APPROXIMATE TRACKING FOR SYSTEMS ON THREE-DIMENSIONAL MATRIX LIE GROUPS VIA FEEDBACK NILPOTENTIZATION

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Abstract: A wide range of dynamical systems from fields as diverse as mechanics, electrical networks and molecular chemistry can be modeled by invariant systems on matrix Lie groups. This paper extends the concept of approximate tracking in the high-frequency limit to non-nilpotent three-dimensional matrix Lie groups by making use of feedback nilpotentization for the local representations of these systems. Further it is shown how to convert these tracking controls involving a state-feedback to an open-loop control law, which can be interpreted as an approximate inverse of the original system.

Keywords: Control systems on matrix Lie groups, nonholonomic control systems, path planning, differential geometric methods, approximate inversion.

1. INTRODUCTION

Invariant systems on matrix Lie groups arise in a wide range of fields encompassing problems as diverse as motion planning and control for autonomous vehicles (Leonard and Krishnaprasad, 1995), power conversion with switching circuits and coherent control of molecular dynamics (Dahleh *et al.*, 1996). The kinematics of the rigid body orientation, as encountered in the attitude control problem for a satellite or an underwater vehicle, can be modeled as an invariant system on the Special Orthogonal group $SO(3)$. We consider the drift-free, controllable setting, with fewer controls than the dimension of the group (state space). This leads to an important sub-class of nonholonomic systems on Lie groups.

In contrast with linear systems where the controllability Gramian can directly be used to construct stabilizing or tracking controls, no methods exist to date for extracting such controls for general nonlinear systems from the controllability Lie algebra. Some progress has been made in this direction for drift-free affine systems (Canny and Li, 1993; Murray and Sastry, 1993) using oscillatory controls which systematically "excite" higher order brackets of the system to steer the system in the desired direction. Leonard and Krishnaprasad considered the problem in a Lie group setting and, applying averaging theory, obtained

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in (Leonard and Krishnaprasad, 1995) results on approximate point-to-point constructive controls for left-invariant systems on arbitrary finite-dimensional Lie groups. Sussmann and Liu gave a very general result for path constructive controllability of drift-free, nilpotent systems on $\Re^n$ (Sussmann and Liu, 1991).

Brockett, achieved in (Brockett, 1993; Brockett, 1996) an appealing synthesis of the problems of open-loop tracking and feedback stabilization via the concept of *approximate inversion* of a system as captured in a simple, three dimensional nilpotent system. Here we investigate Brockett's notion of approximate inversion in the setting of systems on *non*-nilpotent Lie groups. Our main tool will be feedback nilpotentization, which was introduced in the nonlinear control context by Hermes, Lundell, and Sullivan in (Hermes *et al.*, 1984). Also, in (Murray, 1994) necessary and sufficient conditions for feedback stabilization are presented and nilpotentization is applied to nonholonomic motion planning, although not in the context of approximate inversion.

In Section 2 we will introduce exponential representations of systems on matrix Lie groups, exhibit the nilpotentizing input transformations for representatives of controllable, drift-free, invariant systems on three-dimensional matrix Lie groups, and finally show how to obtain open-loop controls for approximate tracking in the high-frequency limit. Section 3 summarizes our results and discusses possible extensions to more general settings.

2. MAIN RESULTS

2.1 *Preliminaries*

For a curve X in G, a real matrix Lie group of dimension n and $\mathcal{G}$ the Lie algebra associated with G having a basis $\{A_1, \ldots, A_n\}$ of constant matrices in $\mathcal{G}$, denote the control U as $U(t) = \sum_{i=1}^{m} u_i(t) A_i,\ m \le n,\ t \ge 0$. The corresponding left-invariant drift-free system on G is

$$\dot{X}(t) = X(t)U(t), \qquad X(0) = X_0. \tag{1}$$

This paper focuses on controllable two-input systems on the matrix Lie groups $H(3)$, $SE(3)$, $SL(2)$, and $SO(3)$, which form a "nearly" complete set of representatives for all controllable two-input systems on three-dimensional matrix Lie groups. For a complete classification of three-dimensional Lie algebras and their two-dimensional left-invariant nonholonomic distributions see (Vershik and Gershkovich, 1994).

An exponential representation of (1) on $\Re^n$ leads to a system of the form

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x) = F(x)U, \tag{2}$$

where $x \in \Re^n$ and $U \in \Re^m$. The *product of exponentials representation* is based on the *canonical coordinates of the second kind* and is characterized in the following theorem, where we use an extended control vector $u = (u_1, \ldots, u_n)^T$ with $u_i = 0, i = m+1, \ldots, n$.

Theorem 1. (Wei and Norman, 1964) Consider the system (1) and its solution $X(t), t \ge 0$. Then, in a neighborhood of $t = 0$ the solution may be expressed in the form

$$X(t) = e^{x_1(t)A_1} e^{x_2(t)A_2} \cdots e^{x_n(t)A_n}. \tag{3}$$

The coordinate functions $x = (x_1(t) \ldots x_n(t))^T$ evolve according to

$$\dot{x}(t) = M(x(t))u(t), \tag{4}$$

where the matrix M is analytic in x. Moreover, if $\mathcal{G}$ is solvable, then there exists a basis and an ordering of this basis for which the representation (3) is global and the x_i can be computed by quadratures.

The computation of $M(x)$ for specific Lie groups or Lie algebras involves differentiating (3), setting this equal to the left-hand side of (1) and using the structure of the corresponding Lie algebra $\mathcal{G}$ (Wei and Norman, 1964).

Example 2. Let $SL(2)$ denote the Special Linear group of 2×2 matrices with determinant one and fix a basis for the corresponding Lie algebra $sl(2)$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = 2\,A_3,\ [A_1, A_3] = -A_1,\ [A_2, A_3] = A_2.$$

Then, the product of exponentials representation turns out to be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} e^{x_3} & 0 & 0 \\ x_2^2 e^{x_3} & e^{-x_3} & 0 \\ -2x_2 e^{x_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

2.2 *Direct Approximate Inversion*

The approach to trajectory tracking for nonholonomic systems taken here is inspired by (Brockett, 1993; Brockett, 1996). It is based on the well-known fact that periodic controls with an appropriate phase and frequency relation create a secular motion in the direction of a certain higher order Lie bracket of the system, while adding only zero-mean perturbations to motions in other directions. Moreover, these perturbations vanish with increasing frequencies of the oscillatory control components which allows, as will be made precise below, to track in the high-frequency limit more states than there are controls available. Modeling such open-loop control laws as time-varying nonlinear systems with the desired trajectories as inputs we obtain the so called *approximate inverse* of the original system. This terminology is motivated by the fact that the concatenation of the approximate inverse and the original system approximates the identity operator on the space of trajectories in the high-frequency limit.

Example 3. Let $H(3)$ denote the Heisenberg group of real 3×3 upper triangular matrices X of the form

$$X = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_i \in \Re,\ i = 1, 2, 3$$

and fix a basis for the corresponding Lie algebra $h(3)$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = 0,\ [A_2, A_3] = 0.$$

The product of exponentials representation of system (1) with $G = H(3)$ can be written as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_2. \end{aligned} \tag{5}$$

Note, that this representation holds globally and is in *chained form.* If we had used the *single* exponential representation (Magnus, 1954) the resulting Magnus equation would have yielded directly the so-called *nonholonomic integrator* for which Brockett presented an approximate inverse in (Brockett, 1993). Similarly, given a desired smooth trajectory $\bar{x} = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T,\ t \geq 0$ the approximate tracking controls for (5) can be written as

$$\begin{aligned} u_1(t) &= \dot{\bar{x}}_1 + \sqrt{|m(t)|}\ \sin(\omega t) \\ u_2(t) &= \dot{\bar{x}}_2 + \mathrm{sign}(m(t))\sqrt{|m(t)|}\ \cos(\omega t), \end{aligned} \tag{6}$$

where $m(t) = 2\omega(\dot{\bar{x}}_3 - \dot{\bar{x}}_1 \bar{x}_2)$ and ω is the carrier frequency. Then, assuming *matched initial conditions*

$$x(0) = \begin{pmatrix} \bar{x}_1(0) - \sqrt{|m|}/\omega \\ \bar{x}_2(0) \\ \bar{x}_3(0) - \bar{x}_2(0)\sqrt{|m(0)|}/\omega \end{pmatrix}$$

the resulting trajectory of (5) converges uniformly in t to the desired trajectory $\bar{x}$ while ω goes to infinity (see proof of Lemma 10 below in Section 2.5). Note, that a smaller tracking error has to be paid for with higher control amplitudes and actuation bandwidth.

In (Struemper and Krishnaprasad, 1997) we have extended Brockett's idea to two-input chained form systems with arbitrary dimension $n \geq 3$. This is significant in this context since a class of not necessarily nilpotent systems and their local representations can be made equivalent to chained form systems with a simple input transformation.

2.3 *Nilpotentization*

Since approximate inversion controls can be found for chained form systems, the question arises if the local representations of systems on matrix Lie groups can be converted to chained form. Given an affine system (resulting e.g. from an exponential representation of (1))

$$\Sigma:\ \dot{x} = f_1(x)u_1 + \cdots + f_m(x)u_m \tag{7}$$

let $\Delta(x) = \mathrm{span}\{f_1(x), \ldots, f_m(x)\}$ denote the *control distribution* of Σ. Recall that a basis of vector fields $\{f_1(x), \ldots, f_m(x)\}$ for Δ is called nilpotent if there exists an integer k such that all iterated Lie brackets $ad_{f_i}^k f_j = 0$ for all $i, j = 1, \ldots, m$. The goal of nilpotentization in the context of approximate inversion is to find a nilpotent basis $\{g_1(x), \ldots, g_m(x)\}$ for Δ and, in particular, one which generates a Lie algebra of vector fields isomorphic to the one generated by the control vector fields of a chained form system.

The problem of finding the corresponding coordinate transformation can be tackled alternatively in a tangent space or in a cotangent setting. Both approaches are treated in (Hermes *et al.*, 1984), where sufficient conditions for the existence of such a transformation are given. Since in the context of two-input systems on three-dimensional matrix Lie groups the control distribution Δ has codimension one it is of advantage to go the cotangent space route. Then the problem is reduced to transforming the differential one-form $\alpha = \Delta^{\perp}$ annihilating Δ into the 1-form $\beta = -y_2 dy_1 + dy_3$ associated with a two-input chained form system of dimension three.

A corollary to Theorem 3 in (Hermes *et al.*, 1984) supplies us with sufficient conditions which can be easily checked in our context.

Theorem 4. Let M be an $(n+1)$-manifold, let $F = \{f_1, \ldots, f_n\}$ be a local basis of vector fields for an n-dimensional distribution Δ in a neighborhood of $p \in M$, and let ω be a 1-form such that $\langle \omega, f_i \rangle = 0$ near p for $i = 1, 2, \ldots, n$. If the rank of the skew-symmetric matrix $S = (\langle \omega(p), [f_i, f_j] \rangle)$ is n for n even or $n-1$ for n odd then Δ has a nilpotent basis.

Now, for the exponential representation $\dot{x} = f_1(x)u_1 + f_2(x)u_2$ of a controllable, two-input systems on three-dimensional matrix Lie groups the Lie bracket $[f_1(x), f_2(x)]$ is independent of $f_1(x)$ and $f_2(x)$ on a neighborhood of the origin. It follows that

$$\langle \omega, [f_1(x), f_2(x)] \rangle = -\langle \omega, [f_2(x), f_1(x)] \rangle \neq 0$$

and consequently S has rank 2, which implies by Theorem 4 that the control distribution Δ admits a nilpotent basis for *all* systems in this class.

The process of actually constructing the required coordinate transform involves in general the solution of a partial differential equation, but in the cases considered here the coordinate transformation $T : \Re^3 \to \Re^3;\ x \mapsto y = T(x)$ can be determined directly from the transformation equation relating α to β:

$$(\alpha_1, \alpha_2, \alpha_3) = (-y_2, 0, 1) \left(\frac{\partial T}{\partial x} \right) \tag{8}$$

Example 5. Consider the product of exponentials representation (2) of a system on $SL(2)$, i.e. we have

$$f_1(x) = \begin{pmatrix} e^{x_3} \\ x_2^2 e^{x_3} \\ -2x_2 e^{x_3} \end{pmatrix}, \ f_2(x) = \begin{pmatrix} 0 \\ e^{-x_3} \\ 0 \end{pmatrix}$$

and $\Delta(x)$= span$\{f_1(x), f_2(x)\}$. The one-form annihilating Δ can then be written as

$$\alpha(x) = -x_2 dx_1 - \frac{1}{2} dx_3.$$

Now, (8) is satisfied with

$$y = T(x) = (x_1, x_2, -\frac{1}{2}x_3)^T$$

which is clearly a diffeomorphism and for which $T(0) = 0$. The nilpotent basis for Δ can be computed using the induced tangent map of T, i.e.

$$g_1(x) = \left(\frac{\partial T}{\partial x} \right)^{-1} g_1(T(x)) = (1, 0, -2x_2)^T \tag{9}$$

and

$$g_2(x) = (0, 1, 0)^T. \tag{10}$$

2.4 *Approximate Tracking Controls for Feedback Nilpotentized Systems*

The affine system

$$\Sigma_{nil} : \ \dot{x} = g_1(x)v_1 + g_2(x)v_2 = G(x)v \tag{11}$$

obtained from the nilpotent basis for Δ = span$\{f_1, f_2\}$ is *trajectory equivalent* to the original system (7). There exists a matrix $H(x)$ invertible in a neighborhood of the origin such that

$$\dot{x} = G(x)v = F(x)H(x)v \tag{12}$$

i.e. a trajectory of Σ_{nil} with input v is identical to the trajectory of the original affine system (7) with input $u = H(x)v$.

On the other hand Σ_{nil} which transforms under T into the chained form system Σ_{cfs}

$$\begin{aligned} \dot{y}_1 &= v_1 \\ \dot{y}_2 &= v_2 \\ \dot{y}_3 &= y_2 v_1. \end{aligned} \tag{13}$$

can be approximately inverted by replacing the desired trajectory $\bar{y}$ in the control equations (6) by $T(x)$:

$$v_1(t) = \frac{d}{dt}(T_1(\bar{x})) + \sqrt{|m(t)|}\ \sin(\omega t) \qquad (14)$$
$$v_2(t) = \frac{d}{dt}(T_2(\bar{x})) + \mathrm{sign}(m(t))\sqrt{|m(t)|}\ \cos(\omega t),$$

with $m(t) = 2\omega(\frac{d}{dt}(T_3(\bar{x})) - \frac{d}{dt}(T_1(\bar{x}))\,T_2(\bar{x}))$.

Thus we have obtained the approximate tracking controls $u = H(x)v$ for (7), i.e. letting the frequency parameter ω in u go to infinity the resulting trajectory converges uniformly to the desired trajectory on any finite interval $[0, T]$.

Example 6. Consider again the single exponential representation (2) of a system on $SL(2)$ and its nilpotent basis (9), (10) for Δ derived above. We have

$$g_1(x) = e^{-x_3} f_1(x) - x_2^2 e^{x_3} f_2(x) \qquad (15)$$
$$g_2(x) = e^{x_3} f_2(x) \qquad (16)$$

and therefore the input transformation

$$H(x) = \begin{pmatrix} e^{-x_3} & 0 \\ -x_2^2 e^{x_3} & e^{x_3} \end{pmatrix} \qquad (17)$$

makes (2) equivalent to the corresponding Σ_{nil}. Then, the approximate inversion controls v for Σ_{nil} are identical to (6) only with $m(t) = 2\omega(-\frac{1}{2}\dot{\bar{x}}_3 - \dot{\bar{x}}_1\bar{x}_2)$. Hence, after applying $H(x)$ to v the controls

$$u_1(t) = e^{-x_3}(\dot{\bar{x}}_1 + \sqrt{2\omega|m(t)|}\ \sin(\omega t)) \qquad (18)$$
$$u_2(t) = -x_2^2 e^{x_3}(\dot{\bar{x}}_1 + \sqrt{2\omega|m(t)|}\ \sin(\omega t))$$
$$+ e^{x_3}(\dot{\bar{x}}_2 + \mathrm{sign}(m(t))\sqrt{2\omega|m(t)|}\ \cos(\omega t)),$$

with m as above achieve approximate tracking for the local representation of the $SL(2)$-system in the high frequency limit (see Fig. 1)

We are now ready to summarize the previous development in the following statement:

Proposition 7. Given a two-input, controllable system on a three-dimensional matrix Lie group and its product of exponentials representation $\dot{x} = f_1(x)u_1 + f_2(x)u_2$, there exist controls of the form $u = H(x)v$ with $H(x)$ as in (12), v as in (14), and matched initial conditions, such that the resulting trajectory converges uniformly in t to a desired trajectory $\bar{x}$ in the high-frequency limit.

In the following we list the results of the above procedure applied to the remaining representatives of our class of systems. Figure 2 shows a block diagram of the proposed control strategy.

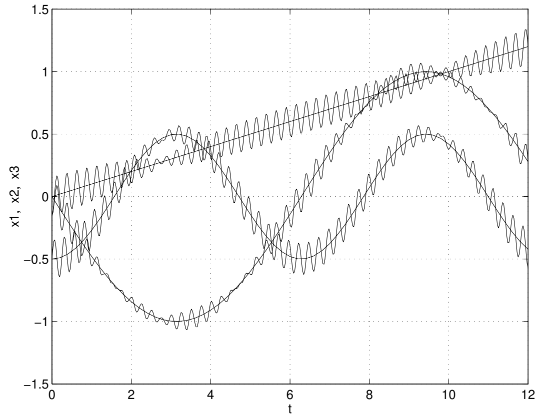

Fig. 1. Approximate tracking for Wei-Norman representation of system on $SL(2)$ with $\omega = 25$

Example 8. Let $SE(2)$ denote the Special Euclidean group representing rigid motions in the plane and fix a basis for the corresponding Lie algebra $se(2)$

$$A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = -A_2,\ [A_2, A_3] = 0.$$

The product of exponentials representation turns out to be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_3 & 1 & 0 \\ -x_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

To retain controllability we choose the two-input system corresponding to $u_3 = 0$. Due to the apparent similarity with the chained form system (13), we can directly read off the required transformations

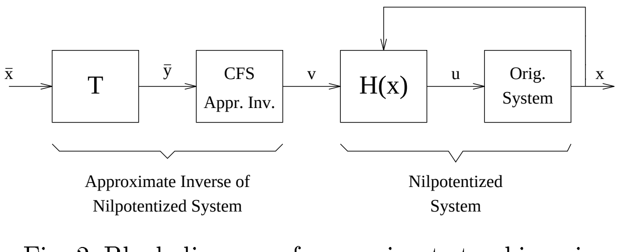

Fig. 2. Block diagram of approximate tracking via feedback nilpotentization

$$T(x) = \begin{pmatrix} -x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (19)$$

$$H(x) = \begin{pmatrix} -1 & 0 \\ x_3 & 1 \end{pmatrix}. \qquad (20)$$

Example 9. Let $SO(3)$ denote the Special Orthogonal group representing rotations in three-dimensional Euclidean space and fix a basis for the corresponding Lie algebra $so(3)$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yielding the following bracket structure:

$$[A_1, A_2] = A_3,\ [A_1, A_3] = -A_2,\ [A_2, A_3] = A_1.$$

The product of exponentials representation turns out to be

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \sec x_2 \cos x_3 & -\sec x_2 \sin x_3 & 0 \\ \sin x_3 & \cos x_3 & 0 \\ -\tan x_2 \cos x_3 & \tan x_2 \sin x_3 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

For computational convenience but without loss of generality we construct a two-input system by choosing the inputs u_1 and u_3, relabeling them with u_1 and u_2, respectively and permutating the state x_2 and x_3 leading to

$$\dot{x} = \begin{pmatrix} \cos x_2 \sec x_3 \\ -\tan x_3 \cos x_2 \\ \sin x_2 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_2$$

Then with the terminology established above one obtains

$$\begin{aligned} \alpha &= -\tan x_2 \cos x_3 \, dx_1 + dx_3 \\ T(x) &= \begin{pmatrix} x_1 \\ \tan x_2 \cos x_3 \\ x_3 \end{pmatrix} \\ H(x) &= \begin{pmatrix} \sec x_2 \cos x_3 & 0 \\ \sin x_3 (1 + \sin^2 x_2) & \cos^2 x_2 \sec x_3 \end{pmatrix}. \end{aligned}$$

Note, that the approximate tracking controls derived for the systems on $SE(2)$, $SL(2)$, and $SO(3)$ require state feedback and are not in open-loop form as is the case for the approximate tracking controls of the system on $H(3)$. Therefore, the former controls can not be interpreted as providing an approximate inverse for the corresponding systems.

2.5 *Open-Loop Controls*

In certain practical applications it might be costly or even impossible to measure the current state of the system. Also the nature of the controls at our disposal might preclude an on-line modification of the controls (see e.g. (Dahleh *et al.*, 1996)). For these cases it would be desirable to obtain an open-loop version of the controls derived above.

Replacing the current state x with the desired state $\bar{x}$ in $H(x)$, does not guarantee that the resulting control would achieve approximate tracking. Even though $H(x)$ converges to $H(\bar{x})$ in an appropriate operator norm, v itself depends on the frequency parameter ω and

$$lim_{\omega \to \infty} (H(\bar{x}) - H(x)) v \neq 0$$

in general. But since Σ_{nil} is state-equivalent to the chained form system (13) which can be integrated by quadratures, we can obtain a better "open-loop" approximation to $H(x)$. The following lemma shows how to obtain an approximation $\hat{x}$ of x up to order ω^{-1}.

Lemma 10. Let Σ_{nil} be an affine system equivalent to (13) under a diffeomorphism T. Assume the desired trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))$ is sufficiently smooth on the time interval considered. Let $x(t)^\omega,\ t \geq 0$ be the trajectory of Σ_{nil} with input v as specified in (14). Let $\hat{x} = T^{-1}(\hat{y})$ with $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)^T$ defined by

$$\begin{aligned} \hat{y}_1(t) &= \bar{y}_1(t) - \omega^{-1} \sqrt{|m(t)|} \cos(\omega t) \\ \hat{y}_2(t) &= \bar{y}_2(t) + \omega^{-1} \mathrm{sign}(m(t)) \sqrt{|m(t)|} \sin(\omega t) \\ \hat{y}_3(t) &= \bar{y}_3(t) - \omega^{-1} \bar{y}_2(t) \sqrt{|m(t)|} \cos(\omega t), \end{aligned}$$

where $m(t) = 2\omega(\dot{\bar{y}}_3(t) - \dot{\bar{y}}_1(t)\bar{y}_2(t))$ and $\bar{y} = T(\bar{x})$. Then

$$x^\omega(t) - \hat{x}(t) = o(\omega^{-1}), \qquad \forall t \geq 0.$$

Proof: Denoting $y^\omega(t)$ the trajectory of Σ_{cfs} (13) with input v (14), we will show that

$$y^\omega(t) - \hat{y}(t) = o(\omega^{-1}), \qquad \forall t \geq 0.$$

from which (10) follows since Σ_{nil} and Σ_{cfs} are equivalent under T. Using integration by parts for (13) with controls (14) yields

$$y_1^\omega(t) = y_1^\omega(0) + \int_0^t \{\dot{\bar{y}}_1(\tau) + \sqrt{|m(\tau)|} \sin(\omega\tau)\} d\tau$$

$$
\begin{aligned}
&= y_1^\omega(0) + \bar{y}_1(t) - \bar{y}_1(0) \\
&\quad +\omega^{-1}\sqrt{|m(0)|} - \omega^{-1}\sqrt{|m(t)|}\cos(\omega t) \\
&\quad +\omega^{-1}\int_0^t \frac{d}{d\tau}(\sqrt{|m(\tau)|})\cos(\omega\tau)d\tau \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
y_2^\omega(t) &= y_2^\omega(0) + \int_0^t \{\dot{\bar{y}}_2(\tau) \\
&\quad +\mathrm{sign}(m(\tau))\sqrt{|m(\tau)|}\cos(\omega\tau)\}d\tau \\
&= y_2^\omega(0) + \bar{y}_2(t) - \bar{y}_2(0) \\
&\quad +\omega^{-1}\mathrm{sign}(m(t))\sqrt{|m(t)|}\sin(\omega t) \qquad (22) \\
&\quad -\omega^{-1}\int_0^t \mathrm{sign}(m(\tau))\frac{d}{d\tau}(\sqrt{|m(\tau)|})\sin(\omega\tau)d\tau.
\end{aligned}
$$

To determine the order in ω of the integrals in (21),(22) note that for a continuously differentiable function $f(t)$, $t \in [0,T]$ with $f(0) = 0$ and $f(t) > 0$, $\forall t > 0$

$$
\begin{aligned}
I &= \int_0^T \frac{d}{d\tau}(\sqrt{f(\tau)})\sin(\omega\tau + \phi)d\tau \\
&= \int_0^{\sqrt{T}} \frac{d}{d\sigma}(\sqrt{f(\sigma^2)})\sin(\omega\sigma^2 + \phi)d\sigma
\end{aligned}
$$

after substituting $\tau = \sigma^2$. For f smooth it follows that $\sqrt{f(\sigma^2)}$ is continuously differentiable and therefore $\frac{d}{d\sigma}(\sqrt{f(\sigma^2)})$ is bounded on $[0,T]$. Using a Second Mean Value Theorem and performing another substitution $\sigma = \omega^{-1/2}\nu$ it can then be verified that $I = o(\omega^{-1/2})$.

This argument applied to the integrals (or the corresponding sums of integrals over subintervals of $[0,T]$) in (21),(22) results in

$$
\begin{aligned}
y_1^\omega(t) &= \bar{y}_1(t) - \omega^{-1}\sqrt{|m(t)|}\cos(\omega t) \\
&\quad +o(\omega^{-1}) \\
y_2^\omega(t) &= \bar{y}_2(t) + \mathrm{sign}(m(t))\omega^{-1}\sqrt{|m(t)|}\sin(\omega t) \\
&\quad +o(\omega^{-1})
\end{aligned}
$$

For the third state one obtains

$$
\begin{aligned}
y_3^\omega(t) &= y_3^\omega(0) + \int_0^t \Big(\bar{y}_2(\tau) + o(\omega^{-1}) \\
&\quad +\mathrm{sign}(m(\tau))\omega^{-1}\sqrt{|m(\tau)|}\sin(\omega\tau)\Big) \\
&\quad \Big(\dot{\bar{y}}_1(\tau) + \sqrt{|m(\tau)|}\sin(\omega\tau)\Big)d\tau \\
&= y_3^\omega(0) + \int_0^t \{\dot{\bar{y}}_1(\tau)\bar{y}_2(\tau) \\
&\quad +(2\omega)^{-1}m(\tau)(1-\cos(2\omega\tau)) \\
&\quad +\omega^{-1}\dot{\bar{y}}_1(\tau)\mathrm{sign}(m(\tau))\sqrt{|m(\tau)|}\sin(\omega\tau) \\
&\quad +\bar{y}_2(\tau)\sqrt{|m(\tau)|}\sin(\omega\tau)\}d\tau + o(\omega^{-1}) \\
&= y_3^\omega(0) + \bar{y}_3(t) - \bar{y}_3(0) + \omega^{-1}\bar{y}_2(0)\sqrt{|m(0)|} \\
&\quad -\omega^{-1}\bar{y}_2(t)\sqrt{|m(t)|}\cos(\omega t) \\
&\quad +\int_0^t \{-(2\omega)^{-1}m(\tau)\cos(2\omega\tau) \\
&\quad +\omega^{-1}\dot{\bar{y}}_1(\tau)\mathrm{sign}(m(\tau))\sqrt{|m(\tau)|}\sin(\omega\tau) \\
&\quad -\omega^{-1}\frac{d}{d\tau}(\bar{y}_2(\tau)\sqrt{|m(\tau)|})\cos(\omega\tau))\}d\tau \\
&\quad +o(\omega^{-1}).
\end{aligned}
$$

Assuming matched initial conditions and using again the argument presented above it follows that

$$
\begin{aligned}
y_3^\omega(t) &= \bar{y}_3(t) - \omega^{-1}\bar{y}_2(t)\sqrt{|m(t)|}\cos(\omega t) \\
&\quad +o(\omega^{-1}).
\end{aligned}
$$

Thus, $y^\omega(t) - \hat{y}(t) = o(\omega^{-1})$, $\forall t \geq 0$, which proves the claim. □

Note, that $\hat{x}(t)$ can be easily computed from $\bar{x}(t)$ via a time-periodic map requiring *no integration* if T is available in closed form. Replacing x in $H(x)$ by $\hat{x}$ we obtain open loop controls achieving approximate trajectory tracking in the high frequency limit.

Theorem 11. Given a two-input, controllable system on a three-dimensional matrix Lie group and its product of exponentials representation Σ : $\dot{x} = F(x)u$, let $H(x)$ be an input transformation achieving nilpotentization and T a diffeomorphism transforming the nilpotentized system Σ_{nil} : $\dot{x} = F(x)H(x)u$ to chained form (13). Let Σ have matched initial conditions and let the controls be of the form $u = H(\hat{x})v$, with $\hat{x}$ as in Lemma 10 and v as in (14).
Then, letting the frequency parameter ω of u go to infinity, the resulting trajectory x^ω converges uniformly in t to the desired trajectory $\bar{x}$.

Proof: We will show that the solution x^ω of Σ with open-loop controls $u = H(\hat{x})v$ will converge to the solution $\tilde{x}^\omega$ of Σ with feedback controls $\tilde{u} = H(x)v$ in the high-frequency limit. Then, since by Proposition 7 $\tilde{x}^\omega$ converges to the desired trajectory $\bar{x}$, also x^ω converges to $\bar{x}$ in the high-frequency limit.

Now for $\lim_{\omega\to\infty} ||x^\omega - \tilde{x}^\omega|| = 0$ it is sufficient to have $\lim_{\omega\to\infty} ||u - \tilde{u}|| = 0$. But, since by Lemma 10 we have $\hat{x} - x^\omega = o(\omega^{-1})$ and since the elements of $H(x)$ are continuously differentiable around the origin, we also have $|||H(\hat{x}) - H(x)||| = o(\omega^{-1})$, where $||| \cdot |||$ is an induced matrix norm, say the maximum column sum associated with the l_1 vector norm. Since by definition $v = o(\omega^{\frac{1}{2}})$ it follows that

$$\begin{aligned} ||u(t) - \tilde{u}(t)|| &= ||\big(H(\hat{x}(t)) - H(x^\omega(t))\big)v|| \\ &\leq |||H(\hat{x}(t)) - H(x^\omega(t))|||\ ||v|| \\ &= o(\omega^{-\frac{1}{2}}),\ \forall t \geq 0, \end{aligned}$$

and therefore $\lim_{\omega\to\infty} ||u(t) - \tilde{u}(t)|| = 0$, $\forall t \geq 0$. □

Theorem 11 clearly holds for general three-dimensional affine systems for which $H(x)$ and T exist, and can straightforwardly be extended to the n-dimensional case, since we are not making use of the fact that Σ stems from a system on a Lie group. Since the controls u do not make use of state-feedback it is now justified to interpret the control law $u = H(\hat{x})v$ as an approximate inverse system to Σ having $\bar{x}$ as input (see Fig. 3).

Example 12. ($SL(2)$). Using T, $H(x)$, and v as derived in Example 5 and 6 we write the open loop approximate tracking controls as $u = H(\hat{x})v$ where

$$\begin{aligned} \hat{x}_1(t) &= \bar{x}_1(t) - \omega^{-1}\sqrt{|m(t)|}\cos(\omega t) \\ \hat{x}_2(t) &= \bar{x}_2(t) + \omega^{-1}\mathrm{sign}(m(t))\sqrt{|m(t)|}\sin(\omega t) \\ \hat{x}_3(t) &= \bar{x}_3(t) + 2\omega^{-1}\bar{x}_2(t)\sqrt{|m(t|}\cos(\omega t)), \end{aligned}$$

with $m(t) = 2\omega(-\frac{1}{2}\dot{\bar{x}}_3 - \dot{\bar{x}}_1\bar{x}_2)$. Performing the otherwise unchanged simulation corresponding to Fig.1 with open-loop instead of feedback controls yields a plot almost indistinguishable from Fig.1. It has to be cautioned, though, that mismatched initial conditions cause increasing deviations in

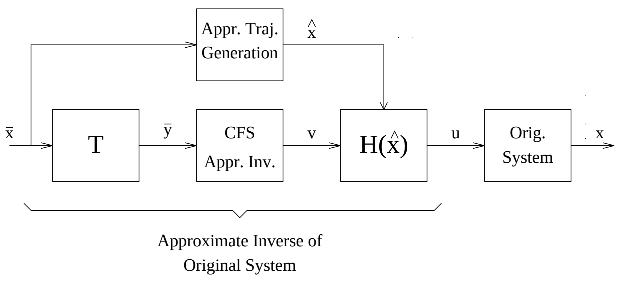

Fig. 3. Block diagram of approximate inversion via nilpotentization

the open-loop case since the approximately pre-computed nilpotentization is not valid anymore. In contrast to this, mismatched initial conditions in the feedback case yield trajectories which are only translated in parallel by the initial mismatch.

Similarly, $\hat{x}$ and $u = H(\hat{x})v$ for the $SE(2)$ and $SO(3)$ cases can be computed using the transformations T and H specified in Examples 8 and 9, respectively.

3. CONCLUSIONS

In this paper the concept of approximate inversion is used to obtain tracking controls for exponential representations of drift-free invariant systems on three-dimensional matrix Lie groups. After computing nilpotent bases for the distributions associated with systems on $SE(2)$, $SL(2)$ and $SO(3)$ we demonstrate how to construct approximate tracking controls from the state and input transformations associated with the nilpotentization. Further it is shown how these controls involving state feedback can be converted into open-loop form, thus obtaining an approximate inverse to the original system.

As was shown in (Brockett, 1996; Struemper and Krishnaprasad, 1997), the systems for which approximate tracking controls were computed via feedback nilpotentization can be exponentially stabilized by simply preceding the tracking control law with an integrator and closing the loop with a negative feedback.

We plan to extend the described method to systems on higher-dimensional Lie groups. A first step in this directions is provided by the sufficient conditions for nilpotentization presented by Murray in (Murray, 1994), guaranteeing already an extension of our results to drift-free two-input systems on *four*-dimensional Lie groups. An obstacle to a straightforward extension to *systems with drift* consists in the loss of trajectory equivalence between original and nilpotentized system due to the fact that the drift vector field is not scalable. But, as pointed out in (Hermes *et al.*, 1984) the nilpotentized system is still *orbit* equivalent to the original system, i.e. they have the same space of feasible paths. Thus, a characterization of the space of feasible paths of (nilpotentizable) nonlinear systems with drift can be drawn from an analysis of the path space of the easier to handle nilpotent systems.

4. REFERENCES

Brockett, R.W. (1993). Pattern generation and feedback control of nonholonomic systems. In: *Conf. on Decision and Contr.: Workshop on Mechanics, Holonomy, and Control*. San Antonio, Texas.

Brockett, R.W. (1996). Characteristic phenomena and model problems in nonlinear control. In: *Proc. of the 1996 IFAC Congress*. Vol. G. pp. 135–140.

Canny, J.F. and Z. Li (1993). *Nonholonomic Motion Planning*. Kluwer. Boston.

Dahleh, M., A. Peirce, H.A. Rabitz and V. Ramakrishna (1996). Control of molecular motion. *Proc. of the IEEE* **84**(1), 7–15.

Hermes, H., A. Lundell and D. Sullivan (1984). Nilpotent bases for distributions and control systems. *J. of Diff. Equations* **55**, 385–400.

Leonard, N. E. and P. S. Krishnaprasad (1995). Motion control of drift-free, left-invariant systems on lie groups. *IEEE Trans. on Automat. Contr.* **40**(9), 1539–1554.

Magnus, W. (1954). On the exponential solution of differential equations for a linear operator. *Comm. on Pure and Appl. Math.* **7**, 649–673.

Murray, R.M. (1994). Nilpotent bases for a class of nonintegrable distributions with application to trajectory generation for nonholonomic systems. *Math. of Contr., Signals, and Syst.* **7**, 58–75.

Murray, R.M. and S.S. Sastry (1993). Nonholonomic motion planning: Steering using sinusoids. *IEEE Trans. on Automat. Contr.* **38**(5), 700–716.

Struemper, H.K. and P.S. Krishnaprasad (1997). Approximate inversion and feedback stabilization for systems on matrix lie groups. Accepted for Amer. Contr. Conf, Albuquerque.

Sussmann, H.J. and W. Liu (1991). Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories. In: *Proc. of the 30th IEEE Conf. on Decision and Contr.*. pp. 437–442.

Vershik, A.M. and V.Y. Gershkovich (1994). *Dynamical Systems VII, Encyclopedia of Mathematical Sciences*. Chap. Nonholonomic Dynamical Systems, Geometry of Distributions and Variational Problems. Vol. 16. Springer Verlag. New York.

Wei, J. and E. Norman (1964). On global representations of the solution of linear differential equations as a product of exponentials. *Proc. of the Amer. Math. Soc.* **15**, 327–334.