Geometry of cyclic pursuit

K. S. Galloway, E. W. Justh, and P. S. Krishnaprasad

***Abstract*— Pursuit strategies (formulated using constant-speed particle models) provide a means for achieving cohesive behavior in systems of multiple mobile agents. In the present paper, we explore an n-agent cyclic pursuit scheme (i.e. agent i pursues agent $i+1$, modulo n) in which each agent employs a *constant bearing pursuit strategy*. We demonstrate the existence of an invariant submanifold, and state necessary and sufficient conditions for the existence of rectilinear and circling relative equilibria on that submanifold. We present a full analysis of steady-state solutions and stability characteristics for two-particle "mutual CB pursuit" and then outline steps to extend the nonlinear stability analysis to the many particle case.**

I. Introduction

Pursuit and evasion phenomena are observed throughout biology, in hunting encounters as well as in mating and play. Pursuit also plays a significant role in the vehicular setting, in military encounters between planes and missiles or between adversarial unmanned vehicles. Though pursuit is often thought of as a competitive phenomenon, it has been shown to serve as a means of achieving cooperative behaviors as well. (See, for instance, Bruckstein's work on ant path-following in [1], [2].) Application of this idea may be relevant to the analysis of starling flock cohesion presented in [3]. In the current work, we focus on a cyclic pursuit scheme with a constant bearing pursuit law to achieve group cohesion for a multi-agent "flock."

Interest in cyclic pursuit dates back to 1877 when Edouard Lucas originally posed his question asking what trajectories would be traced out by three "dogs" which started at the vertices of an equilateral triangle and pursued their next neighbor at a constant speed.[1] Since that time, mathematicians have analyzed various facets and extensions of the problem, demonstrating, for instance, that mutual capture is assured in the three "dog" (or "bug") problem but non-mutual capture is possible in the more general case (i.e. for $n > 3$). (See [7] and [12] respectively.) Recently, there has been a renewed interest in studying cyclic pursuit from a control-theoretic perspective, particularly for decentralized control of groups of autonomous agents. In [8], Marshall, Broucke and Francis present an analysis of wheeled vehicles (modeled as kinematic unicycles) engaged in cyclic pursuit, moving at a fixed common vehicle speed and employing a steering law dependent on linear feedback of the relative bearing error. The authors prove the existence of $2(n-1)$ relative equilibrium formations and conduct local stability analysis of the equilibria based on linearization of the shape dynamics. In [9], the authors extend their analysis to incorporate feedback control of the vehicle speeds as well, presenting a global stability analysis for the two-particle case and a local stability analysis for the general case. In [13], Sinha and Ghose present a generalization of this previous work which involves heterogeneous formations of agents with differing speeds as well as differing controller gains. Variations on the theme are presented in [14] (a hierarchical cyclic pursuit scheme), [15] (with applications to rendezvous problems and curve-shortening), and [11] (with applications to coverage).

The contribution of the current work lies in the nonlinear nature of both the steering law and the stability analysis. We begin by briefly describing the particle model dynamics (section II) and introducing the constant bearing pursuit law (section III). This constant bearing pursuit law (originally developed in [16]) involves the relative bearing error as well as a term similar to the "motion camouflage" law in [5]. In section IV we discuss the reduction to "shape space," providing an explicit parametrization and describing an invariant submanifold with reduced dynamics. After stating propositions concerning conditions for the existence of special solutions (e.g. relative equilibria) on the invariant submanifold, we present the full two-particle analysis and sketch an extension to the many particle case in section VII.

II. Modeling interactions

We describe the movement of agents in our system as unit-mass particles tracing out twice continuously-differentiable curves in $\mathbb{R}^2$, deriving our dynamics from the natural Frenet frame equations (see, e.g., [4] for details). (A three-dimensional analysis of cyclic pursuit formulated in terms of the natural Frenet frame equations is a topic of ongoing work.) As is depicted in figure 1, we let $\mathbf{r}_i$ denote the position of the i^{th} particle (with respect to a fixed inertial frame), $\mathbf{x}_i$ denote the unit tangent vector to the curve, and $\mathbf{y}_i$ the unit vector normal to $\mathbf{x}_i$. An n-agent system then evolves according to the particle dynamics given by

$$\begin{aligned} \dot{\mathbf{r}}_i &= \nu_i \mathbf{x}_i, \\ \dot{\mathbf{x}}_i &= \nu_i \mathbf{y}_i u_i, \\ \dot{\mathbf{y}}_i &= -\nu_i \mathbf{x}_i u_i, \; i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

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[1] See [12] for a nice historical summary of the cyclic pursuit problem.

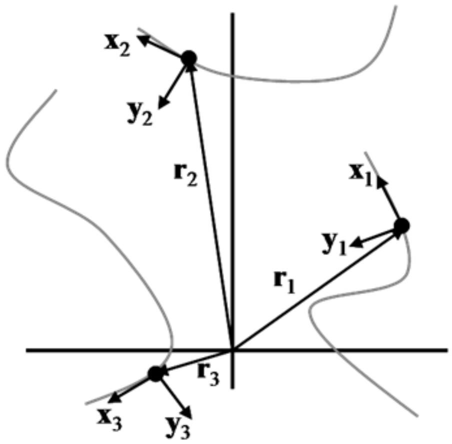


Fig. 1. Illustration of particle positions and corresponding natural Frenet frames for three particles in the plane.

Here $\mathbf{y}_i = \mathbf{x}_i^{\perp}$, by which we mean that rotating $\mathbf{x}_i$ counter-clockwise in the plane by $\pi/2$ radians gives $\mathbf{y}_i$. Note that ν_i, the speed of particle i, could possibly be given by a time-varying function, but here it is constant and equal to 1. Our controls, u_i, can be viewed as curvature controls or steering controls in the planar setting. We also define the "baseline vectors" $\mathbf{r}_{i,i+1}$ by $\mathbf{r}_{i,i+1} = \mathbf{r}_i - \mathbf{r}_{i+1}$, $i = 1, 2, ..., n$ (interpreted modulo n throughout this paper).

System (1) evolves on the manifold M_{state} defined by

$$M_{state} = \Big\{ (\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n) \in \mathbb{R}^{6n} \;\Big|\; \mathbf{r}_i \neq \mathbf{r}_{i+1}, \; |\mathbf{x}_i| = 1, \; \mathbf{y}_i = \mathbf{x}_i^{\perp}, \; i = 1, 2, \ldots, n \Big\}. \tag{2}$$

Note that we have only disallowed "sequential collocation", i.e. the state manifold does not include states for which $\mathbf{r}_i = \mathbf{r}_{i+1}$. In terms of pursuit, this means that we restrict our analysis away from the point of actual capture/rendezvous, allowing well-posedness of the feedback laws of section III.

III. Pursuit strategies and steering laws

We describe pursuit interactions in terms of the geometry of the encounter, typically in terms of relative positions and velocities. The analysis in [16] concerns several of these geometric depictions of pursuit (or *pursuit strategies*), including Classical Pursuit (CP), Constant Bearing Pursuit (CB), and Motion Camouflage Pursuit (MC), and demonstrates that appropriate cost functions and associated pursuit manifolds can be defined for each pursuit strategy. For a two-particle system with particle i pursuing particle $i+1$, we define cost functions for Motion Camouflage Pursuit and Constant Bearing Pursuit, respectively, by $\Gamma_i : M_{state} \to \mathbb{R}$ and $\Lambda_i : M_{state} \to \mathbb{R}$,

$$\Gamma_i = \frac{\frac{d}{dt}|\mathbf{r}_{i,i+1}|}{|\frac{d\mathbf{r}_{i,i+1}}{dt}|} = \left(\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \cdot \frac{\dot{\mathbf{r}}_{i,i+1}}{|\dot{\mathbf{r}}_{i,i+1}|} \right), \tag{3}$$

$$\Lambda_i = R(\alpha_i)\mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \tag{4}$$

where $\alpha_i \in [0, 2\pi)$ is the offset angle for the constant bearing pursuit[2] and $R(\alpha_i) \in SO(2)$ is the rotation matrix defined by

$$R(\alpha_i) = \begin{pmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{pmatrix}. \tag{5}$$

(Since Classical Pursuit is a special case of Constant Bearing Pursuit, we can define the associated cost functions by substituting $\alpha_i = 0$ into (4).) Then, as in [16], we have the associated pursuit manifolds defined by $\Gamma_i = -1$ and $\Lambda_i = -1$ and we can derive steering laws to drive the system toward the desired pursuit manifold. In particular, we have for MC,

$$u_{MC(i)} = -\mu_i \left(\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \cdot \dot{\mathbf{r}}_{i,i+1}^{\perp} \right), \tag{6}$$

and for CB,

$$u_{CB(\alpha_i)} = -\mu_i \left(R(\alpha_i)\mathbf{y}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \right) - \frac{1}{|\mathbf{r}_{i,i+1}|} \left(\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \cdot \dot{\mathbf{r}}_{i,i+1}^{\perp} \right), \tag{7}$$

where $\mu_i > 0$ is a control gain.

We will focus on the CB pursuit law (7) for our analysis of cyclic pursuit.

Remark: The classical work on cyclic pursuit (e.g., as in [7]), and the more recent work of Bruckstein et al. [2] and Richardson [12], presuppose that the CP strategy is exactly realized (with no reference to a feedback law). On the other hand, [9] is based on a linear feedback law motivated by the CP strategy.

IV. Symmetry, shape and reduction

Pursuit laws (6) and (7) leave our system dynamics (1) invariant under the action of the special Euclidean group SE(2) and therefore permit reduction to the quotient manifold $M_{state}/SE(2)$, also known as the shape space. The shape space describes the *relative* positions and velocities of the agents in the system.

A particularly useful parametrization of this shape space is given by (c.f. [4])

$$\begin{aligned} \rho_i &= |\mathbf{r}_{i,i+1}|, \\ \phi_i &= \mathbf{x}_i \cdot \mathbf{x}_{i+1}, \\ \gamma_i &= \mathbf{x}_i \cdot \mathbf{y}_{i+1}, \\ \beta_i &= \mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \\ \delta_i &= \mathbf{y}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}, \; i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Note that ϕ_i and γ_i are dot products of unit vectors, as are β_i and δ_i, and they satisfy

$$\phi_i^2 + \gamma_i^2 = 1, \;\; \beta_i^2 + \delta_i^2 = 1, \;\; i = 1, 2, ..., n. \tag{9}$$

[2]Note that $\alpha_i = 0$ corresponds to classical pursuit. For $\pi/2 < \alpha_i < 3\pi/2$, the pursuer's feedback law drives its velocity vector away from the "pursuee" so it may be more appropriate to describe the strategy as constant bearing "evasion" as opposed to "pursuit".

We can derive the associated shape dynamics by first calculating the derivative of ρ_i, as follows:

$$\begin{aligned}\dot{\rho}_i &= \frac{d}{dt}\left|\mathbf{r}_{i,i+1}\right| = \frac{1}{2}(\mathbf{r}_{i,i+1}\cdot\mathbf{r}_{i,i+1})^{-1/2}(2\dot{\mathbf{r}}_{i,i+1}\cdot\mathbf{r}_{i,i+1}) \\ &= (\mathbf{x}_i - \mathbf{x}_{i+1})\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} = \beta_i - (\phi_i\beta_i - \gamma_i\delta_i) \\ &= \beta_i\left(1-\phi_i\right) + \gamma_i\delta_i. \qquad (10)\end{aligned}$$

(Note that the simplification in the second line is made possible by expressing $\mathbf{x}_{i+1}$ in terms of $\mathbf{x}_i$ and $\mathbf{y}_i$ components.)

Remark: It is possible to derive the following expressions for the CB cost function (4) and the CB pursuit law (7) in terms of the shape variables:

$$\Lambda_i = \cos(\alpha_i)\beta_i + \sin(\alpha_i)\delta_i, \qquad (11)$$

$$u_{CB(\alpha_i)} = \mu_i\left[\sin(\alpha_i)\beta_i - \cos(\alpha_i)\delta_i\right] - \frac{1}{\rho_i}\left[\delta_i(1-\phi_i) - \gamma_i\beta_i\right]. \qquad (12)$$

Noting that

$$\begin{aligned}\frac{d}{dt}\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} &= \frac{d}{dt}\frac{\mathbf{r}_{i,i+1}}{\rho_i} = \frac{\dot{\mathbf{r}}_{i,i+1}\rho_i - \mathbf{r}_{i,i+1}\dot{\rho}_i}{\rho_i^2} \\ &= \frac{1}{\rho_i}\left[\mathbf{x}_i - \mathbf{x}_{i+1} - \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|}\left[\beta_i\left(1-\phi_i\right) + \gamma_i\delta_i\right]\right], \qquad (13)\end{aligned}$$

it is a straightforward (though somewhat lengthy) exercise to show that our resultant shape dynamics can be expressed as

$$\begin{aligned}\dot{\rho}_i &= \beta_i\left(1-\phi_i\right) + \gamma_i\delta_i, \\ \dot{\phi}_i &= -\gamma_i(u_i - u_{i+1}), \\ \dot{\gamma}_i &= \phi_i(u_i - u_{i+1}), \\ \dot{\beta}_i &= u_i\delta_i + \frac{1}{\rho_i}\left[\delta_i^2(1-\phi_i) - \beta_i\gamma_i\delta_i)\right], \\ \dot{\delta}_i &= -u_i\beta_i + \frac{1}{\rho_i}\left[\gamma_i\beta_i^2 - \beta_i\delta_i\left(1-\phi_i\right)\right], \qquad (14)\end{aligned}$$

$i = 1, 2, \ldots, n$, with (9) as constraints on the initial conditions.

Remark: The dynamics given by (14) hold for *any* $SE(2)$-invariant (feedback) control u_i.

If we further impose a cyclic pursuit scheme (i.e. i pursues $i+1$, modulo n) with each particle using the CB pursuit law given by (7), our closed-loop shape dynamics are given by

$$\begin{aligned}\dot{\rho}_i &= \beta_i\left(1-\phi_i\right) + \gamma_i\delta_i, \\ \dot{\phi}_i &= -\gamma_i(u_{CB(\alpha_i)} - u_{CB(\alpha_{i+1})}), \\ \dot{\gamma}_i &= \phi_i(u_{CB(\alpha_i)} - u_{CB(\alpha_{i+1})}), \\ \dot{\beta}_i &= \mu_i\left[\sin(\alpha_i)\beta_i\delta_i - \cos(\alpha_i)\delta_i^2\right], \\ \dot{\delta}_i &= \mu_i\left[\cos(\alpha_i)\beta_i\delta_i - \sin(\alpha_i)\beta_i^2\right], \qquad (15)\end{aligned}$$

$i = 1, 2, \ldots, n$, with (9) again providing constraints on the initial conditions. (The expressions for $\dot{\phi}_i$ and $\dot{\gamma}_i$ do not simplify greatly with the substitution of (7) or (12), and therefore we have not expressed them in their full explicit form.) One should note that under these closed-loop dynamics, the evolution of the subsystem $\left(\dot{\beta}_i, \dot{\delta}_i\right)$ is dependent only on β_i and δ_i, an important fact which will be discussed in greater detail later in the paper.

A. An invariant submanifold

We can now define an important submanifold of the state space corresponding to system states for which each agent has achieved CB pursuit of the next agent. As discussed previously, $\Lambda_i = -1$ if and only if agent i has achieved CB pursuit of agent $i+1$, and therefore we *define* the CB Pursuit Manifold $M_{CB(\alpha)} \subset M_{state}$ by

$$\begin{aligned}M_{CB(\alpha)} = \Big\{&(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1, \ldots, \mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n) \in M_{state} \mid \\ &\Lambda_i = -1, i = 1, 2, \ldots, n\Big\}, \qquad (16)\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. (Note that we could equally as well define $M_{CB(\alpha)}$ as a submanifold of the shape space, as is made evident by (11).) In [16], it is demonstrated that the derivative of Λ_i under pursuit law (7) can be expressed as $\dot{\Lambda}_i = -\mu_i\left(1-\Lambda_i^2\right)$. Noting that this holds regardless of the maneuver of the pursuee (i.e. agent $i+1$) and that $\dot{\Lambda}_i = 0$ for $\Lambda_i = -1$, we can state that $M_{CB(\alpha)}$ is invariant under cyclic pursuit dynamics with pursuit law (7).

We can formulate reduced dynamics on $M_{CB(\alpha)}$ in terms of the shape variables (8) as follows. By expressing $\mathbf{x}_i$ in the basis $\{R(\alpha_i)\mathbf{x}_i, R(\alpha_i)\mathbf{y}_i\}$ and making use of the fact that $\Lambda_i = R(\alpha_i)\mathbf{x}_i\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} = -1$ (i.e. $R(\alpha_i)\mathbf{y}_i\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} = 0$) on $M_{CB(\alpha)}$, we have

$$\begin{aligned}\beta_i &= \mathbf{x}_i\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \\ &= \Big[\Big(\mathbf{x}_i\cdot R(\alpha_i)\mathbf{x}_i\Big)R(\alpha_i)\mathbf{x}_i \\ &\qquad + \Big(\mathbf{x}_i\cdot R(\alpha_i)\mathbf{y}_i\Big)R(\alpha_i)\mathbf{y}_i\Big]\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \\ &= \cos(\alpha_i)R(\alpha_i)\mathbf{x}_i\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} - \sin(\alpha_i)R(\alpha_i)\mathbf{y}_i\cdot\frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} \\ &\equiv -\cos(\alpha_i). \qquad (17)\end{aligned}$$

Similar calculations yield

$$\delta_i \equiv -\sin(\alpha_i), \qquad (18)$$

and therefore we have the following reduced system dynamics on $M_{CB(\alpha)}$:

$$\begin{aligned}\dot{\rho}_i &= -\left(1-\phi_i\right)\cos(\alpha_i) - \gamma_i\sin(\alpha_i), \\ \dot{\phi}_i &= -\gamma_i\Big[\frac{1}{\rho_i}\Big(\left(1-\phi_i\right)\sin(\alpha_i) - \gamma_i\cos(\alpha_i)\Big) \\ &\qquad - \frac{1}{\rho_{i+1}}\Big(\left(1-\phi_{i+1}\right)\sin(\alpha_{i+1}) - \gamma_{i+1}\cos(\alpha_{i+1})\Big)\Big], \\ \dot{\gamma}_i &= \phi_i\Big[\frac{1}{\rho_i}\Big(\left(1-\phi_i\right)\sin(\alpha_i) - \gamma_i\cos(\alpha_i)\Big) \\ &\qquad - \frac{1}{\rho_{i+1}}\Big(\left(1-\phi_{i+1}\right)\sin(\alpha_{i+1}) - \gamma_{i+1}\cos(\alpha_{i+1})\Big)\Big], \\ \dot{\beta}_i &= 0, \\ \dot{\delta}_i &= 0, \quad i = 1, 2, \ldots, n. \qquad (19)\end{aligned}$$

Remark: Richardson's model ([12]) is confined to the CP manifold and obeys equation (19) with $\alpha_i = 0, \forall i$.

Existence of the invariant submanifold $M_{CB(\alpha)}$ on which the system shape evolves according to these reduced shape dynamics suggests a two-part approach for characterization of the solution space and subsequent stability analysis. The first part of our approach will entail analysis of the solutions and stability properties of the reduced dynamics *on* the CB Pursuit Manifold, and the other part will focus on the evolution of the system on the full shape space, unrestricted to (but possibly converging to) the CB Pursuit Manifold. We sketch this latter step in section VII.

V. Special solutions

Here we characterize possible special solutions for the system dynamics and determine criteria for their existence. We will discuss relative equilibria as well as a more general type of special solution.

A. Relative equilibria

Equilibria of the shape space dynamics (14) correspond to relative equilibria of the full system dynamics (1). As is demonstrated in [4], system dynamics of the form (1) permit only two types of relative equilibria: rectilinear and circling. For a rectilinear relative equilibrium, all the particle velocities are aligned (i.e. $\phi_i = 1, \ i = 1, 2, \ldots, n$) and $u_1 = u_2 = \cdots = u_n = 0$. For a circling relative equilibrium, the particles travel on a common closed circular trajectory, separated by fixed chordal distances. At this relative equilibrium, we have $u_1 = u_2 = \cdots = u_n = \frac{1}{r_c} \neq 0$, where r_c is the radius of the circular orbit.

We now state propositions concerning the existence of relative equilibria in terms of conditions on the CB angles $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Note that we restrict our attention to relative equilibria existing on the submanifold $M_{CB(\alpha)}$ due to stability properties of this particular submanifold which will be discussed in section VII. The first proposition deals with rectilinear motion, and the second with circling motion.

Proposition 1: Given $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a relative equilibrium corresponding to rectilinear motion on $M_{CB(\alpha)}$ exists for system (1) under cyclic pursuit with CB(α) control law (7) if and only if there exists a set of constants $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $\sigma_i > 0, \ i = 1, 2, \ldots, n$ and

$$\sum_{i=1}^{n} \sigma_i e^{j\alpha_i} = 0, \tag{20}$$

where $j = \sqrt{-1}$.

Proof: See Appendix.

Proposition 2: Given $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a relative equilibrium corresponding to circling motion on a common orbit on $M_{CB(\alpha)}$ exists for system (1) under cyclic pursuit with CB(α) control law (7) if and only if

i. $\sin(\alpha_i) > 0 \ \ \forall i \in \{1, 2, \ldots, n\}$ or
$\sin(\alpha_i) < 0 \ \ \forall i \in \{1, 2, \ldots, n\}$,

ii. $$\sin\left(\sum_{i=1}^{n} \alpha_i\right) = 0. \tag{21}$$

Proof: See Appendix.

Remark: Note that the condition of **Proposition 1** and condition (i) of **Proposition 2** are mutually exclusive, and therefore the set of all possible $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for which rectilinear equilibria exist is disjoint from the set of all possible $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for which circling equilibria exist. Also, the two cases in condition (i) of **Proposition 2** correspond to the choice of either clockwise or counter-clockwise circling equilibria.

B. Other special solutions

A collection of particles in cyclic pursuit can be viewed as a planar polygon with labeled vertices (i.e. the sides connect particle i to particle $i + 1$). It is possible to discuss the "shape" and "size" of a particle polygon with the usual geometric meanings of these terms. A relative equilibrium is then a system trajectory for which the size and shape of the associated particle polygon remain invariant. Our continued analysis and numerical simulations have demonstrated that a more general type of special solution exists, for which the *shape* of the associated particle polygon remains invariant but the *size* is time-varying. Typically, these solutions are characterized by either spiraling out or spiraling in. Explicit examples are discussed for the two-particle case in section VI.

VI. Mutual CB pursuit

In the two-particle case, cyclic pursuit is more aptly described as *mutual pursuit*. In this case, we are able to perform a global stability analysis and fully describe the steady-state solutions as parametrized by α_1 and α_2. We note that the $n = 2$ case provides valuable insight into methods for attempting stability analysis of higher-dimensional systems (i.e. for $n > 2$).

In the two-particle case, we are able to derive a simplified form of the shape dynamics (15) as follows. First, the obvious symmetries in the two-particle case allow for the assignment

$$\phi = \phi_1 = \phi_2, \ \ \gamma = \gamma_1 = -\gamma_2, \ \ \rho = \rho_1 = \rho_2. \tag{22}$$

Then, making use of the orthogonal decomposition of $\mathbf{x}_2$ (in terms of $\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$ and $\frac{(\mathbf{r}_1 - \mathbf{r}_2)^\perp}{|\mathbf{r}_1 - \mathbf{r}_2|}$) and the identity $\mathbf{a} \cdot \mathbf{b}^\perp = -\mathbf{a}^\perp \cdot \mathbf{b}$, we have

$$\phi = -\beta_1\beta_2 - \delta_1\delta_2, \ \ \gamma = -\beta_1\delta_2 + \delta_1\beta_2. \tag{23}$$

Substituting these expressions into our shape dynamics (15), we then have the simplified two-particle version given by

$$\begin{aligned}
\dot{\rho} &= \beta_1 + \beta_2, \\
\dot{\beta}_1 &= \mu_1 \left[\sin(\alpha_1)\beta_1\delta_1 - \cos(\alpha_1)\delta_1^2\right], \\
\dot{\beta}_2 &= \mu_2 \left[\sin(\alpha_2)\beta_2\delta_2 - \cos(\alpha_2)\delta_2^2\right], \\
\dot{\delta}_1 &= \mu_1 \left[\cos(\alpha_1)\beta_1\delta_1 - \sin(\alpha_1)\beta_1^2\right], \\
\dot{\delta}_2 &= \mu_2 \left[\cos(\alpha_2)\beta_2\delta_2 - \sin(\alpha_2)\beta_2^2\right],
\end{aligned} \tag{24}$$

with the auxiliary algebraic equations

$$\begin{aligned}
&\phi = -\beta_1\beta_2 - \delta_1\delta_2, \;\; \gamma = -\beta_1\delta_2 + \delta_1\beta_2, \\
&\beta_1^2 + \delta_1^2 = 1, \;\; \beta_2^2 + \delta_2^2 = 1.
\end{aligned} \tag{25}$$

As noted previously, since $\left(\dot{\beta}_i, \dot{\delta}_i\right)$ depends only on β_i and δ_i (for $i = 1, 2$) we can separately analyze these sub-systems. Due to the orthogonality condition $\beta_i^2 + \delta_i^2 = 1$, it can readily be shown that each subsystem has two equilibrium points at $(\beta_i, \delta_i) = \pm\Big(\cos(\alpha_i), \sin(\alpha_i)\Big)$.

Proposition 3: For $i = 1, 2$, the point $(\beta_i, \delta_i) = -\Big(\cos(\alpha_i), \sin(\alpha_i)\Big)$ is an asymptotically stable equilibrium point for the subsystem $\left(\dot{\beta}_i, \dot{\delta}_i\right)$, with region of convergence given by $\left\{(\beta_i, \delta_i) \in S^1 \middle| (\beta_i, \delta_i) \neq \Big(\cos(\alpha_i), \sin(\alpha_i)\Big)\right\}$.

Proof: Use Lyapunov's indirect method to show that the equilibrium point at $(\beta_i, \delta_i) = \Big(\cos(\alpha_i), \sin(\alpha_i)\Big)$ is unstable. Then use LaSalle's invariance principle([6]) with Lyapunov function $\Lambda_i = \cos(\alpha_i)\beta_i + \sin(\alpha_i)\delta_i$ to prove asymptotic stability of the other equilibrium point. □

Note from (17) and (18) that $(\beta_i, \delta_i) = -\Big(\cos(\alpha_i), \sin(\alpha_i)\Big)$, $i = 1, 2$, corresponds to our definition of $M_{CB(\alpha)}$ for the two-particle case, so **Proposition 3** establishes asymptotic convergence to $M_{CB(\alpha)}$. The rate of convergence is governed by the control gains μ_1 and μ_2. We therefore seek to describe the system dynamics on $M_{CB(\alpha)}$ as parametrized by the particular values of α_1, α_2, i.e. the asymptotic behavior of the system.

On $M_{CB(\alpha)}$, the only (possibly) nonzero portion of the shape dynamics is given by

$$\dot{\rho} = \beta_1 + \beta_2 = -\left[\cos(\alpha_1) + \cos(\alpha_2)\right]. \tag{26}$$

We can also derive an expression for the controls $u_{CB(\alpha_i)}(t)$ on $M_{CB(\alpha)}$, starting from (12) and substituting for $\phi, \gamma_i, \beta_i, \delta_i$ to get

$$u_{CB(\alpha_i)}(t) = \frac{1}{\rho(t)}\left[\sin(\alpha_1) + \sin(\alpha_2)\right], \; i = 1, 2. \tag{27}$$

Since the respective signs of $\dot{\rho}$ and $u(t) = u_{CB(\alpha_1)}(t) = u_{CB(\alpha_2)}(t)$ are determined by the respective signs of the quantities $[\cos(\alpha_1) + \cos(\alpha_2)]$ and $[\sin(\alpha_1) + \sin(\alpha_2)]$, we can concisely state our characterization of the manifold in terms of vector addition in the complex plane, noting that

$$\begin{aligned}
e^{j\alpha_1} + e^{j\alpha_2} &= \left[\cos(\alpha_1) + \cos(\alpha_2)\right] + j\left[\sin(\alpha_1) + \sin(\alpha_2)\right] \\
&= -\dot{\rho} + j\rho u.
\end{aligned} \tag{28}$$

The results are summarized in table I and illustrated in figure 2.

Remark: For $\alpha_1 = \alpha_2$, the center of mass (i.e. $\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$) remains fixed. If $\alpha_1 = \alpha_2$ and $e^{j\alpha_1} + e^{j\alpha_2}$ does not lie on one of the axes, then the particles will spiral out or spiral in around the center of mass.

Remark: In work to appear [10], Mischiati and Krishnaprasad have investigated mutual pursuit under motion camouflage.

VII. Stability for many particles

As was previously alluded to at the end of section IV-A, the form of the cyclic CB closed-loop shape dynamics (15) suggests an approach to stability analysis which involves a separate investigation of convergence *to* $M_{CB(\alpha)}$ and convergence to special solutions *on* $M_{CB(\alpha)}$, in a manner somewhat reminiscent of center manifold theory. Here we sketch an outline of this approach, with the intent of presenting a more complete analysis in future work.

A. Convergence to the invariant manifold

We suggest that the path for proving system convergence to the invariant manifold $M_{CB(\alpha)}$ lies in following an approach analogous to that demonstrated in the two-particle analysis presented in section VI. In the two-particle case, the cost function Λ_i (for $i = 1, 2$) was used as a Lyapunov function to prove (by way of the invariance principle) that the $\left(\dot{\beta}_1, \dot{\delta}_1\right)$ and $\left(\dot{\beta}_2, \dot{\delta}_2\right)$ subsystems converge to equilibrium points corresponding to the definition of $M_{CB(\alpha)}$. In an analogous approach for the $n > 2$ case, we define the projection function $\Lambda : M_{state}/SE(2) \longrightarrow [-1, 1]^n$ by

$$\begin{aligned}
\Lambda(\phi_1, \gamma_1, \beta_1, \delta_1, \rho_1, \ldots, \phi_n, \gamma_n, \beta_n, \delta_n, \rho_n)& \\
= (\Lambda_1, \Lambda_2, \ldots, \Lambda_n),&
\end{aligned} \tag{29}$$

with Λ_i as defined by (11), and then discuss convergence to $M_{CB(\alpha)}$ in terms of convergence of Λ to the corresponding point $(-1, -1, \ldots, -1)^T \in [-1, 1]^n$. (A Lyapunov function such as $V = \sum_{i=1}^n \Lambda_i$ should serve the purpose.) Further analysis is required to complete the technical proof (note, for instance, that care must be taken since the point $(-1, -1, \ldots, -1)^T$ lies on the boundary of $[-1, 1]^n$), but the outlined approach seems to point the way to a nonlinear stability analysis for the many particle case.

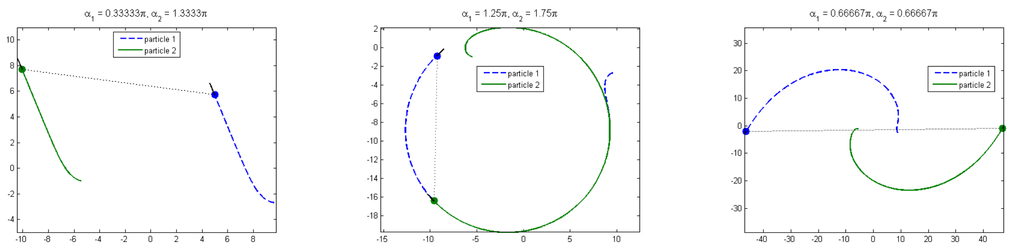
B. Convergence to special solutions on the invariant manifold

The second portion of the stability analysis focuses on the reduced dynamics (19) which govern the system shape evolution on $M_{CB(\alpha)}$. This analysis requires both the characterization of all steady-state solutions corresponding to a given set of CB angles $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ as well as a determination of the stability properties of these solutions. The beginning steps in this process were described in section V with the identification of necessary and sufficient conditions for the existence of relative equilibria and the description

TABLE I

CHARACTERIZATION OF THE ASYMPTOTIC BEHAVIOR FOR MUTUAL CB PURSUIT

Case	$\dot{\rho}(t);\ u(t)$	$e^{j\alpha_1} + e^{j\alpha_2}$	**Description**
I.	$\dot{\rho}(t) \equiv 0;\ u(t) \equiv 0$	0	rectilinear equilibrium
II.	$\dot{\rho}(t) \equiv 0;\ u(t) > 0$	positive imaginary axis	CCW circling equilibrium
III.	$\dot{\rho}(t) \equiv 0;\ u(t) < 0$	negative imaginary axis	CW circling equilibrium
IV.	$\dot{\rho}(t) < 0;\ u(t) \equiv 0$	positive real axis	"straight flight rendezvous"
V.	$\dot{\rho}(t) > 0;\ u(t) \equiv 0$	negative real axis	"straight flight retreat"
VI.	$\dot{\rho}(t) < 0;\ u(t) > 0$	quadrant I	CCW "inward sweep"
VII.	$\dot{\rho}(t) > 0;\ u(t) > 0$	quadrant II	CCW "outward sweep"
VIII.	$\dot{\rho}(t) > 0;\ u(t) < 0$	quadrant III	CW "outward sweep"
IX.	$\dot{\rho}(t) < 0;\ u(t) < 0$	quadrant IV	CW "inward sweep"



(a) Case I: Rectilinear equilibrium $(\pi/3,\ 4\pi/3)$

(b) Case III: CW circling equilibrium $(5\pi/4,\ 7\pi/4)$

(c) Case VII: CCW "outward sweep" $(2\pi/3,\ 2\pi/3)$

Fig. 2. These MATLAB simulations illustrate a selection of the different cases described in table I. The particular choices of $(\alpha_1,\ \alpha_2)$ for each simulation are listed in the captions associated with each figure.

of other special solutions; the classification presented in the two-particle case may provide the framework for characterizing *all* possible steady-state solutions. Convergence to these particular solutions can then be addressed in terms of Lyapunov analysis with the reduced dynamics (19).

VIII. CONCLUSION

In this work we have presented a formulation of a cyclic pursuit scheme with a constant bearing pursuit law. By an explicit reduction to shape space, we have demonstrated the existence of the invariant submanifold $M_{CB(\alpha)}$ and provided necessary and sufficient conditions for existence of relative equilibria on that submanifold. For the two-particle case, we have proved asymptotic convergence to $M_{CB(\alpha)}$ and developed a full characterization of the possible steady-state solutions in terms of the CB angles. Finally, we have outlined an approach for nonlinear stability analysis in the many particle case.

Work in progress (in addition to that which was outlined in section VII) includes a generalization to the full three-dimensional case (with cyclic pursuit on the sphere as an intermediate step) as well as allowing for time-varying CB angles α_i.

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APPENDIX

Proof of Proposition 1: ($\Rightarrow$) At a rectilinear relative equilibrium on $M_{CB(\alpha)}$ we have ρ_i constant, $i = 1, 2, ..., n$, and therefore we can make the assignment $\sigma_i = \rho_i = |\mathbf{r}_{i,i+1}|$. Furthermore, by definition of a rectilinear relative equilibrium, there exists a unit vector $\mathbf{x}_{com}$ such that $\mathbf{x}_1 = \mathbf{x}_2 = \ldots = \mathbf{x}_n = \mathbf{x}_{com}$.

Note that the closure constraint

$$\sum_{i=1}^{n} \mathbf{r}_{i,i+1} = \mathbf{0} \tag{30}$$

always holds, implying that

$$\begin{aligned} 0 &= \mathbf{x}_{com} \cdot \sum_{i=1}^{n} \mathbf{r}_{i,i+1} = \sum_{i=1}^{n} \mathbf{x}_i \cdot \mathbf{r}_{i,i+1} \\ &= \sum_{i=1}^{n} |\mathbf{r}_{i,i+1}| \, \beta_i = \sum_{i=1}^{n} \sigma_i \ \cos(\alpha_i), \end{aligned} \tag{31}$$

where the last step follows from the definition of $M_{CB(\alpha)}$. An analogous chain of logic starting with $\mathbf{x}_{com}^{\perp} \cdot \sum_{i=1}^{n} \mathbf{r}_{i,i+1} = 0$ yields $\sum_{i=1}^{n} \sigma_i \ \sin(\alpha_i) = 0$, from which (20) follows.

($\Leftarrow$) Assume that there exists a set of constants $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ which satisfy the conditions of **Proposition 1**. Then a rectilinear relative equilibrium can be constructed as follows:

1) Place $\mathbf{r}_1$ at the origin with velocity vector $\mathbf{x}_1$ aligned with the horizontal axis.
2) Assign the positions and velocities of the remaining $n-1$ particles in an iterative fashion by

$$\mathbf{x}_i = \mathbf{x}_1, \ i = 2, 3, \ldots, n, \tag{32}$$

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \sigma_i R\left(\alpha_i\right) \mathbf{x}_i, \ i = 1, 2, \ldots, n-1. \tag{33}$$

We must show that our constructed state is on $M_{CB(\alpha)}$. Using (4) with (33), we compute

$$\begin{aligned} \Lambda_i &= R\left(\alpha_i\right) \mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} = R\left(\alpha_i\right) \mathbf{x}_i \cdot \frac{-\sigma_i R\left(\alpha_i\right) \mathbf{x}_i}{|\sigma_i R\left(\alpha_i\right) \mathbf{x}_i|} \\ &= -R\left(\alpha_i\right) \mathbf{x}_i \cdot R\left(\alpha_i\right) \mathbf{x}_i \\ &= -1, \ i = 1, 2, \ldots, n-1. \end{aligned} \tag{34}$$

This shows that the first $n-1$ particles are on $M_{CB(\alpha)}$, and we must now show that $\Lambda_n = -1$, also.

Summing up expressions (33), and substituting $\mathbf{x}_i = \mathbf{x}_1$ per (32), we have

$$\mathbf{r}_k = \left(\sum_{i=1}^{k-1} \sigma_i R(\alpha_i) \right) \mathbf{x}_1, \ k = 2, 3, \ldots, n. \tag{35}$$

Therefore we can express $\mathbf{r}_n$ by

$$\begin{aligned} \mathbf{r}_n &= \left(\sum_{i=1}^{n-1} \sigma_i R(\alpha_i) \right) \mathbf{x}_1 \\ &= \left(\left(\sum_{i=1}^{n} \sigma_i R(\alpha_i) \right) - \sigma_n R(\alpha_n) \right) \mathbf{x}_1 \\ &= \left(\begin{bmatrix} \sum_{i=1}^{n} \sigma_i \cos(\alpha_i) & -\sum_{i=1}^{n} \sigma_i \sin(\alpha_i) \\ \sum_{i=1}^{n} \sigma_i \sin(\alpha_i) & \sum_{i=1}^{n} \sigma_i \cos(\alpha_i) \end{bmatrix} \right. \\ &\qquad \left. - \sigma_n R(\alpha_n) \right) \mathbf{x}_1 \\ &= -\sigma_n R(\alpha_n) \mathbf{x}_1, \end{aligned} \tag{36}$$

where the last step follows from the assumptions of **Proposition 1**. Calculations analogous to (34) can then be used to show that $\Lambda_n = -1$, and therefore we conclude that the state lies in $M_{CB(\alpha)}$.

By (32) we have $\phi_i = 1$ (and $\gamma_i = 0$), and direct substitution into (19) shows that this corresponds to an equlibrium for the reduced dynamics (i.e. a relative equilibrium for the full dynamics.) Since the velocity vectors are aligned, this is necessarily a rectilinear equilibrium. □

Remark: The next proof uses the following identities (derived from common trigonometric identities and the definition of the rotation matrix in (5)):

$$\text{i. } \sin(\theta) R\left(\theta + \frac{\pi}{2}\right) = \frac{1}{2}\Big(R(2\theta) - I\Big), \tag{37}$$

$$\text{ii. } R(2\theta) = I \Leftrightarrow \sin(\theta) = 0. \tag{38}$$

Proof of Proposition 2: ($\Rightarrow$) Without loss of generality, assume that the circling relative equilibrium (with radius r_c) rotates in a CCW direction and has its center at the origin, i.e.

$$|\mathbf{r}_1| = |\mathbf{r}_2| = \ldots = |\mathbf{r}_n| = r_c. \tag{39}$$

Since the particles are moving in a CCW direction with velocities tangent to the circle, we have

$$\mathbf{x}_i = \frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|} = \frac{\mathbf{r}_i^{\perp}}{r_c}. \tag{40}$$

Our circling equilibrium lies on $M_{CB(\alpha)}$, and therefore it is possible to show that

$$R(\alpha_i) \mathbf{x}_i = -\frac{\mathbf{r}_i - \mathbf{r}_{i+1}}{|\mathbf{r}_i - \mathbf{r}_{i+1}|} \tag{41}$$

by making use of the definition of Λ_i and the orthogonal decomposition of the vector $R(\alpha_i)\mathbf{x}_i$ in terms of $\frac{\mathbf{r}_i - \mathbf{r}_{i+1}}{|\mathbf{r}_i - \mathbf{r}_{i+1}|}$ and its corresponding orthogonal unit vector. By substitution of (40) into (41) it follows that

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \rho_i R(\alpha_i) \frac{\mathbf{r}_i^{\perp}}{r_c}, \tag{42}$$

which establishes the necessity of condition (i) since we must have $\sin(\alpha_i) > 0$.[3] .

Taking the inner product of each side of (42) with itself, we can solve for ρ_i as follows:

$$\mathbf{r}_{i+1} \cdot \mathbf{r}_{i+1} = \left(\mathbf{r}_i + \rho_i R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{r_c}\right) \cdot \left(\mathbf{r}_i + \rho_i R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{r_c}\right)$$
$$\implies r_c^2 = r_c^2 + \rho_i^2 + 2\rho_i \mathbf{r}_i \cdot R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{r_c}$$
$$\implies \rho_i = -2\mathbf{r}_i \cdot R(\alpha_i + \tfrac{\pi}{2})\frac{\mathbf{r}_i}{r_c} = -2\,|\mathbf{r}_i|\,\frac{|\mathbf{r}_i|}{r_c}\cos(\alpha_i + \tfrac{\pi}{2})$$
$$= 2r_c \sin(\alpha_i). \tag{43}$$

Substituting this result into (42) and making use of (37) yields

$$\begin{aligned}\mathbf{r}_{i+1} &= \mathbf{r}_i + 2r_c\sin(\alpha_i)R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{r_c} \\ &= \mathbf{r}_i + 2\sin(\alpha_i)R(\alpha_i + \tfrac{\pi}{2})\mathbf{r}_i \\ &= \left[I + 2\left(\frac{1}{2}\Big(R(2\alpha_i) - I\Big)\right)\right]\mathbf{r}_i \\ &= R(2\alpha_i)\mathbf{r}_i,\end{aligned} \tag{44}$$

which can be expressed in an alternate form as

$$\mathbf{r}_j = R\left(2\sum_{k=0}^{j-1}\alpha_k\right)\mathbf{r}_1,\ j = 1, 2, \ldots, n, \tag{45}$$

if we use the convention $\alpha_0 \equiv 0$.

Combining (45) with the closure condition (30) yields

$$\begin{aligned}\mathbf{0} &= \sum_{i=1}^{n}(\mathbf{r}_i - \mathbf{r}_{i+1}) \\ &= \sum_{i=1}^{n}\left[R\left(2\sum_{k=0}^{i-1}\alpha_k\right)\mathbf{r}_1 - R\left(2\sum_{k=0}^{i}\alpha_k\right)\mathbf{r}_1\right] \\ &= \left[I - R\left(2\sum_{k=0}^{n}\alpha_k\right)\right]\mathbf{r}_1 \implies \sin\left(\sum_{k=0}^{n}\alpha_k\right) = 0,\end{aligned} \tag{46}$$

where the last step follows from (38).

($\Leftarrow$) Assume that the conditions of **Proposition 2** hold. (Without loss of generality, we'll take $\sin(\alpha_i) > 0, \quad i = 1, 2, \ldots, n$)[4]. We claim that the following construction leads to a circling relative equilibrium:

1) Place $\mathbf{r}_1$ on the horizontal axis with $|\mathbf{r}_1| = r_{com} > 0$ and assign the positions of the remaining $n-1$ particles by

$$\mathbf{r}_i = R\left(2\sum_{k=1}^{i-1}\alpha_k\right)\mathbf{r}_1,\ i = 2, 3, \ldots, n. \tag{47}$$

2) Specify the velocities by

$$\mathbf{x}_i = \frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|},\ i = 1, 2, \ldots, n. \tag{48}$$

(Note that (47) implies that $|\mathbf{r}_i| = |\mathbf{r}_1| = r_{com},\ i = 1, 2, \ldots, n$.) As in the previous proof, we must show that this state lies on $M_{CB(\alpha)}$ and that it corresponds to an equilibrium point for the reduced dynamics (19). We will start with the former.

Note from (47) that we have

$$\begin{aligned}\mathbf{r}_{i,i+1} &= R\left(2\sum_{k=1}^{i-1}\alpha_k\right)\mathbf{r}_1 - R\left(2\sum_{k=1}^{i}\alpha_k\right)\mathbf{r}_1 \\ &= \left[I - R(2\alpha_i)\right]R\left(2\sum_{k=1}^{i-1}\alpha_k\right)\mathbf{r}_1 \\ &= -2\sin(\alpha_i)R\left(\alpha_i + \frac{\pi}{2}\right)\mathbf{r}_i,\ i = 1, 2, \ldots, n-1,\end{aligned} \tag{49}$$

where we have used identity (37). Therefore it holds that

$$|\mathbf{r}_{i,i+1}| = 2\,|\sin(\alpha_i)|\,|\mathbf{r}_i|\,,\ i = 1, 2, \ldots, n-1. \tag{50}$$

Since $\sin(\alpha_i) > 0$ (by assumption), (48) implies that

$$\begin{aligned}R(\alpha_i)\mathbf{x}_i \cdot \frac{\mathbf{r}_{i,i+1}}{|\mathbf{r}_{i,i+1}|} &= R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|} \cdot \frac{-2\sin(\alpha_i)R\left(\alpha_i + \frac{\pi}{2}\right)\mathbf{r}_i}{2\sin(\alpha_i)\,|\mathbf{r}_i|} \\ &= -R(\alpha_i)\frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|} \cdot R\left(\alpha_i\right)\frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|} \\ &= -1,\ \ i = 1, 2, \ldots, n-1.\end{aligned} \tag{51}$$

Also, by (47) we have

$$\begin{aligned}\mathbf{r}_n &= R\left(2\sum_{k=1}^{n-1}\alpha_k\right)\mathbf{r}_1 = R\left(2\sum_{k=1}^{n}\alpha_k\right)R\left(-2\alpha_n\right)\mathbf{r}_1 \\ &= R\left(-2\alpha_n\right)\mathbf{r}_1,\end{aligned} \tag{52}$$

(where the last simplification is made possible by condition (ii) of our proposition in conjunction with identity (38)), and therefore $\mathbf{r}_1 = R\left(2\alpha_n\right)\mathbf{r}_n$, i.e.

$$\mathbf{r}_n - \mathbf{r}_1 = \left[I - R(2\alpha_n)\right]\mathbf{r}_n. \tag{53}$$

Then calculations analogous to (51) yield $R(\alpha_n)\mathbf{x}_n \cdot \frac{\mathbf{r}_{n,1}}{|\mathbf{r}_{n,1}|} = -1$, demonstrating that our state lies on $M_{CB(\alpha)}$.

From (47) and (52) it is clear that

$$\mathbf{r}_{i+1} = R\left(2\alpha_i\right)\mathbf{r}_i,\ i = 1, 2, \ldots, n, \tag{54}$$

and therefore we can express ϕ_i for our constructed state as

$$\begin{aligned}\phi_i &= \frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|} \cdot \frac{\mathbf{r}_{i+1}^{\perp}}{|\mathbf{r}_{i+1}|} = \frac{\mathbf{r}_i^{\perp}}{r_{com}} \cdot R\left(2\alpha_i\right)\frac{\mathbf{r}_i^{\perp}}{r_{com}} \\ &= \cos(2\alpha_i),\ i = 1, 2, \ldots, n.\end{aligned} \tag{55}$$

[3]To see this, note that $\frac{\mathbf{r}_i^{\perp}}{r_c}$ denotes the unit vector tangent to the circle ("pointing" counter-clockwise) at $\mathbf{r}_i$. The unit vector that results from rotating $\frac{\mathbf{r}_i^{\perp}}{r_c}$ counter-clockwise by α_i must therefore indicate the direction toward particle $i+1$. If $\sin(\alpha_i) \leq 0$, then this vector will point outside the circle.

[4]In order to prove **Proposition 2** for the alternative case, i.e. for $\sin(\alpha_i) < 0, \quad i = 1, 2, \ldots, n$, we replace (48) with $\mathbf{x}_i = -\frac{\mathbf{r}_i^{\perp}}{|\mathbf{r}_i|},\ i = 1, 2, \ldots, n$, and proceed in the same fashion.

(Similar calculations show that $\gamma_i = -\sin(2\alpha_i)$.) For these particular values of (ϕ_i, γ_i), we note the following:

$$\begin{aligned}(1-\phi_i)\sin(\alpha_i) - \gamma_i\cos(\alpha_i) \\ &= (1-\cos(2\alpha_i))\sin(\alpha_i) + \sin(2\alpha_i)\cos(\alpha_i) \\ &= \sin(\alpha_i) - \cos(2\alpha_i)\sin(\alpha_i) + \sin(2\alpha_i)\cos(\alpha_i) \\ &= \sin(\alpha_i) + \sin(2\alpha_i - \alpha_i) \\ &= 2\sin(\alpha_i),\ i = 1, 2, \ldots, n. \qquad (56)\end{aligned}$$

Substitution of (50), (55) and (56) into (19) demonstrates explicitly that our constructed state is in fact a relative equilibrium. $\square$