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2–MODULE NONHOLONOMIC VARIABLE GEOMETRY TRUSS ASSEMBLY: MOTION CONTROL

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Abstract: The nonholonomic motion planning problem is considered for a novel class of modular mobile manipulators, where each module is implemented as a planar parallel manipulator with *idler* wheels. This assembly is actuated by shape changes of its modules, which, under the influence of the nonholonomic constraints on the wheels, induce a global snake–like motion of the assembly. The kinematics for a 2–module assembly of this type are formulated and the corresponding motion planning problem is studied.

Key Words: Nonholonomic Motion Planning; Variable Geometry Truss; Parallel Manipulators

1. INTRODUCTION

In this paper, a class of Variable Geometry Truss (VGT) assemblies is considered, which are structures consisting of longitudinal repetition of truss modules. In the present instance, each module is implemented as a planar parallel manipulator consisting of two platforms connected by legs whose lengths can vary under the control of linear actuators. Each platform is equipped with a pair of wheels, so that it can move on the plane that supports the structure (Fig. 1). The wheels of each platform are free and not actuated and their motion is independent of each other, while it is assumed that the wheels roll without slipping on the plane. This imposes a nonholonomic constraint on the motion of each platform, namely the requirement that its velocity is perpendicular to the axis connecting the wheels. When the legs of the individual modules are expanded or contracted, the shape of the whole VGT assembly changes. As a consequence of the nonholonomic constraints imposed by the rolling–without–slipping assumption on the wheels, this shape change induces a global motion of the VGT assembly.

The motion planning problem for such an assembly is of the nonholonomic variety (Li and Canny (1993); Murray *et al.* (1994)). The main characteristic of the system presented here is the prominence of shape changes as the means which, together with the action of the nonholonomic constraints, induces global motion. This is analogous to the idea of reorientation in free–floating multibody systems, induced by closed joint space trajectories under the nonholonomic constraint of conservation of angular momentum (Krishnaprasad (1990); Marsden *et al.* (1990)).

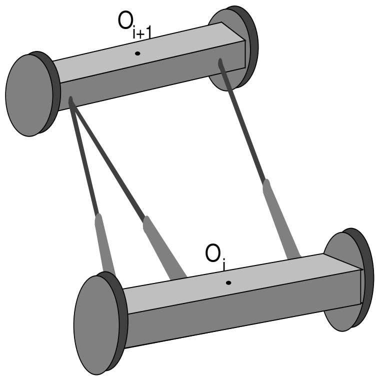

Fig. 1. One Module of the VGT assembly

VGT assemblies of the type discussed here have been examined in the past (see (Chirikjian and Burdick (1991)) and references there), but the emphasis was on its capabilities as a redundant manipulator and on locomotion using snake–like motions, not on the special problems introduced by nonholonomic constraints. A system similar to the one described here was built by (Chirikjian and Burdick (1993)) *using castors instead of wheels* in the platforms of the modules and therefore the nonholonomic constraints that are considered here were not present.

In section 2, the kinematics of a VGT assembly with 2 modules are examined. Consider the $(i, i+1)$–th module (Fig. 1): Its *shape* can be described by the relative position and orientation of a coordinate frame centered at the point O_{i+1} with respect to a coordinate frame centered at the point O_i. Then, the shape of each module corresponds to an element of the Special Euclidean group $SE(2)$ that describes rigid motions on the plane and, as a result, the shape of the 2–module VGT can be described by 2 elements of $SE(2)$. The *configuration* of the whole VGT assembly can be described by the shape of its 2 modules and by the position and orientation of the assembly with respect to some fixed (world) coordinate system, thus by a total of 3 elements of $SE(2)$. In section 2.1, the group theoretical tools and notation used are introduced, in particular the left–invariant dynamical system that describes the evolution of curves in $SE(2)$, their Wei–Norman representation and the adjoint action of $SE(2)$ on its algebra. The shape of each module is expressed, using the Wei–Norman representation of $SE(2)$, as a product of the one–parameter subgroups of $SE(2)$. Then, the configuration of the whole assembly can be expressed as a product of such one–parameter subgroups. In section 2.2, the kinematics of the 2–module VGT are derived. Using the notion of the adjoint action of $SE(2)$ on its Lie algebra, it is possible to determine how the motion of a module relates to the motion of the other modules of the assembly and to express the nonholonomic constraints in a compact form that will be used to make explicit the dependence of the assembly configuration on the shape of its modules. A basic result is presented in Proposition 2.2.3, where it is shown that, because of the nonholonomic constraints, the velocity vector can be partitioned in two parts, one of which constitutes the independent shape controls and the other being velocities which characterize the global motion of the assembly with respect to the world coordinate system. Those latter depend only on the shape of the assembly and the shape controls. In this work, the implementation of each module as a planar parallel manipulator is considered. The shape of each module is determined by the lengths of the legs of the parallel manipulator. From the velocity kinematics of the parallel manipulator it can be seen that motion planning schemes for the VGT assembly can disregard the particular details of the implementation of the modules and only consider the shape of each module. Thus, instead of considering the changes in leg lengths as controls for the VGT assembly, the corresponding shape controls of each module can be used.

In section 3, the way shape changes induce a global snake–like motion is demonstrated. The motion planning problem is considered under a specific shape actuation scheme, where one of the two modules is responsible for the motion of the assembly by periodic changes of its shape and the other module is responsible for steering. It is shown how to generate primitive “straight line motion” and “turning” behaviors, which can be synthesized into more complex ones, like avoidance of obstacles.

In section 4, possible extensions of this work are discussed.

It is found convenient to employ the language of matrix Lie groups throughout, which leads to compact notation and enables the immediate extension of those results to more complex VGT assemblies. For the sake of brevity most proofs are omitted. The interested reader should consult (Krishnaprasad and Tsakiris (1994b)).

2. KINEMATICS

2.1. *The Components of the 2–Module VGT*

The instantaneous shape of a module of the VGT assembly, the position and orientation of each platform of the assembly or the position and orientation of the whole assembly with respect to the world coordinate system correspond, as was discussed in section 1, to an element χ of the matrix Lie group $G = SE(2)$. Given a curve $\chi(.) \subset G = SE(2)$, there is a curve $V(.) \subset \mathcal{G} = se(2)$, the Lie algebra of $SE(2)$, such that:

$$\dot{\chi} = \chi V \ . \tag{1}$$

Let $\{\mathcal{A}_i,\ i = 1, 2, 3\}$ be the following basis of $\mathcal{G}$:

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\} = \\ = \left\{ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2}$$

with $[\ ,\ ]$ being the usual Lie bracket on G. Then:

$$[\mathcal{A}_1\ ,\ \mathcal{A}_2] = \mathcal{A}_3\ ,\ [\mathcal{A}_1\ ,\ \mathcal{A}_3] = -\mathcal{A}_2\ , \\ [\mathcal{A}_2\ ,\ \mathcal{A}_3] = 0\ . \tag{3}$$

Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$, i.e. the space of linear functions from $\mathcal{G}$ to $I\!R$. Let $\{\mathcal{A}_i^\flat,\ i = 1,2,3\}$ be a basis of $\mathcal{G}^*$ such that $\mathcal{A}_i^\flat(\mathcal{A}_j) = \delta_i^j$, for $i,j = 1,2,3$, where δ_i^j is the Kronecker symbol. Then the curve $V(.) \subset \mathcal{G}$ can be represented as:

$$V = \sum_{i=1}^{3} v_i \mathcal{A}_i = \sum_{i=1}^{3} \mathcal{A}_i^\flat(V)\mathcal{A}_i \ . \tag{4}$$

Here v_i is a scalar function of t, the curve parametrization.

Proposition 2.1.1. (Wei and Norman (1964)) Let $\chi(0) = I$, the identity of G. There exists a *global* representation of the solution $\chi(.) \subset G = SE(2)$ of (1) of the form:

$$\chi(t) = e^{\gamma_1(t)\mathcal{A}_1} e^{\gamma_2(t)\mathcal{A}_2} e^{\gamma_3(t)\mathcal{A}_3} \ . \tag{5}$$

The coefficients $\gamma_i \in I\!R$ are related to the coefficients v_i in (4) by:

$$\begin{pmatrix} \dot\gamma_1 \\ \dot\gamma_2 \\ \dot\gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \ . \tag{6}$$

Equation (6) can be solved by quadratures.

For $\chi \in G$, define the *adjoint action* of G on $\mathcal{G}$ denoted $Ad_\chi : \mathcal{G} \to \mathcal{G}$ by:

$$Ad_\chi V \stackrel{\text{def}}{=} \chi V \chi^{-1}, \text{ for } V \in \mathcal{G} \ . \tag{7}$$

From (4):

$$Ad_\chi V = \sum_{i=1}^{3} v_i Ad_\chi \mathcal{A}_i = \sum_{i=1}^{3} \mathcal{A}_i^\flat(V) Ad_\chi \mathcal{A}_i \ . \tag{8}$$

Proposition 2.1.2. Consider the Wei–Norman representation (5) of χ. Then:

$$\begin{aligned} Ad_{\chi^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3 \ , \\ Ad_{\chi^{-1}}\mathcal{A}_2 &= \cos\gamma_1\mathcal{A}_2 - \sin\gamma_1\mathcal{A}_3 \ , \\ Ad_{\chi^{-1}}\mathcal{A}_3 &= \sin\gamma_1\mathcal{A}_2 + \cos\gamma_1\mathcal{A}_3 \ . \end{aligned} \tag{9}$$

Using (2) and (5):

$$\chi(t) = e^{\gamma_1\mathcal{A}_1} e^{\gamma_2\mathcal{A}_2} e^{\gamma_3\mathcal{A}_3} = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

where:

$$\begin{aligned} x &\stackrel{\text{def}}{=} \gamma_2\cos\gamma_1 - \gamma_3\sin\gamma_1 \ , \\ y &\stackrel{\text{def}}{=} \gamma_2\sin\gamma_1 + \gamma_3\cos\gamma_1 \ , \\ \phi &\stackrel{\text{def}}{=} \gamma_1 \ . \end{aligned} \tag{11}$$

2.2. The 2–module VGT

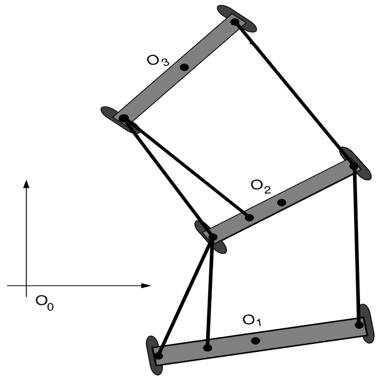

Fig. 2. The 2–module VGT assembly

A chain of $\ell = 2$ modules of the type shown in Fig. 1 is considered (Fig. 2). This system has $n = 3(\ell+1) = 9$ degrees–of–freedom, its configuration space is $Q = SE(2) \times SE(2) \times SE(2)$, it is subject to $3\ell = 6$ holonomic constraints from the parallel manipulator legs and to $p = \ell + 1 = 3$ nonholonomic constraints from the rolling–without–slipping wheel motion. The configuration of the assembly can be determined by its shape (which is an element of the shape space $S = SE(2) \times SE(2)$) *and* by the position and orientation of the assembly with respect to the world coordinate system (which is an element of $G = SE(2)$). Then $Q = G \times S$.

Consider a world coordinate system centered at O_0 and platform coordinate systems centered at $O_i,\ i = 1,2,3$.

Let $\chi_i \in G = SE(2)$ be the configuration matrix of the i–th platform with respect to the world coordinate system. Let $V_i \subset \mathcal{G} = se(2)$ be the corresponding curve in equation (1). Also define the vector $v^i = (v_1^i\, v_2^i\, v_3^i)^\top = (\ \mathcal{A}_1^\flat(V_i)\ \mathcal{A}_2^\flat(V_i)\ \mathcal{A}_3^\flat(V_i)\ )^\top$. Let $\chi_{i,j} \in G$ be the configuration matrix of the j–th platform with respect to the coordinate system of the i–th platform. This subsystem will be called the (i,j)*–th module*. Let $V_{i,j} \subset \mathcal{G} = se(2)$ be the corresponding curve in (1) and let $v^{i,j}$ be the vector of elements of $V_{i,j}$.

The *shape* of the VGT assembly is determined by $\{\chi_{1,2}, \chi_{2,3}\}$. The velocities $\{V_{1,2}, V_{2,3}\}$ are called *shape controls* for reasons to become obvious by the end of this section.

Proposition 2.2.1. From the system kinematics:

$$\begin{aligned} \chi_2 &= \chi_1\chi_{1,2} \ , \\ \chi_3 &= \chi_2\chi_{2,3} = \chi_1\chi_{1,2}\chi_{2,3} \ , \\ \chi_{13} &= \chi_{1,2}\chi_{2,3} \ . \end{aligned} \tag{12}$$

For the corresponding velocities:

$$\begin{aligned} V_2 &= Ad_{\chi_{1,2}^{-1}} V_1 + V_{1,2} \ , \\ V_3 &= Ad_{\chi_{2,3}^{-1}} V_2 + V_{2,3} \\ &= Ad_{\chi_{1,3}^{-1}} V_1 + Ad_{\chi_{2,3}^{-1}} V_{1,2} + V_{2,3} \ , \\ V_{1,3} &= Ad_{\chi_{2,3}^{-1}} V_{1,2} + V_{2,3} \ . \end{aligned} \tag{13}$$

The $p = 3$ nonholonomic constraints of rolling–without–slipping on the wheels of each platform can be expressed, for $i = 1, 2, 3$, as:

$$v_2^i = \mathcal{A}_2^\flat(V_i) = \dot{x}_i \cos\phi_i + \dot{y}_i \sin\phi_i = 0 \ . \tag{14}$$

Proposition 2.2.2. The $p = 3$ nonholonomic constraints can be written in matrix form as:

$$A(\chi_{1,2}, \chi_{2,3})\, v = 0 \ , \tag{15}$$

where $v = \begin{pmatrix} v^1 \\ v^{1,2} \\ v^{2,3} \end{pmatrix}$. The $p \times n$ matrix A is a function of only the shape variables $\chi_{1,2}$, $\chi_{2,3}$ of the chain. It is a block lower triangular matrix of maximal rank $p = 3$ of the form:

$$A(\chi_{1,2}, \chi_{2,3}) = \begin{pmatrix} \star_{1,1} & 0 & 0 \\ \star_{1,2} & \star_{2,2} & 0 \\ \star_{1,3} & \star_{2,3} & \star_{3,3} \end{pmatrix} \tag{16}$$

with the diagonal blocks $\star_{i,i}$, for $i = 1, 2, 3$, defined as $\star_{i,i} = (\, 0\ 1\ 0 \,)$ and the off–diagonal blocks $\star_{i,j}$, for $i < j$, $i = 1, 2$ and $j = 2, 3$, defined as:

$$\begin{aligned} \star_{i,j} &= \Big(\, \mathcal{A}_2^\flat(Ad_{\chi_{i,j}^{-1}} \mathcal{A}_1) \quad \mathcal{A}_2^\flat(Ad_{\chi_{i,j}^{-1}} \mathcal{A}_2) \\ &\qquad\qquad \mathcal{A}_2^\flat(Ad_{\chi_{i,j}^{-1}} \mathcal{A}_3) \, \Big) \\ &= (\, -\gamma_3^{i,j} \quad \cos\gamma_1^{i,j} \quad \sin\gamma_1^{i,j} \,) \ , \end{aligned}$$

where $\gamma_k^{i,j}$ are the Wei–Norman parameters of the configuration of the (i, j)–th module that were introduced in Proposition 2.1.1.

The null space $\mathcal{N}(A)$ has always dimension $m \stackrel{\text{def}}{=} n - p = 6$. Thus, from equation (15) it is possible to show:

Proposition 2.2.3. The (global) velocity of the 2–module VGT assembly with respect to the world coordinate system, as it is characterized by v^1, can be expressed as a function of only the shape variables of the assembly:

$$v^1 = -A_2^{-1}(\chi_{1,2}, \chi_{2,3}) A_1(\chi_{2,3}) \begin{pmatrix} v^{1,2} \\ v^{2,3} \end{pmatrix} \ , \tag{17}$$

by partitioning A as $(A_1 \ A_2)$ with

$$A_1(\chi_{2,3}) = \begin{pmatrix} 0 & 0 \\ \star_{2,2} & 0 \\ \star_{2,3} & \star_{3,3} \end{pmatrix} \tag{18}$$

and a locally invertible $p \times p$ matrix

$$A_2(\chi_{1,2}, \chi_{2,3}) = \begin{pmatrix} \star_{1,1} \\ \star_{1,2} \\ \star_{1,3} \end{pmatrix} \ . \tag{19}$$

Notice that, since A depends only on the shape, so does A_2. As the shape of the assembly changes, A_2 may become singular. The corresponding configurations of the VGT assembly shall be referred to as *nonholonomic singularities.* It is easy to see that the matrix A_2 is singular whenever the axes of all three platforms intersect at the same point or are parallel. However, even in this case, the 3 nonholonomic constraints *remain independent* (c.f. equation (15), where $rank(A) = 3$), but, since the platforms have a common instantaneous center of rotation, equation (15) cannot be recast in the form of (17). Therefore, the system's motion cannot be controlled by the shape controls alone and the dynamics of the system ought to be considered. This is analogous to what practising engineers refer to as loss of control authority.

Unlike previous work on nonholonomic motion planning, in this case the shape controls in equation (17) do not correspond directly to the controls of the system and, thus, are not at the disposal of the designer to alter at will. The real controls are the leg velocities $\dot{\sigma}$ of the parallel manipulator modules. However, off the *kinematic* singularities of the parallel manipulators, the shape controls can easily determine the corresponding leg velocities (Krishnaprasad and Tsakiris (1994b)). Therefore, in order to simplify the discussion of motion planning, the particulars of the implementation of the modules will be disregarded and only actuation under the shape controls will be considered.

From $\dot{\chi}_1 = \chi_1 V_1$ and the properties (3) of the basis $\{\mathcal{A}_i\}$ of $\mathcal{G}$, it is easy to see that, away from the nonholonomic singularities, *controllability* is guaranteed for a generic set of shape controls whenever the 1st and 3rd rows of the matrix $A_2^{-1} A_1$ are linearly independent. However, the latter always holds away from the nonholonomic singularities.

3. MOTION PLANNING

In this section, the motion planning problem for the 2-module VGT assembly is examined. There are several possible *actuation strategies* for this system. A simple one will be considered here, where the first module (module–(1, 2)) "steers" the system, while the second (module–(2, 3)) provides the translation mechanism through periodic variations of its shape parameters. More specifically, the special case of motions that are generated by keeping the shape of the first module fixed, i.e. $v^{1,2} = 0$, and vary the shape controls $v^{2,3}$ of the second module periodically will be considered. As can be seen from (17), the *global* motion of the system is determined completely, at least away from the nonholonomic singularities, by those two sets of *shape controls*.

A *qualitative* description of this motion is given by the following result:

Proposition 3.1. *i)* If $v^{1,2} = 0$ and $\gamma_1^{1,2} = 0$, the 2–module VGT instantaneously *translates* along an axis perpendicular to platforms 1 and 2.

ii) If $v^{1,2} = 0$ and $\gamma_1^{1,2} \neq 0$, the 2–module VGT instantaneously *rotates* around the intersection of the axes of platforms 1 and 2.

A *quantitative* description of the system's motion will be provided next, under a set of shape controls where $v^{1,2} = 0$ and where $v^{2,3}$ is the following periodic control:

$$\begin{aligned} v_1^{2,3} &= \alpha_1 \omega \cos \omega t \ , \\ v_2^{2,3} &= \alpha_2 \omega \sin \omega t \cos \gamma_1^{2,3} \ , \\ v_3^{2,3} &= -\alpha_2 \omega \sin \omega t \sin \gamma_1^{2,3} \ . \end{aligned} \tag{20}$$

As can be seen by (6) and (11), this $v^{2,3}$ forces the shape of the module–$(2,3)$ to trace a closed elliptical path in $(x_{2,3}, \phi_{2,3})$–space.

In Proposition 3.1 it was shown that the instantaneous global motion of the VGT assembly induced by those shape controls, as characterized by the position and orientation γ^1 of platform 1, is a translation whenever $\gamma_1^{1,2} = 0$ or a rotation whenever $\gamma_1^{1,2} \neq 0$. The question is whether, after a period $T = \frac{2\pi}{\omega}$ of the shape controls, there is a net motion $\Delta\gamma^1 \stackrel{\text{def}}{=} \gamma^1(T) - \gamma^1(0)$ of the VGT assembly which would correspond to the "stride length" of the motion induced by those shape controls. This is equivalent to the geometric phase idea of (Krishnaprasad (1990)).

3.1. *Translation*

Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) = 0$. In Proposition 3.1 it was shown that the instantaneous motion of the 2–module VGT in this case is a translation along the perpendicular to platforms 1 and 2. The position of the assembly on this axis can be characterized by the parameter γ_3^1. From (17):

$$\begin{aligned} \Delta\gamma_3^1(t) = \gamma_3^1(t) - \gamma_3^1(0) &= \int_0^t v_3^1(\tau) d\tau \\ &= -\alpha_2 \int_0^t \frac{\omega \sin \omega\tau}{\tan(\gamma_1^{2,3}(0) + \alpha_1 \sin \omega\tau)} d\tau \ . \end{aligned} \tag{21}$$

Using Mathematica, equation (21) can be integrated numerically and it can be verified that after a period of the shape controls, the 2–module VGT assembly has moved forward by a distance specified by $\Delta\gamma_3^1(\frac{2\pi}{\omega}) = \gamma_3^1(\frac{2\pi}{\omega}) - \gamma_3^1(0)$ (Fig. 3).

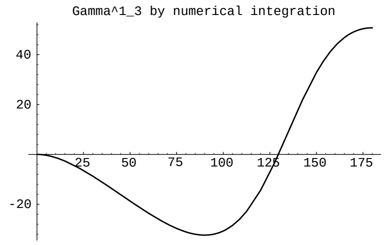

Fig. 3. Geometric phase of translating 2–VGT

3.2. *Rotation*

Let $\gamma_1^{1,2} = \gamma_1^{1,2}(0) \neq 0$. In Proposition 3.1 it was shown that the instantaneous motion of the 2–module VGT in this case is a rotation around the intersection of the axes of platforms 1 and 2. The position of the assembly with respect to this point can be characterized by the angle γ_1^1. From (17):

$$\begin{aligned} \Delta\gamma_1^1(t) = \gamma_1^1(t) - \gamma_1^1(0) &= \int_0^t v_1^1(\tau) d\tau \\ &= -\sin \gamma_1^{1,2}(0) \int_0^t \frac{v_2^{2,3}(\tau)}{det(A_2(\tau))} d\tau \ . \end{aligned} \tag{22}$$

Using Mathematica, equation (22) can be integrated numerically and it can be verified that for e.g. $\gamma_1^{1,2} = -\frac{\pi}{4}$, after a period of the shape controls, the 2–module VGT assembly rotates clockwise around the intersection of the first module platforms' axes by an angle specified by $\Delta\gamma_1^1(\frac{2\pi}{\omega}) = \gamma_1^1(\frac{2\pi}{\omega}) - \gamma_1^1(0)$.

If the closed shape–space path described by equation (20) is traced in the reverse direction, the assembly will translate backwards by the same distance or will rotate counter–clockwise by the same angle.

The nonholonomic kinematics of the 2–module VGT were simulated on a Silicon Graphics Indigo 2 graphics workstation. As can be seen from Fig. 3, during a period of the shape controls the 2–VGT assembly first moves "backwards" and then "forward". This gives the impression of a snake–like motion. The primitive straight line and rotational motions described above can be synthesized to display more complex behaviors of the system, like obstacle avoidance.

4. CONCLUSIONS

In this paper, Variable Geometry Truss assemblies *with nonholonomic constraints* were introduced. Their kinematics were derived and motion planning was examined by showing how periodic shape changes induce global translation or rotation of the assembly under the influence of the nonholonomic constraints. This discussion can be extended to VGT assemblies of ℓ–modules for $\ell > 2$ and to non–serial tree–like module arrangements. It is noteworthy that the nonholonomic kinematics of those assemblies can be analyzed by a straightforward application of the tools introduced here (Krishnaprasad and Tsakiris (1994b)).

The framework discussed here is an instance of a class of nonholonomic systems referred to as *G–snakes*. Those are nonholonomic kinematic chains evolving on a Lie group G. The configuration and shape spaces of the chain form a G–bundle, while the nonholonomic constraints specify a *connection* on it (this generalizes Proposition 2.2.3). The connection gives rise to a holonomy whenever shape–space loops are traversed. Details appear in (Krishnaprasad and Tsakiris (1994a)).

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The machines that Joel Burdick and his students have built inspired this work. It is a pleasure to acknowledge the creative energies behind those machines and, especially, Burdick's willingness to communicate freely his ideas and his enthusiasm.

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