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On Approximate Inversion and Feedback Stabilization for Systems on Matrix Lie Groups

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Abstract

This paper studies open-loop tracking and feedback stabilization for nonholonomic systems. Using the concept of approximate inversion, results for drift-free, left-invariant systems on specific matrix Lie groups are presented.

1. Introduction

Invariant systems on matrix Lie groups arise in problems as diverse as motion control for autonomous underwater vehicles and satellites [8], power conversion with switching circuits [16] and coherent control of molecular dynamics [6]. We consider the drift-free, controllable setting, with fewer controls than the dimension of the group (state space). This leads to an important sub-class of nonholonomic systems on Lie groups.

In contrast with linear systems where the controllability Gramian can directly be used to construct stabilizing or tracking controls, no methods exist to date for extracting such controls for general nonlinear systems from the controllability Lie algebra. Some progress has been made in this direction for drift-free affine systems [4, 10] using oscillatory controls which systematically "excite" higher order brackets of the system to steer the system in the desired direction. Leonard and Krishnaprasad considered the problem in a Lie group setting and, applying averaging theory, obtained in [8] results on approximate point-to-point constructive controls for left-invariant systems on arbitrary finite-dimensional Lie groups. Sussmann and Liu gave a very general result for path constructive controllability of drift-free, nilpotent systems on $\Re^n$ [14]. Parallel to these developments in motion planning, Coron [5] showed the existence of time-varying stabilizing controls for a general class of drift-free nonlinear systems while Pomet presented in [11] a method to explicitly construct these control laws for a more restrictive class of systems.

Brockett, achieved in [2, 3] an appealing synthesis of the problems of open-loop tracking and feedback stabilization via the concept of *approximate inversion* of a system as captured in a simple, three dimensional nilpotent system. Here we investigate Brockett's notion of approximate inversion in the setting of systems on Lie groups.

2. Preliminaries

For a curve X in G, a real matrix Lie group of dimension n and $\mathcal{G}$ the Lie algebra associated with G having a basis $\{A_1, \ldots, A_n\}$ of constant matrices in $\mathcal{G}$, denote the control U as $U(t) = \sum_{i=1}^m u_i(t)A_i,\ m \leq n,\ t \geq 0$. The corresponding left-invariant drift-free system on G is

$$\dot{X}(t) = X(t)U(t), \qquad X(0) = X_0. \tag{1}$$

An exponential representation of (1) on $\Re^n$ leads to a drift-free affine system of the form $\dot{x} = \sum_{i=1}^m u_i f_i(x)$, where $x \in \Re^n$ and $U \in \Re^m$. The *single exponential representation* is based on the fact that the exponential map $\exp : \mathcal{G} \to \mathcal{G}$ is a local diffeomorphism for finite-dimensional Lie groups G on a reasonably large neighborhood $U \subset \mathcal{G}$ of the origin. For $H(3)$ and its higher-dimensional nilpotent generalizations mentioned below the exponential map is actually a *global* diffeomorphism. This suggests that we locally represent the solution of (1) as $X(t) = \exp Z(t) = e^{Z(t)}$, where $Z(t) = \sum_{i=1}^n z_i(t)A_i \in \mathcal{G}$ and call the z_i, $i = 1, 2, \ldots, n$ the *canonical coordinates of the first kind* for G.

Theorem 1 (Magnus [9]) *Consider the left-invariant, drift-free system (1) on a matrix Lie group G and the single exponential representation $Z(t)$, $t \geq 0$ of its solution. Then, if certain unspecified conditions of convergence are satisfied, $Z(t)$ can be written in the form*

$$\begin{aligned}
\dot{Z}(t) &= \frac{ad_{Z(t)}}{1-\exp(-ad_{Z(t)})}\, U(t) \\
&= (I + \frac{1}{2}ad_{Z(t)} + \sum_{p=1}^{\infty} \frac{\beta_{2p}}{(2p)!} ad_{Z(t)}^{2p})\, U(t) \\
&= U(t) + \frac{1}{2}[Z(t), U(t)] + \frac{1}{12}[Z(t), [Z(t), U(t)]] \\
&\quad - \frac{1}{720}[Z(t), [Z(t), [Z(t), U(t)]]] \pm \ldots,
\end{aligned}$$

where the β_{2p} are Bernoulli numbers.

The *product of exponentials representation* is based on the *canonical coordinates of the second kind* and is characterized in the following theorem, where we use an extended control vector $u = (u_1, \ldots, u_n)^T$ with $u_i = 0, i = m+1, \ldots, n$.

Theorem 2 (Wei–Norman [15]) *Consider the system (1) and its solution $X(t)$, $t \geq 0$. Then, in a neighborhood of $t = 0$ the solution may be expressed in the form*

$$X(t) = e^{x_1(t)A_1} e^{x_2(t)A_2} \cdots e^{x_n(t)A_n}. \tag{2}$$

The coordinate functions $x = (x_1(t) \dots x_n(t))^T$ evolve according to

$$\dot{x}(t) = M(x(t))u(t), \tag{3}$$

where the matrix M is analytic in x. Moreover, if $\mathcal{G}$ is solvable, then there exists a basis and an ordering of this basis for which the representation (2) is global and the x_i can be computed by quadratures.

The computation of $M(x)$ for specific Lie groups or Lie algebras involves differentiating (2), setting this equal to the left-hand side of (1) and using the structure of the corresponding Lie algebra $\mathcal{G}$ ([15]).

3. Approximate Inversion

Our approach to open-loop tracking for nonholonomic systems on Lie groups is inspired by [2, 3] and based on the well-known fact that periodic controls with appropriate phase and frequency relation create a motion in the direction of a certain higher order Lie bracket of the system. If we want to move the system in several higher order Lie bracket directions simultaneously we have to additively superimpose several of these oscillatory control components, namely some carrier wave modulated by a function specifying the desired velocity in the corresponding Lie bracket direction. It turns out that if we let the carrier frequencies go to infinity these motions are increasingly independent of each other and even though the underlying system is nonlinear an *approximate high-frequency superposition principle* [14] emerges.

Modeling these open-loop control laws as time-varying nonlinear systems with the desired trajectories as inputs we obtain the so called approximate inverse of the original system. This terminology is motivated by the fact, that concatenating the approximate inverse and the original system results in an approximate identity operator on the space of trajectories.

3.1. Approximate Inversion on a Subgroup of Unipotent Matrices

In this section we will consider two-input drift-free systems on a particular subgroup of the unipotent matrices (see [12]) which is of special interest due to its simple Lie algebra structure. It turns out that the product of exponentials representation of this system is of chained form:

$$\begin{array}{rclcrcl} \dot{x}_1 & = & u_1 & \qquad & \dot{x}_4 & = & u_1 x_3 \\ \dot{x}_2 & = & u_2 & & & \vdots & \\ \dot{x}_3 & = & u_1 x_2 & & \dot{x}_k & = & u_1 x_{k-1} \end{array} \tag{4}$$

Chained form systems have immediate practical relevance as local descriptions for the kinematic car or a tractor and trailer system [10]. Note that the product of exponentials representation (4) and the following result hold globally since the matrix Lie group under consideration is solvable.

Due to its lower triangular structure we can integrate (4) by quadratures and the solution for the n^{th} state can be written as

$$x_n(t) = \int_0^t u_1(\tau_1) \int_0^{\tau_1} u(\tau_2) \dots \int_0^{\tau_{n-2}} u_2(\tau_{n-1}) d\tau_{n-1} \cdots d\tau_1 \tag{5}$$

assuming $x(0) = 0$. The succession of integrations and multiplications with u_1 in (5) suggests to use oscillating control components in u_2 with integrally distributed frequencies which are “down-modulated” by multiplications with the fundamental harmonic in u_1.

Theorem 3 *Given a $(n+2)$-dimensional chained-form system (4), let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \dots, \bar{x}_{n+2}(t))^T$, $t \in [0,T]$ denote the desired trajectory. Assume that $\bar{x}$ is twice differentiable on $[0,T]$ and that $x(0) = 0$, $\bar{x}(0) = 0$, $\dot{\bar{x}}(0) = 0$. Define a sequence $\{u^{(\omega)}\}_{\omega=1}^{\infty}$ of controls where*

$$\begin{aligned} u_1^{(\omega)} &= \dot{\bar{x}}_1(t) + 2\,\omega^{\frac{n}{n+1}} \cos(\omega t) \qquad (6) \\ u_2^{(\omega)} &= \dot{\bar{x}}_2(t) + \sum_{m=1}^{n} \alpha_m(t)\, \omega^{-\frac{mn}{n+1}} \frac{m!}{m^m} \frac{d^m}{dt^m}(\cos(m\omega t)), \end{aligned}$$

where $\alpha_m(t) = \dot{\bar{x}}_{m+2}(t) - \dot{\bar{x}}_1(t)\bar{x}_{m+1}(t)$. Let $x^{(\omega)}(t)$, $0 \le t \le T$ be the solution of (4) with $u^{(\omega)}$ as input.

Then $\lim_{\omega \to \infty} x^{(\omega)}(t) = \bar{x}(t)$, $\forall t \in [0,T]$ where convergence is uniform with respect to t.

The proof of this theorem is lengthy but straightforward. Applying integration by parts and the Riemann Lebesgue Theorem to the quadrature solution of states $x_i, i = 1,2,3$ it can be shown that $x_i^{(\omega)}$ converges to $\bar{x}_i$ uniformly. The general result for arbitrary n then follows by induction. The complete proof can be found in [12].

Since the above convergence is uniform in t, also the tracking error, for instance defined as $E = \int_0^T \|\bar{x}(\tau) - x^{(\omega)}(\tau)\|\, d\tau$, goes to zero with $\omega \to \infty$. Note also that because the residual terms are of order $-\frac{1}{n+1}$ in ω the convergence properties w.r.t. the frequency parameter ω worsen with increasing dimension $n+2$ of the chained form system.

3.2. Approximate Tracking on $SE(2)$

Below we will present tracking controls for the system (1) on $G = SE(2)$ with the Lie algebra characterized by $[A_1, A_2] = A_3$, $[A_1, A_3] = -A_2$,$[A_2, A_3] = 0$. This result is remarkable since $SE(2)$ is not nilpotent as are $H(3)$ or the matrix groups mentioned in Section 3.1.. Since the controls involve feedback they do not define an approximate inverse for the given system (see remark below). We will state the theorem for the Wei-Norman representation of (1), where it follows from the solvability of $SE(2)$ and Theorem 2 that this result is globally valid.

Theorem 4 *Consider the drift-free controllable system (1) on $G = SE(2)$ with associated Wei-Norman repre-*

sentation

$$\begin{aligned}\dot{x}_1(t) &= u_1(t)\\ \dot{x}_2(t) &= u_2(t) + u_1(t)x_3(t)\\ \dot{x}_3(t) &= -u_1(t)x_2\end{aligned} \tag{7}$$

and a twice differentiable desired trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t))^T$*,* $t \in [0,T]$*, such that* $x(0) = \bar{x}(0)$*. Define the parameterized family of controls*

$$\begin{aligned}u_1^{(\omega)}(t) &= \dot{\bar{x}}_1(t) + \sqrt{2\omega}\sin(\omega t) \qquad (8)\\ u_2^{(\omega)}(t) &= \dot{\bar{x}}_2(t) - \sqrt{2\omega}\,m(t)\cos(\omega t) - u_1^{(\omega)}(t)x_3(t)\end{aligned}$$

with $m(t) = (\dot{\bar{x}}_3(t) + \dot{\bar{x}}_1(t)\bar{x}_2(t))$ *and let* $x^{(\omega)}(t), t \geq 0$ *be the solution of (7) with controls* $u_1^{(\omega)}(t), u_2^{(\omega)}(t)$ *as input. Then* $\lim_{\omega\to\infty} x^{(\omega)}(t) = \bar{x}(t)$*,* $\forall t \in [0,T]$*, uniformly in* t*.*

Proof: Rewrite the controls $u = (u_1 u_2)^T$ as $u = H(x)v$ with

$$\begin{aligned}H(x) &= \begin{pmatrix} 1 & 0 \\ -x_3 & 1\end{pmatrix}\\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} \dot{\bar{x}}_1(t) + \sqrt{2\omega}\sin(\omega t) \\ \dot{\bar{x}}_2(t) - \sqrt{2\omega}\,m(t)\cos(\omega t)\end{pmatrix}\end{aligned}$$

and express equation (7) accordingly as

$$\dot{x} = F(x)u = F(x)H(x)v = \tilde{F}(x)v \tag{9}$$

System (7) with controls (8) thus assumes the form

$$\begin{aligned}\dot{x}_1 &= v_1\\ \dot{x}_2 &= v_2\\ \dot{x}_3 &= -v_1 x_2\end{aligned} \tag{10}$$

i.e. it is of chained form. Using integration by parts we get

$$\begin{aligned}x_1(t) &= x_1(0) + \bar{x}_1(t) - \bar{x}_1(0) + \sqrt{2/\omega}\,(1-\cos(\omega t))\\ x_2(t) &= x_2(0) + \bar{x}_2(t) - \bar{x}_2(0) - \sqrt{2/\omega}\,m(t)\sin(\omega t)\\ &\quad + \sqrt{2/\omega}\int_0^t \dot{m}(\tau)\sin(\omega\tau)d\tau\end{aligned}$$

For $x(0) = \bar{x}(0)$ and by a version of the Riemann-Lebesgue Theorem this results in

$$\begin{aligned}x_1(t) &= \bar{x}_1(t) + o(\omega^{-\frac{1}{2}})\\ x_2(t) &= \bar{x}_2(t) + o(\omega^{-\frac{1}{2}})\end{aligned}$$

Integrating the third state using again a version of the Riemann-Lebesgue Theorem and $x(0) = \bar{x}(0)$ we obtain

$$\begin{aligned}x_3(t) = &\ x_3(0) - \int_0^t \Big(\dot{\bar{x}}_1(\tau) + \sqrt{2\omega}\,\sin(\omega\tau)\Big)\Big(\bar{x}_2(\tau) - \\ &\sqrt{2/\omega}\,m(\tau)\sin(\omega\tau) + o(\omega^{-\frac{3}{2}})\Big)d\tau\\ = &\ x_3(0) - \int_0^t \big\{\dot{\bar{x}}_1(\tau)\bar{x}_2(\tau) - \dot{\bar{x}}_1(\tau)\sqrt{2/\omega}\,m(\tau)\sin(\omega\tau)\\ &-m(\tau) + m(\tau)\cos(2\omega\tau) + \bar{x}_2(\tau)\sqrt{2\omega}\sin(\omega\tau)\\ &+o(\omega^{-1})\big\}d\tau\\ = &\ \bar{x}_3(t) - \int_0^t \big\{-\dot{\bar{x}}_1(\tau)\sqrt{2/\omega}\,m(\tau)\sin(\omega\tau)\\ &+m(\tau)\cos(2\omega\tau) + \bar{x}_2(\tau)\sqrt{2\omega}\sin(\omega\tau) + o(\omega^{-1})\big\}d\tau\\ = &\ \bar{x}_3(t) + o(\omega^{-\frac{1}{2}}).\end{aligned}$$

□

The above result is an application of the concept of nilpotentization since the input transformation $H(x)$ is introduced to make the distribution spanned by the control vector fields of (7) nilpotent. In [13] we how show this approach can be applied to other non-nilpotent matrix Lie groups and how the state feedback can be replaced by a state estimate, resulting in *open-loop* control laws defining an approximate inverse for the systems considered.

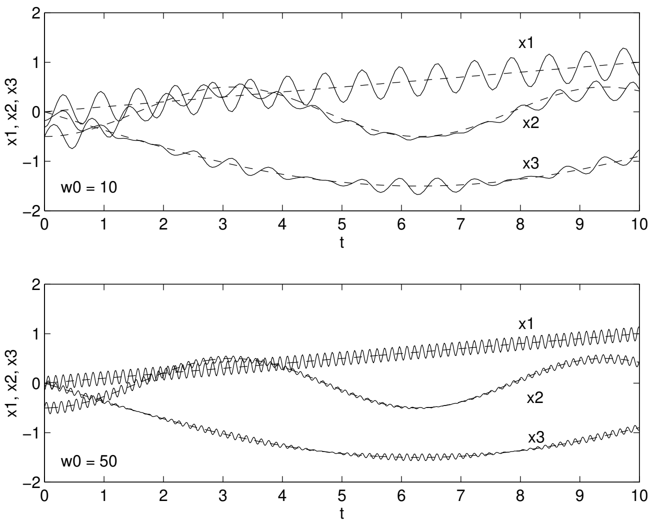

Fig.1: Approximate tracking for system on SE(2)

4. Feedback Stabilization

Ever since the existence of smooth time-*invariant* feedback laws for the asymptotic stabilization of nonholonomic systems was ruled out [1], a good deal of attention has been devoted to proving existence [5] and constructing [11] time-*varying*, periodic feedback laws for these systems. Following Brockett [2, 3] our approach consists of composing the approximate inverse with the original system yielding an approximate identity operator, which can be robustly stabilized feeding back the negative integrals of the state. Consequently the resulting control law achieves exponential rather than just uniform stability of the origin.

The control law we present here is essentially the same as in [2, 3] and is based on the approximate inverse of the

Magnus representation of a system on $H(3)$. We will give a modified proof and show furthermore that the same control law also asymptotically stabilizes the left–invariant systems on $SE(2)$, $SO(3)$, and $SL(2)$. We use the Magnus equation representation of (1) in this context, since it allows us to interpret the systems on $SE(2)$, $SO(3)$, and $SL(2)$ as perturbations of the system on $H(3)$. The feedback law derived for $H(3)$ turns out to be robust enough to locally stabilize the perturbed systems as well. Note also, that since the Magnus equation is globally valid for the system on $H(3)$ and the linearization (15) is in this case equivalent to the system itself, we actually have *global* exponential stability for $H(3)$.

Theorem 5 *Consider the left-invariant drift-free systems on the three-dimensional real matrix Lie groups $H(3)$, $SE(2)$, $SO(3)$, and $SL(2)$ and their corresponding Magnus equations*

$$\dot{Z} = U + \frac{1}{2}[Z,U] + \frac{1}{12}[Z,[Z,U]] \mp \dots, \qquad (11)$$

where $Z = z_1A_1 + z_2A_2 + z_3A_3$.

Then the control $U = u_1A_1 + u_2A_2$ with

$$\begin{aligned} u_1 &= -z_1 + \sqrt{2\omega|z_3|}\,sign(z_3)\sin(\omega t) \\ u_2 &= -z_2 + \sqrt{2\omega|z_3|}\cos(\omega t) \end{aligned} \qquad (12)$$

makes the zero solution of the systems (11) exponentially stable for ω sufficiently high.

Proof: We will show that after partitioning the state space and applying a coordinate transform the linearizations of the resulting systems are exponentially stable. Thus, we can conclude that also the zero solutions of the original systems are exponentially stable.

First, note that with $z_3 = 0$ we have for the closed loop

$$\begin{aligned} [Z,U] &= (z_1u_2 - z_2u_1)[A_1,A_2] + z_3u_1[A_3,A_1] \\ &\quad + z_3u2[A_3,A_2] \\ &= \left(z_1\sqrt{2\omega|z_3|}\cos(\omega t) - z_2\sqrt{2\omega|z_3|}\mathrm{sign}(z_3)\sin(\omega t)\right) \\ &\quad \times[A_1,A_2] + z_3u_1[A_3,A_1] + z_3u_2[A_3,A_2] \\ &= 0 \end{aligned}$$

such that $\dot{Z} = U$, $\forall Z \in S_0 = \{Z|z_3 = 0\}$. It follows that $\dot{Z}$ has no component in the direction of A_3 and hence S_0 is an invariant set of the closed loop system (11). Moreover, trajectories starting outside of S_0 do not enter S_0 in finite time and hence we have a partition of the state space into three invariant sets S_0, $S_1 = \{Z|z_3 > 0\}$, and $S_2 = \{Z|z_3 < 0\}$. Thus, we can carry out the stability analysis on each of the sets separately and study if trajectories starting in a neighborhood of the origin in the augmented sets S_0, $S_1\cup\{0\}$, and $S_2\cup\{0\}$ converge to it exponentially.

For $Z(0) \in S_0$ the closed loop system system (11) written in the coordinates of the Lie algebras $\mathcal{G}$ simplifies to

$$\dot{z}_1 = -z_1; \qquad \dot{z}_2 = -z_2; \qquad \dot{z}_3 = 0 \qquad (13)$$

and is clearly exponentially stable on S_0.

Now, for $Z(0)$ in S_1 or S_2 write (11) as

$$\begin{aligned} \dot{z}_1 &= -z_1 + \sqrt{2\omega|z_3|}\,\mathrm{sign}(z_3)\sin(\omega t) + p_1(z,t) \\ \dot{z}_2 &= -z_2 + \sqrt{2\omega|z_3|}\cos(\omega t) + p_2(z,t) \\ \dot{z}_3 &= z_1\sqrt{2\omega|z_3|}\cos(\omega t) - z_2\sqrt{2\omega|z_3|}\,\mathrm{sign}(z_3)\sin(\omega t) + p_3(t,z) \end{aligned}$$

where p_1 and p_2 represent the contributions of the second and higher order terms on the right hand side of (11), while p_3 represents the contribution of the third and higher order terms in (11). Define a coordinate map $x_1 = z_1$, $x_2 = z_2$, and $x_3 = \mathrm{sign}(z_3)\sqrt{2|z_3|}$, which is a diffeomorphism away from S_0 and apply it to (14) resulting in

$$\begin{aligned} \dot{x}_1 &= -x_1 + \sqrt{\omega}\,x_3\sin(\omega t) + \tilde{p}_1(x,t) \\ \dot{x}_2 &= -x_2 + \sqrt{\omega}\,\mathrm{sign}(x_3)x_3\cos(\omega t) + \tilde{p}_2(x,t) \\ \dot{x}_3 &= \frac{1}{2}(x_1\sqrt{\omega}\cos(\omega t) - x_2\sqrt{\omega}\,\mathrm{sign}(x_3)\sin(\omega t)) + \tilde{p}_3(x,t) \end{aligned} \qquad (14)$$

Linearizing (14) at the origin and assuming $Z(0) \in S_1$ yields

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & \sqrt{\omega}\sin(\omega t) \\ 0 & -1 & \sqrt{\omega}\cos(\omega t) \\ \frac{1}{2}\sqrt{\omega}\cos(\omega t) & -\frac{1}{2}\sqrt{\omega}\sin(\omega t) & 0 \end{pmatrix} x \\ &= A_1(t)x, \end{aligned} \qquad (15)$$

a time-varying system system with periodic coefficients of period $T = 2\pi/\omega$. By a standard Floquet argument the origin of (15) will be exponentially stable if the Floquet multipliers, i.e. the eigenvalues of the transition matrix $\Phi_{A_1}(T,0)$ all lie in the unit circle. Writing out the Peano-Baker-Series for $\Phi_{A_1}(T,0)$ and using $T = 2\pi/\omega$ we obtain

$$\begin{aligned} \Phi_{A_1}(T,0) &= I_3 + \int_0^T A_1(\sigma_1)d\sigma_1 + \int_0^T A_1(\sigma_1)\int_0^{\sigma_1} A_1(\sigma_2)d\sigma_2 d\sigma_1 + \dots \\ &= I_3 + \begin{pmatrix} -T & 0 & 0 \\ 0 & -T & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{T}{4} + \frac{T^2}{2} & 0 & -\frac{T^{3/2}}{\sqrt{2\pi}} \\ 0 & \frac{T}{4} + \frac{T^2}{2} & 0 \\ 0 & -\frac{T^{3/2}}{2\sqrt{2\pi}} & -\frac{T}{2} \end{pmatrix} + o(T^2) \\ &= \begin{pmatrix} 1 - \frac{3T}{4} & 0 & 0 \\ 0 & 1 - \frac{3T}{4} & 0 \\ 0 & 0 & 1 - \frac{T}{2} \end{pmatrix} + o(T^{3/2}). \end{aligned}$$

If we choose T sufficiently small or ω sufficiently large the eigenvalues of $\Phi_{A_1}(T,0)$ are guaranteed to be within the unit circle. Therefore the linearization (15) is exponentially stable and consequently also trajectories starting in a neighborhood of the origin intersected with S_1 converge to the origin exponentially.

The argument for $Z(0)$ in an intersection of a neighborhood of the origin and S_2 is analogous and the details are left to the reader. Thus, given a sufficiently large ω there exists a neighborhood U of the origin such that trajectories starting in each intersection of U with S_i, $i = 1,2,3$ converge to the origin exponentially. □

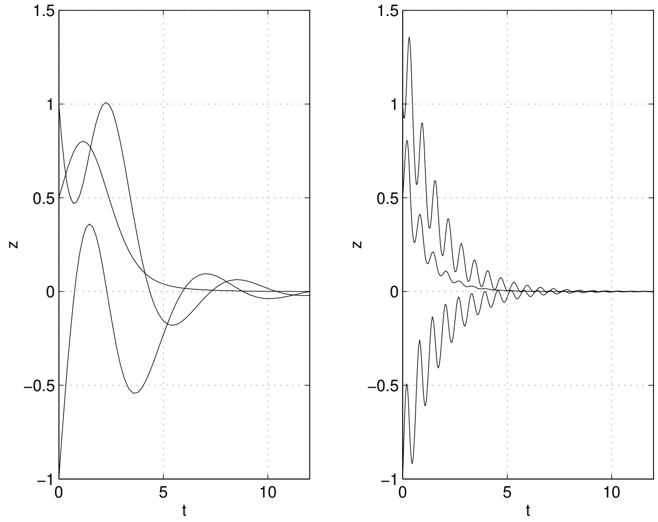

Fig. 2: Exponential feedback stabilization of Magnus equation of system on $SO(3)$ with $\omega = 1$ (left) and $\omega = 10$ (right)

5. Discussion

We have presented approximate tracking controls for left-invariant drift-free systems on $SE(2)$ and a nilpotent subgroup of the unipotent matrices. Further a feedback based on the idea of approximate inversion has been shown to exponentially stabilize the origin for systems on $H(3)$, $SE(2)$, $SO(3)$, and $SL(2)$. These solutions for specific matrix Lie groups should serve as examples to point out the significance of the approximate inversion approach to open-loop tracking and feedback stabilization of nonholonomic systems.

We plan to generalize these results, heading toward a general method to construct an approximate inverse for a system on a matrix Lie group based solely on the structure of the Lie algebra and the nature and number of the inputs. Numerical solutions might have to be taken into account where closed form results are not available.

A rigorous characterization of the approximate inverse together with the original system as a time–varying perturbation of an identity operator needs to be developed for a more unified approach to feedback stabilization of nonholonomic systems. Robust stability methods could then be applied drawing only on the characterization of the approximate identity operator rather than the properties of the original system. Eventually we would like to extend our results to systems with drift.

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