Hence the Lagrangian $L : TSE(2) \rightarrow \mathbb{R}$ for this case is simply the kinetic energy, i.e.

$$L(R, r, \dot{R}, \dot{r}) = \frac{1}{2} < \Omega, I\Omega > + \frac{m}{2} \|\dot{r}\|^2 \qquad (2)$$

Let G be a Lie Group. We now define the G action (in this case left translation by $g \in G$) of $g = (\bar{R}, \bar{r}) \in SE(2)$ on an element of the configuration space $Q \equiv$

$SE(2)$ as

$$\begin{array}{l}\phi : SE(2) \times Q \to SE(2) \\ \quad (\bar{R}, \bar{r}, R, r) \mapsto (\bar{R}R, \bar{R}r + \bar{r})\end{array}$$

Similarly the lifted action of G on $TSE(2)$ is defined as

$$\begin{array}{l}\phi_{g^*} : TSE(2) \to TSE(2) \\ \quad (R, r, \dot{R}, \dot{r}) \mapsto (\bar{R}R, \bar{R}r + \bar{r}, \bar{R}\dot{R}, \bar{R}\dot{r})\end{array}$$

A Lagrangian $L : TQ \to \mathbb{R}$ is G-invariant if

$$L(\phi_{g^*}(q, \dot{q})) = L(q, \dot{q})$$

It is easily verified that the action defined above leaves the Lagrangian (2) invariant. In other words the Lagrangian does not change if the inertial frame is translated or rotated. The corresponding Hamiltonian on $T^*SE(2)$ is given by

$$H = \frac{1}{2} < \Pi, I^{-1}\Pi > + \frac{\|p\|^2}{2m} \tag{3}$$

where $\Pi = I\Omega$ is the body angular momentum and $p = m\dot{r}$ is the spatial linear momentum. From the Newton-Euler balance Laws we have

$$\dot{p} = f^r, \quad \text{and} \quad \dot{\pi} = d^r \times f^r$$

for the spatial angular momentum (π) and the linear momentum (p) of the rigid body. Collecting together the balance laws one can write dynamics in spatial (inertial) variables (R, r, π, p) as

$$\begin{aligned}
\dot{R} &= R\widehat{\Omega} && \text{(4-a)} \\
\dot{r} &= p/m && \text{(4-b)} \\
\dot{\pi} &= d^r \times f^r && \text{(4-c)} \\
\dot{p} &= f^r && \text{(4-d)}
\end{aligned}$$

3 Symmetry and Reduction

Since the Hamiltonian (3) is $SE(2)$ invariant one can induce a Hamiltonian on the quotient space, $T^*SE(2)/SE(2)$, and express the dynamics in terms of the appropriate reduced variables. In the present case it is easy to see that the quotient space $T^*SE(2)/SE(2)$ is isomorphic to $se(2)^*$, the dual of the Lie algebra of $SE(2)$ and the reduced variables are:

Π, the body angular momentum,
$P = R^T p$, the convected linear momentum.

In terms of these reduced (convected) variables the dynamics take the form

$$\begin{aligned}
\dot{P} &= \dot{R}^T p + R^T \dot{p} \\
&= -\widehat{\Omega} P + R^T f^r \\
\dot{\Pi} &= R^T((d^r - r) \times f^r) \\
&= d^b \times F
\end{aligned}$$

where $F = R^T f^r$ (cf. [8]). The reduced Hamiltonian $\widetilde{H}$ is given by

$$\widetilde{H} = \frac{1}{2} < \Pi, I^{-1}\Pi > + \frac{\|P\|^2}{2m} \tag{5}$$

Let $|F|$ denote the magnitude of the force f^r and $|d|$ denote the length of the vector d^b. Since the motion of the hovercraft is restricted to the $e_1^r e_2^r$ plane the reduced equations simplify to

$$\begin{aligned}
\dot{P}_1 &= P_2\Pi/I + |F|\cos\phi && \text{(6-a)} \\
\dot{P}_2 &= -P_1\Pi/I + |F|\sin\phi && \text{(6-b)} \\
\dot{\Pi} &= |F|\,|d|\sin\phi && \text{(6-c)}
\end{aligned}$$

These reduced equations could also also be interpreted as being derived from the reduced Hamiltonian $\widetilde{H}$ as

$$\dot{x} = \{x, \widetilde{H}\} + f_{ext} \tag{7}$$

where $x = (P_1, P_2, \Pi)^T$, $\{\cdot, \widetilde{H}\}$ is the *Lie-Poisson* bracket induced on the quotient space and f_{ext} is the external force projected appropriately. Recall [4] that a left invariant Hamiltonian on T^*G gives rise to reduced dynamics on $\mathcal{G}^*_-$ (the space $\mathcal{G}^*$ associated with the minus Poisson structure). Suppose $\widetilde{H} : T^*SE(2)/SE(2) = \mathcal{G}^*_- \to \mathbb{R}$ is a reduced Hamiltonian. Let$\{X_1, X_2, X_3\}$ and $\{X_1^b, X_2^b, X_3^b\}$ be the basis for the Lie algebra $\mathcal{G}$ and the dual basis for $\mathcal{G}^*$ respectively, i.e. $< X_i^b, X_j >= \delta^i_j$. Any $\mu \in \mathcal{G}^*_-$ can be expressed as $\mu = \sum_{i=1}^3 \mu_i X_i^b$. The differential equations satisfied by the coordinate function are then

$$\dot{\mu}_i = \{\mu_i, \widetilde{H}\}_- \quad i = 1, 2, 3$$

where the bracket $\{\cdot, \cdot\}_-$ is now the minus Lie-Poisson bracket, defined in its coordinate free form by

$$\{F, H\}_-(x) = - < x, [\nabla F(x), \nabla H(x)] >$$

where $F, G \in C^\infty(\mathcal{G}^*)$ and $x \in \mathcal{G}^*$, or as

$$\{F, H\}_-(x) = - \sum_{i,j,k=1}^{3} c_{ij}^k x^k \frac{\partial F}{\partial x^j} \frac{\partial H}{\partial x^j}$$

where $c_{ij}^k, i, j, k = i, 2, 3$ are the structure constants of $\mathcal{G}$ relative to a basis $\{X_1, X_2, X_3\}$. In our case choosing

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$
$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the commutation relations $[X_1, X_2] = 0$, $[X_3, X_1] = X_2$ and $[X_2, X_3] = X_1$, the Lie-Poisson equations on $se(2)^*$ are

$$\dot{P}_1 = P_2 \frac{\partial \widetilde{H}}{\partial \Pi} + |F|\cos\phi \tag{8-a}$$

$$\dot{P}_2 = -P_1 \frac{\partial \widetilde{H}}{\partial \Pi} + |F| \sin\phi \quad (8\text{-b})$$

$$\dot{\Pi} = -P_1 \frac{\partial \widetilde{H}}{\partial P_1} + P_2 \frac{\partial \widetilde{H}}{\partial P_2} + |F|\,|d| \sin\phi \quad (8\text{-c})$$

Depending on the control authority we distinguish two versions of the problem.

Case 1: The Jet-Puck Problem: Here we assume that the line of action of the force is fixed (i.e. ϕ is fixed) but its direction can be reversed. Written as a control system equations (6) take the form

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + \alpha u & (9\text{-a}) \\ \dot{P}_2 &= -P_1\Pi/I + \beta u & (9\text{-b}) \\ \dot{\Pi} &= d\beta u & (9\text{-c}) \end{aligned}$$

where $\alpha = \cos\phi, \beta = \sin\phi$ and $u \in [1, -1]$.

Case 2: The Hovercraft Problem: Here we assume that we now have control over both the magnitude of the thrust and ϕ. The equations now take the form

$$\begin{aligned} \dot{P}_1 &= P_2\Pi/I + u_1\cos(u_2) & (10\text{-a}) \\ \dot{P}_2 &= -P_1\Pi/I + u_1\sin(u_2) & (10\text{-b}) \\ \dot{\Pi} &= u_1\sin(u_2) & (10\text{-c}) \end{aligned}$$

where $u_1 \in [-1, 1]$ and $u_2 \in [\phi_{min}, \phi_{max}]$

In the rest of the discussion each of these cases will be discussed separately.

Remark: We would like to point out that the jet puck equations (9) obtained here are similar to those of the controlled Euler equations with one control. Recall that the equations describing the dynamics of the spacecraft attitude control with gas jets (basically those of a rotating rigid body with external torque control) are given by

$$\dot{R} = -RS(\omega) \quad (11)$$

$$J\dot{\omega} = S(\omega)J\omega + \sum_{i=1}^{m} b_i u_i \quad (12)$$

where $R \in SO(3)$, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector and $S(\omega)$ is a 3×3 skew symmetric matrix defined by

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

and $J = \mathrm{diag}(j_1, j_2, j_3)$ is the inertia matrix. Assuming we have only one control and $j_2 = j_1$, (12) can be rewritten as

$$\begin{aligned} \dot{\omega_1} &= \frac{(j_1 - j_3)}{j_1}\omega_2\omega_3 + \alpha u & (13\text{-a}) \\ \dot{\omega_2} &= -\frac{(j_1 - j_3)}{j_1}\omega_1\omega_3 + \beta u & (13\text{-b}) \\ \dot{\omega_3} &= \gamma u & (13\text{-c}) \end{aligned}$$

Even though the jet-puck and the spacecraft with a single control are two entirely different systems, observe the similar structure of the base-space equations for the jet-puck (9) and those of the controlled Euler equations (13) given above. This similarity, would not have been otherwise obvious if one did not look at the reduced dynamics for the jet-puck.

4 Controllability

Given a system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \;\; u = (u_1, \cdots u_m) \in U \quad (14)$$

where $U \subset \mathbb{R}^m$, let $L(x) = \{\mathrm{span}\{X(x) \mid X \text{ vectorfield in } \mathcal{L}, x \in M\}$ where $\mathcal{L}$, the accessibility algebra, is the smallest subalgebra of the Lie algebra of vector fields on M that contains $f, g_1, \cdots, g_m$. The system is said to satisfy the Lie algebra rank condition (LARC) if

$$L(x) = T_x M, \; \forall x \in M \quad (15)$$

We know that if a system of the form (14) satisfies LARC then it is locally accessible, and in addition if $f = 0$, i.e. the system is symmetric, then LARC implies that the system is controllable. While the kinematic equations of motion can often be written as a drift free system, once dynamics are included symmetry is lost and LARC does not imply controllability. Proving controllability is usually much harder than proving accessibility. In [9] sufficient conditions are given, in terms of the "group action", that a locally accessible system is also locally reachable. In [10] sufficient conditions for the controllability of a conservative dynamical polysystem on a compact Riemannian manifold are presented. More recently this result was extended by [11] to dynamical polysystems where only the drift vector field was required to be weakly positively Poisson stable. We use this result to show base-space controllability of the jet-puck and the hovercraft. Before we present related theorems we introduce some definitions and related theorems regarding Poisson stable systems. We follow the development in [11, 12, 13, 14].

Let X be a smooth complete vector field on M and let $\phi_t^X(p)$ denote its flow.

Definition: A point $p \in M$ is called *positively Poisson stable* for X if for all $T > 0$ and any neighborhood V_p of p, there exists a time $t > T$, such that $\phi_t^X(p) \in V_p$. The vector field X is called positively Poisson stable if the set of Poisson stable points for X is dense in M.

Definition: A point $p \in M$ is called nonwandering point of X if for all $T > 0$, any neighborhood V_p of p, there exists a time $T > 0$ such that $\phi_t^X(V_p) \bigcap V_p \neq \phi$, where $\phi_t^X(V_p) = \{\phi_t^X(q) \mid q \in V_p\}$.

One should observe here that though it is a sufficient condition that the nonwandering set of a positively Poisson stable vector field is the entire manifold M, but there could exist weaker conditions under which the nonwandering set is M. This gives rise to the definition of a weakly positively Poisson stable (WPPS).

Definition: The vector field X is called weakly positively Poisson stable if its nonwandering set is M.

Theorem 1 (B. Bonnard) *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \cdots u_m) \in U \subset \mathbb{R}^m$$

where U contains $\{u \mid |u_i| \leq M_i \neq 0, i, \cdots, m\}$ and f is a Poisson stable vector field, then the system is controllable if LARC is satisfied.

Corollary 1 *If the system*

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \cdots u_m) \in U$$

is such that f is a Poisson stable vector field, and LARC is satisfied, then the system with controls constrained by $u_i \in \{-M_i, M_i\}, M_i > 0,\ i = 1, \cdots, m$ is controllable.

Remark: In [11] it is shown that for the statement of theorem 1 to be true it is sufficient that the drift vectorfield f be WPPS.

Proposition 1: The jet-puck dynamics defined by (9) are controllable.

Proof: We first show that LARC is satisfied. To show that

$$\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 3, \quad \forall p \in se(2)^*$$

where $f = (P_2\Pi/I, -P_1\Pi, 0)^T$ and $g = (\alpha, \beta, d\beta)^T$, observe that

$$\begin{aligned}
&\det(g, [[f,g],g], [[f,g], [[f,g],g]]) \\
&= det \begin{bmatrix} \alpha & 2\frac{d}{I}\beta^2 & -2\frac{d^2}{I^2}\beta^2\alpha \\ \beta & -2\frac{d}{I}\beta\alpha & -2\frac{d^2}{I^2}\beta^3 \\ d\beta & 0 & 0 \end{bmatrix} \\
&= -4\frac{(d\beta)^4}{I^3}(\beta^2 + \alpha^2) \\
&= -4\frac{(d\beta)^4}{I^3} \ \ (\text{since } \alpha^2 + \beta^2 = 1)
\end{aligned}$$

Hence $\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 3\ \forall p \in se(2)^*$ as long as $\beta = \sin\phi \neq 0$, i.e. as long as the line of action of F does not pass through the center of mass.

Also since every orbit of f is periodic it is Poisson stable. Thus by theorem 1 the jet-puck dynamics are controllable. □

Proposition 2: The hovercraft dynamics defined by (10) are controllable.

Proof: In (10) setting $u_2 = k$, where k is some constant not equal to zero, the equations reduce to those of the jet-puck and hence from Proposition 1 the dynamics are controllable.

4.1 Cotangent Space Controllability

We now investigate the controllability in the cotangent space. We show that the complete (unreduced) system is locally strongly accessible. The complete set of equations for the jet-puck take the form

$$\begin{aligned}
\dot{x} &= p_1/m && \text{(16-a)} \\
\dot{y} &= p_2/m && \text{(16-b)} \\
\dot{\theta} &= \Pi/I && \text{(16-c)} \\
\dot{p_1} &= (\cos(\theta+\phi))u && \text{(16-d)} \\
\dot{p_2} &= (\sin(\theta+\phi)u && \text{(16-e)} \\
\dot{\Pi} &= (d\sin\phi)u && \text{(16-f)}
\end{aligned}$$

Proposition 3 : For the system defined by (16) the cotangent space $T^*SE(2)$ is locally strongly accessible.

Proof: To show that

$$\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 6, \quad \forall p \in T^*SE(2)$$

where $f = (p_1/m, p_2/m, \Pi/I, 0, 0, 0)^T$,
and $g = (0, 0, 0, \cos(\theta+\phi), \sin(\theta+\phi), d\sin\phi)^T$, observe that

$$\begin{aligned}
&det(g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \\
&= 16K(\cos^2(\theta+\phi) + \sin^2(\theta+\phi))^2 \\
&= 16K
\end{aligned}$$

where

$$\begin{aligned}
&\xi_1 = [f,g],\ \xi_2 = [[f,g],g],\ \xi_3 = [[[f,g],f],g], \\
&\xi_4 = [[[[f,g],f],g],[f,g]], \\
&\xi_5 = [[[[f,g],f],g],[[f,g],g]] \text{ and} \\
&K = (\tfrac{1}{m})^2(\tfrac{1}{I})^7(d\sin\phi)^8
\end{aligned}$$

Hence again if $\sin\phi \neq 0$, $\dim(\mathrm{span}\mathcal{L}_{\{f,g\}})(p) = 6, \quad \forall p \in T^*SE(2)$. Also $[f, X] \in \mathrm{span}(g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5), \forall X \in \{g, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$. Hence the complete system is locally strongly accessible. □

5 Conclusions and Future Work

The controllability of the base system and the fact that the complete system is locally strongly accessible suggest that the complete system is controllable [15]. Details on small-time local controllability and controllability are currently being studied and details can be found in [16]. Some future research includes control laws to steer the hovercraft and hybrid architectures for generation of "scripts" for obstacle avoidance and navigation.

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