Geometry of Steering Laws for Swarming

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Outline

• Motivation: UAV formation control

• Planar model based on unit-speed motion with steering control

• Nonlinear analysis of two-vehicle formation control laws

- Connection to gyroscopic systems

- Contrast with synthetic potential methods

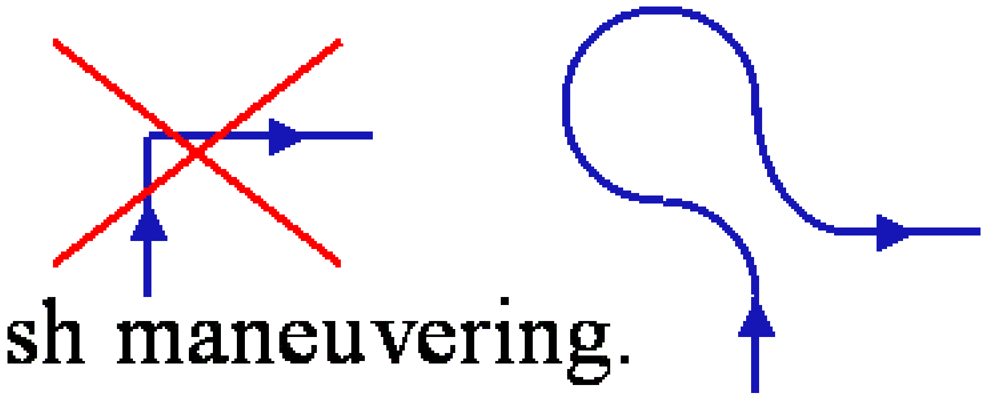
• Generalizations to *n* vehicles

• Future research directions

UAV Modeling

Dragon Eye

(Photo credit: Jonathan Finer, The Washington Post)

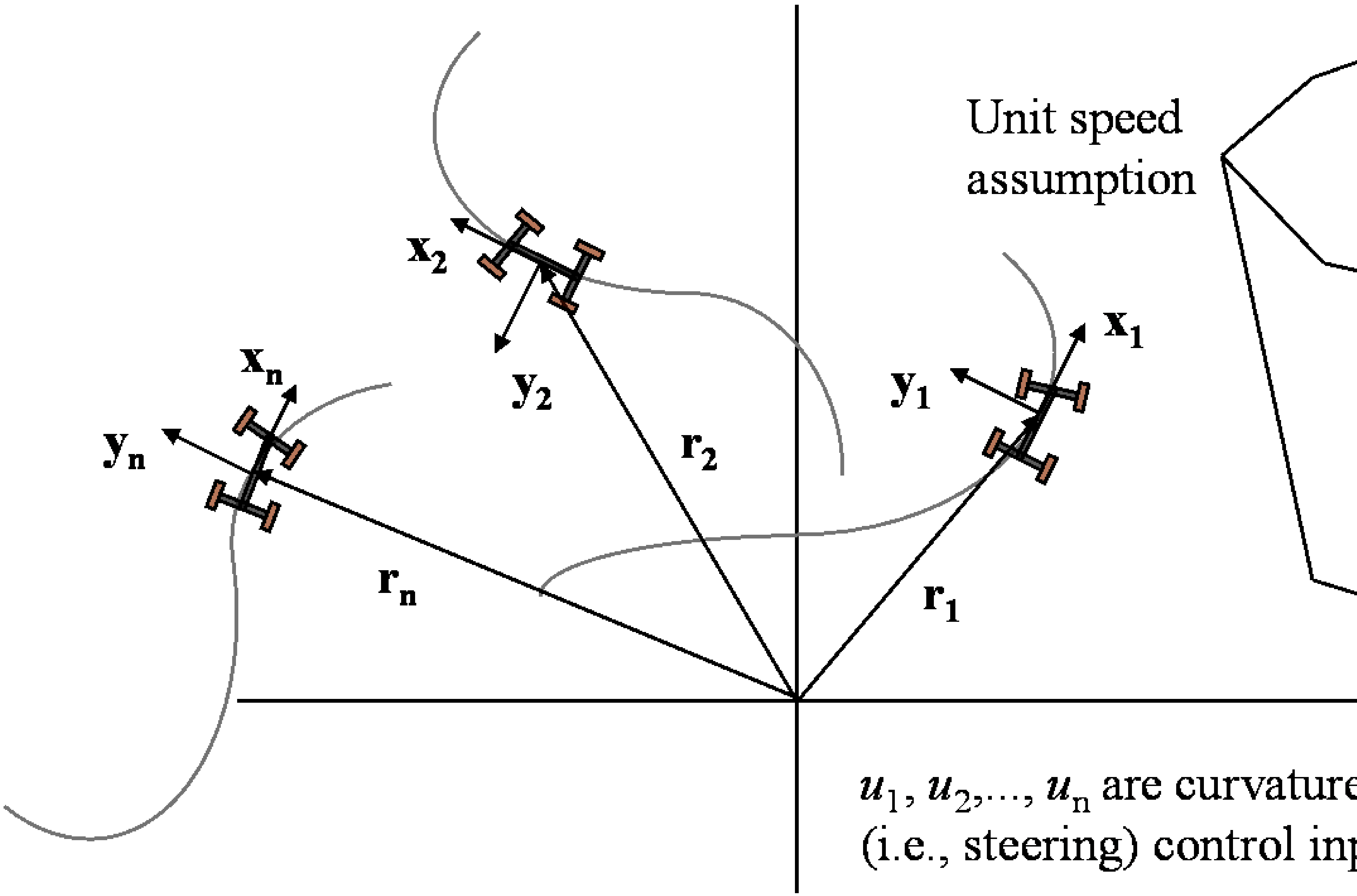


- Features of UAV model:
  - High speed ⇔ sluggish maneuvering.
  - Turning ⇒ significant energy penalty (due to side slip).
  - An autopilot takes into account the detailed vehicle kinematics.
- Vehicles modeled as point particles moving at **unit speed** and subject to **steering control**.
- A **formation control law** is a feedback law which specifies these steering controls.
- This modeling may be appropriate in other settings in which there are high speeds and penalties associated with turning (e.g. loss of dynamic stability).

Dragon Runner

(Photo from U.S. Marine Corps website)

Planar Model (Frenet-Serret Equations)



Unit speed assumption

$$\dot{\mathbf{r}}_1 = \mathbf{x}_1$$
$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1$$
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1$$

$$\dot{\mathbf{r}}_2 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$
$$\dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

$$\vdots$$

$$\dot{\mathbf{r}}_n = \mathbf{x}_n$$
$$\dot{\mathbf{x}}_n = \mathbf{y}_n u_n$$
$$\dot{\mathbf{y}}_n = -\mathbf{x}_n u_n$$

$u_1, u_2, \ldots, u_n$ are curvature (i.e., steering) control inputs.

Specifying $u_1, u_2, \ldots, u_n$ as feedback functions of $(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{r}_2, \mathbf{x}_2, \mathbf{y}_2), \ldots, (\mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n)$ defines a **control law**.

An Intuitive Two-Vehicle Control Law

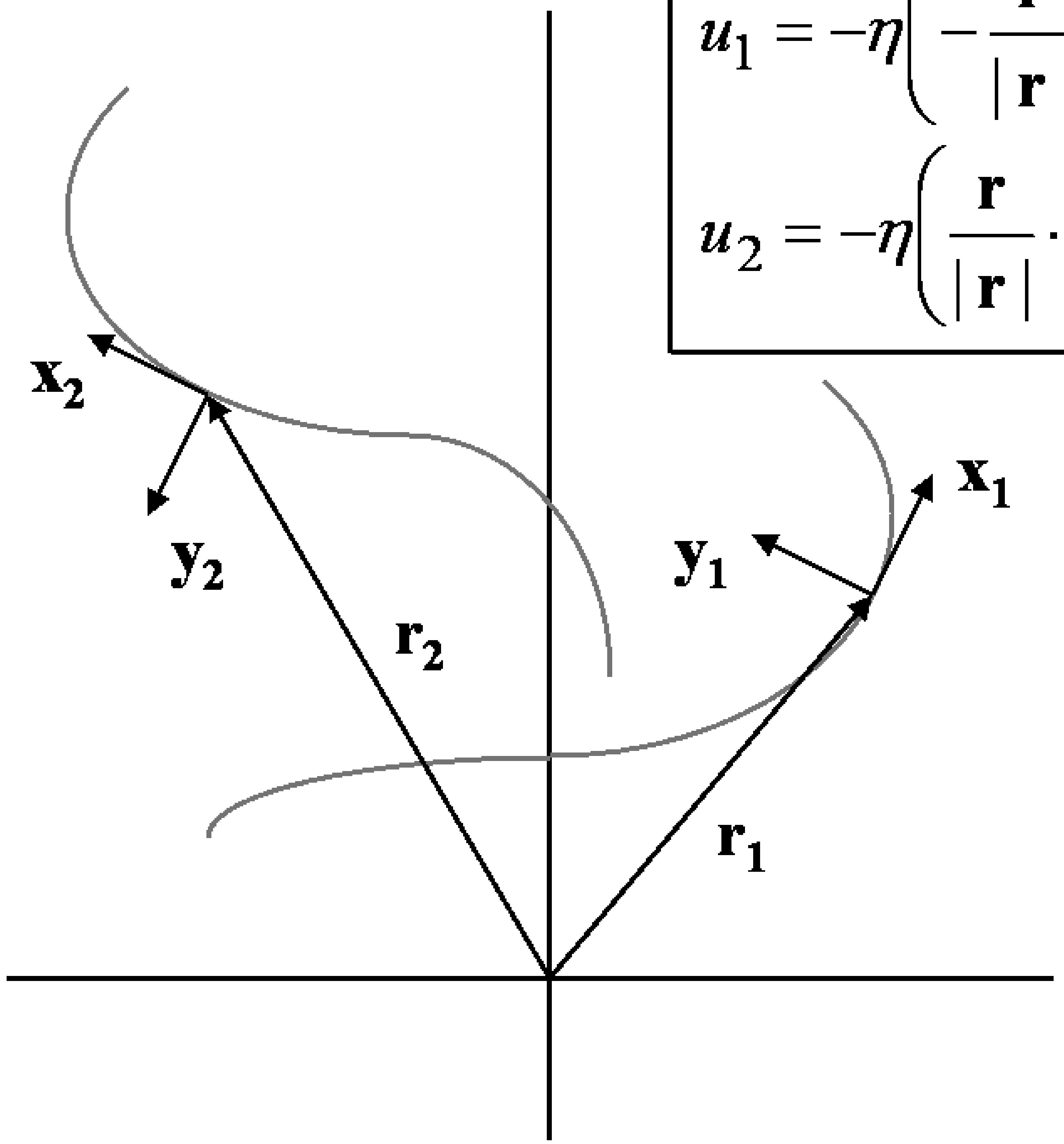
$$\dot{\mathbf{r}}_1 = \mathbf{x}_1 \qquad \dot{\mathbf{r}}_2 = \mathbf{x}_2 \qquad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1 \qquad \dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$

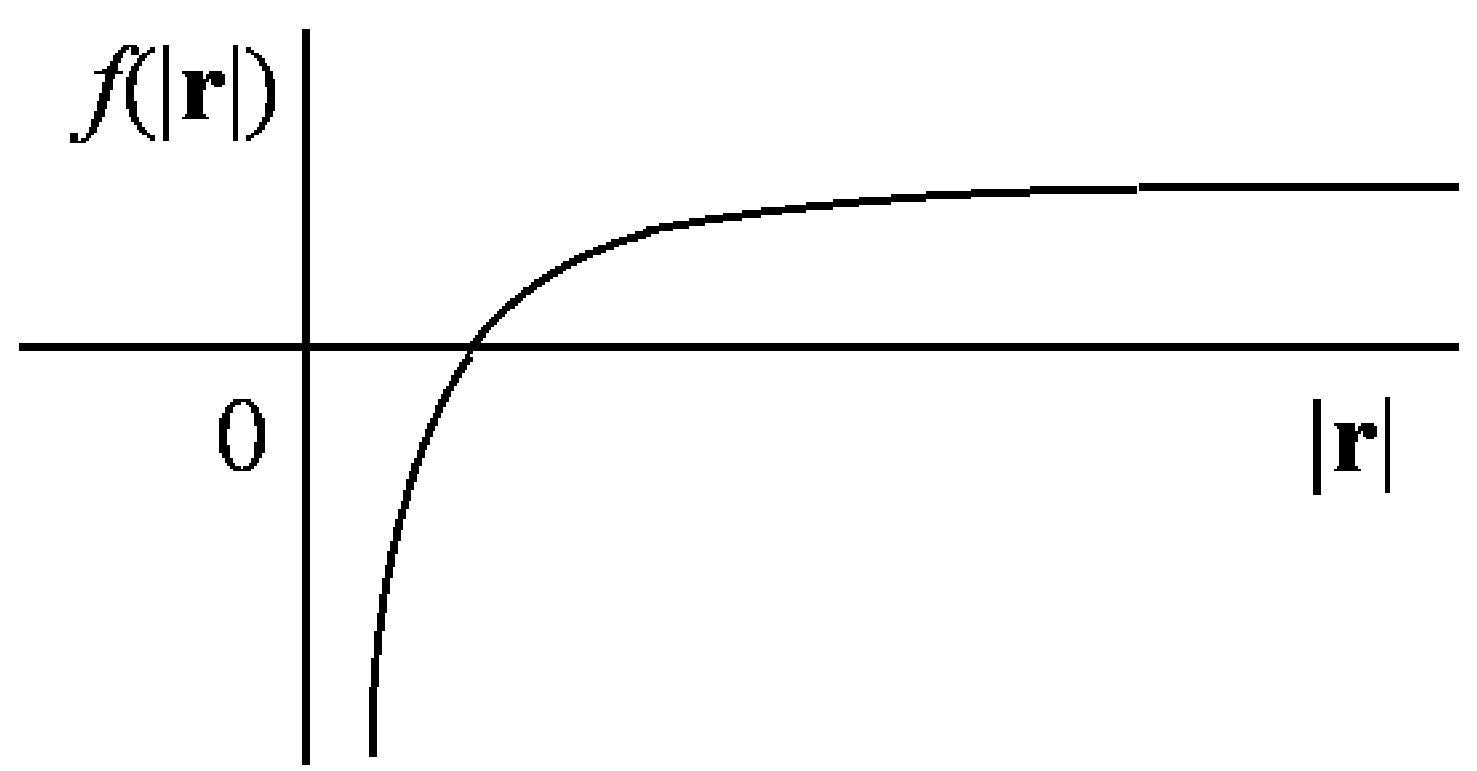
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1 \qquad \dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

$$u_1 = -\eta\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1\right)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) + \mu\mathbf{x}_2\cdot\mathbf{y}_1$$

$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$



$$\mu > \frac{\eta}{2} > 0$$



Two-Vehicle Law: Intuition

Steering control equation for UAV #2:

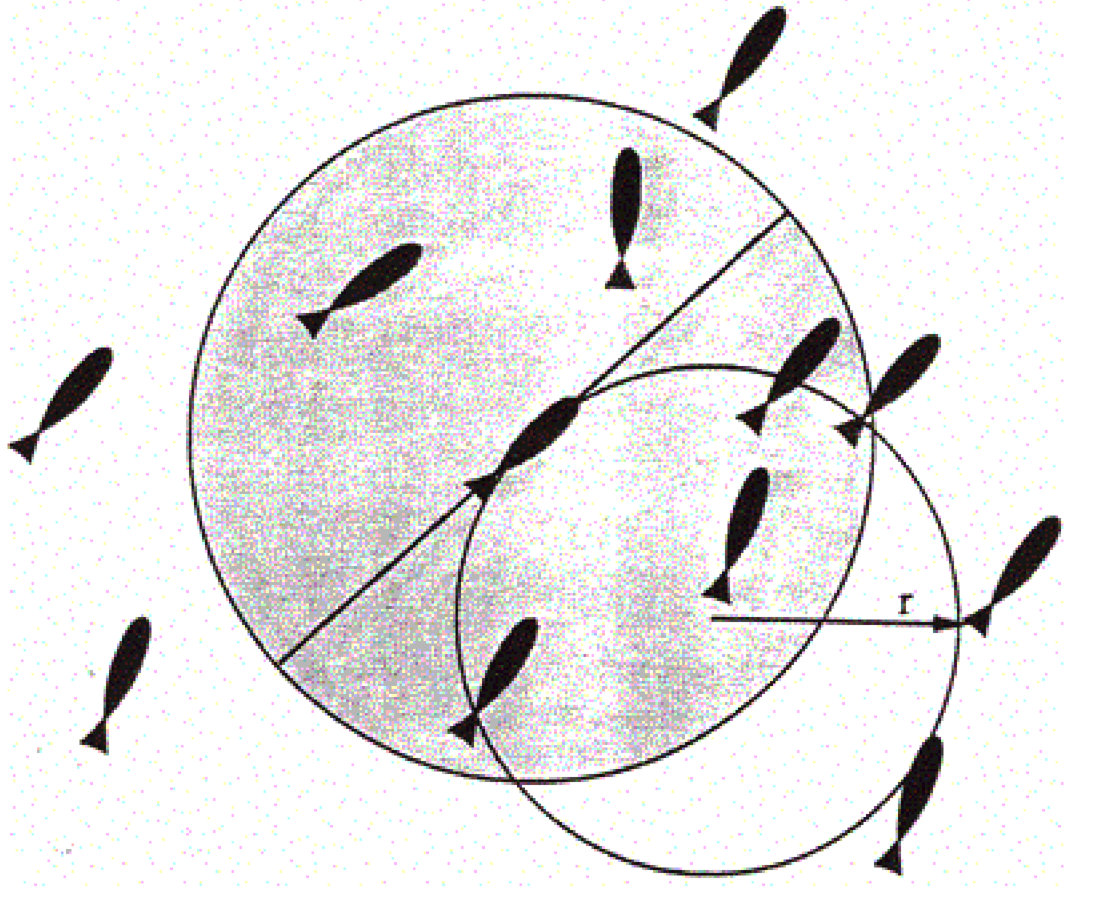
$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$

Align each vehicle perpendicular to the baseline between the vehicles.

Steer toward or away from the other vehicle to maintain appropriate separation.

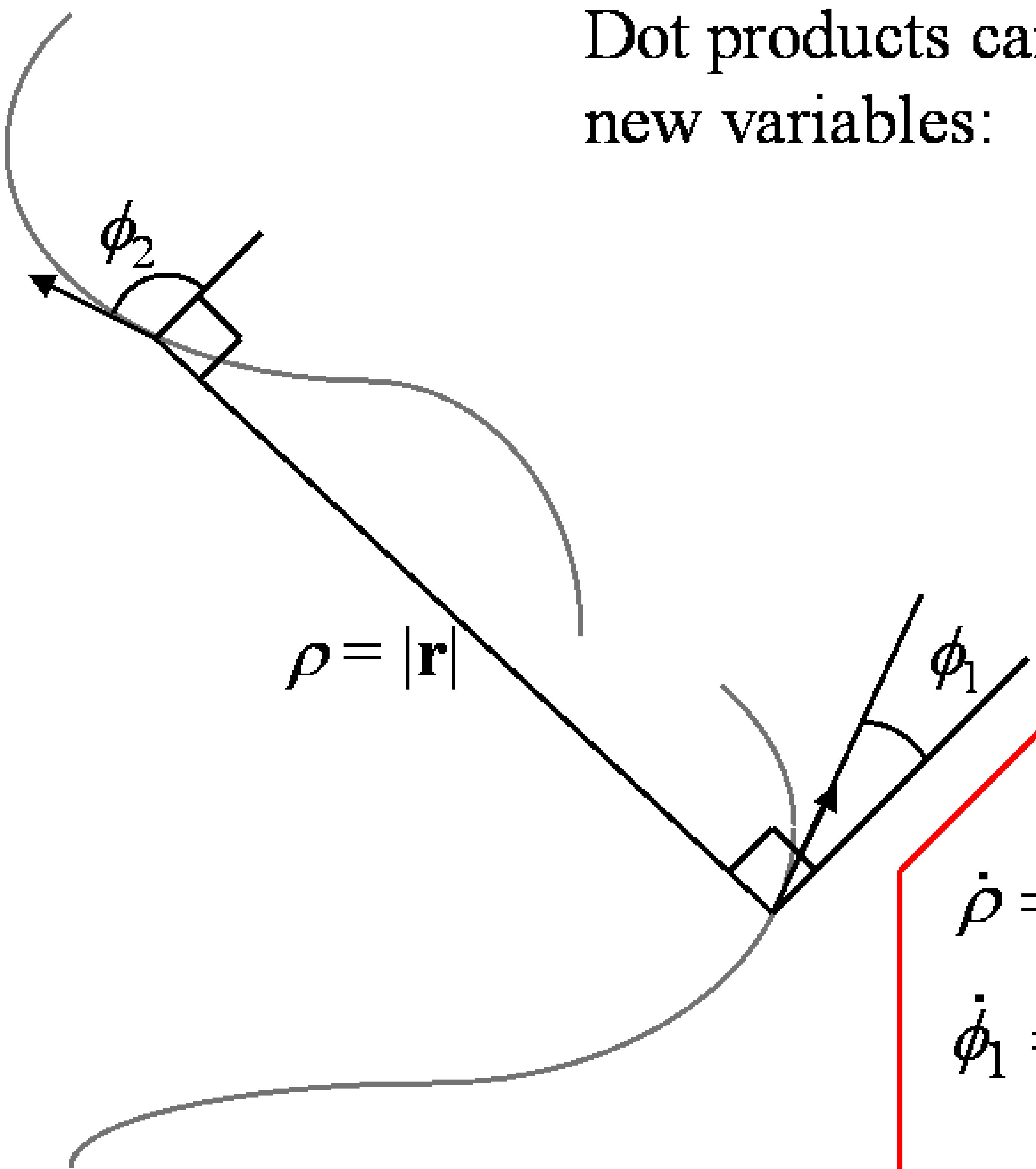
Align with the other vehicle's heading.

- Biological analogy (swarming, schooling):
  - Decreasing responsiveness at large separation distances.
  - Switch from attraction to repulsion based on separation distance or density.
  - Mechanism for alignment of headings.



D. Grünbaum, "Schooling as a strategy for taxis in a noisy environment," in *Animal Groups in Three Dimensions*, J.K. Parrish and W.M. Hamner, eds., Cambridge University Press, 1997.

Two-Vehicle Law: Change of Variables



Dot products can be expressed as sines and cosines in the new variables:

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1 = \sin\phi_1 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1 = \cos\phi_1$$

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2 = \sin\phi_2 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2 = \cos\phi_2$$

$$\mathbf{x}_2\cdot\mathbf{y}_1 = \sin(\phi_2-\phi_1)$$

$$\mathbf{x}_1\cdot\mathbf{y}_2 = \sin(\phi_1-\phi_2)$$

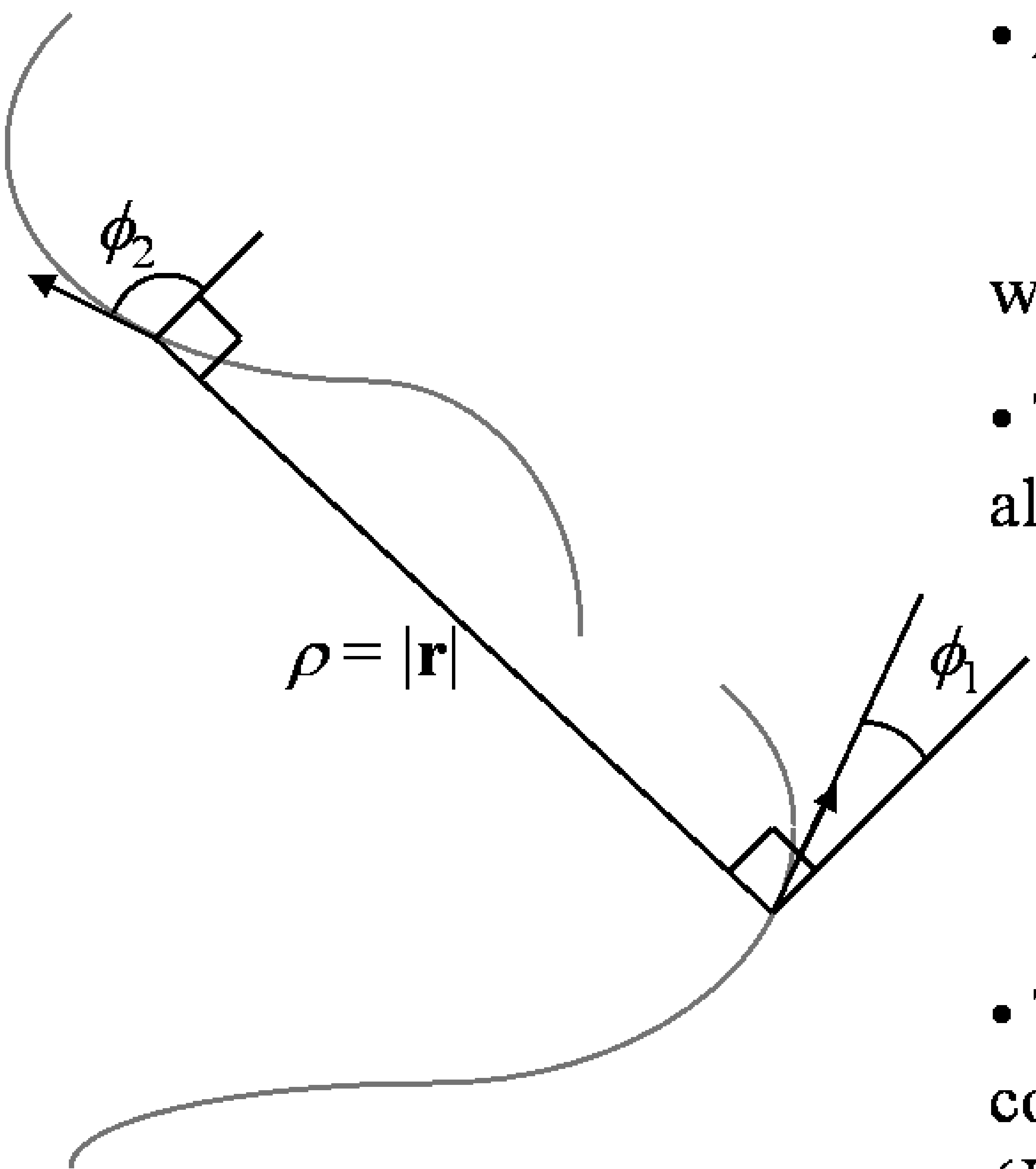
System after change of variables:

$$\dot\rho = \sin\phi_2 - \sin\phi_1$$

$$\dot\phi_1 = -\eta\sin\phi_1\cos\phi_1 + f(\rho)\cos\phi_1 + \mu\sin(\phi_2-\phi_1) + (1/\rho)(\cos\phi_2 - \cos\phi_1)$$

$$\dot\phi_2 = -\eta\sin\phi_2\cos\phi_2 - f(\rho)\cos\phi_2 + \mu\sin(\phi_1-\phi_2) + (1/\rho)(\cos\phi_2 - \cos\phi_1)$$

Two-Vehicle Law: Lyapunov Function



• A Lyapunov function is

$$V_{pair} = -\ln(1+\cos(\phi_2 - \phi_1)) + h(\rho)$$

where $f(\rho) = dh/d\rho$.

• The derivative of V_{pair} with respect to time along trajectories of the system is

$$\begin{aligned}\dot{V}_{pair} &= \frac{\partial V_{pair}}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial V_{pair}}{\partial \phi_2}\dot{\phi}_2 + \frac{\partial V_{pair}}{\partial \rho}\dot{\rho} \\ &\vdots \\ &\leq 0.\end{aligned}$$

• This Lyapunov function leads to a **global** convergence result for the two-UAV system (Justh, Krishnaprasad, 2002).

Note: V_{pair} is **not** to be thought of as a synthetic potential (commonly used in robotics for directing motion toward a target or away from obstacles). V_{pair} is a Lyapunov function for the **shape dynamics** of the two-vehicle formation.

Gyroscopic Forces and Vector Potentials

Kinetic energy — Scalar potential

Consider the Lagrangian:

$$L = \tfrac{1}{2}\dot{x}\cdot M\dot{x} + y(x)\cdot\dot{x} - x\cdot Kx\,,$$

$$x \in \mathbb{R}^n,\;\; M = M^T > 0,\;\; K = K^T.$$

Vector potential (linear-in-velocity term)

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\big(M\dot{x} + y(x)\big) - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + \left(\frac{\partial y}{\partial x}\right)\dot{x} - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + Q(x)\dot{x} + Kx$$

$$= 0,$$

$$(*)\quad Q(x) = \left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial y}{\partial x}\right)^T \Rightarrow Q(x) = -Q^T(x).$$

Note: Lagrangian with linear-in-velocity term $\Rightarrow$ skew term in the dynamics, but the converse only holds if $Q(x)$ can be expressed as in (*) for some $y(x)$.

Gyroscopically Interacting Particles

For a single particle in the plane:

$$\mathbf{r} = \text{position}, \quad \mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \ddot{\mathbf{r}},$$
$$m = \text{mass} = 1,$$
$$H = \text{kinetic energy} = \tfrac{1}{2}\|\mathbf{v}\|^2,$$
$$m\mathbf{a} = \mathbf{F} \Leftrightarrow \ddot{\mathbf{r}} = \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} \dot{\mathbf{r}}.$$

Note: F is a **gyroscopic force**
(Recall Lorentz force law for a charged particle in a magnetic field)

$\Longrightarrow$

$$\dot{H} = 0,$$
$$\dot{\mathbf{r}} = \begin{pmatrix} \sqrt{2H}\cos\theta \\ \sqrt{2H}\sin\theta \end{pmatrix},$$
$$\dot{\theta} = u.$$

$\Downarrow$ Restrict to the level-set of H given by H=1/2.

$$\dot{\mathbf{r}} = \mathbf{x}$$
$$\dot{\mathbf{x}} = \mathbf{y}u$$
$$\dot{\mathbf{y}} = -\mathbf{x}u$$

Frenet-Serret equations

Note that u may be a complicated function of time, and may involve feedback. $\longrightarrow$

For multiple particles, the kinetic energy of **each particle** is conserved, and the particles interact via **gyroscopic forces alone**.

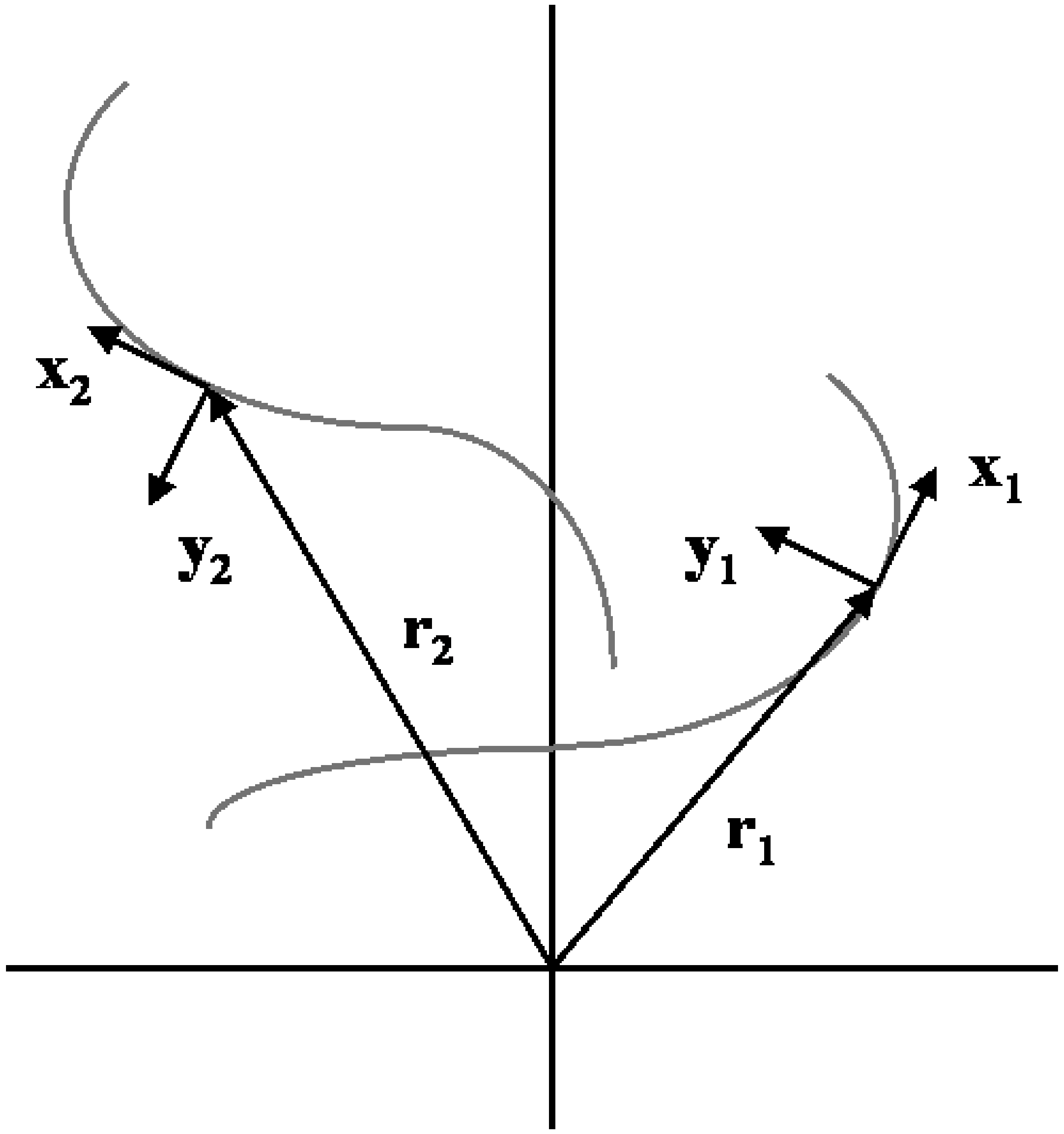
References:

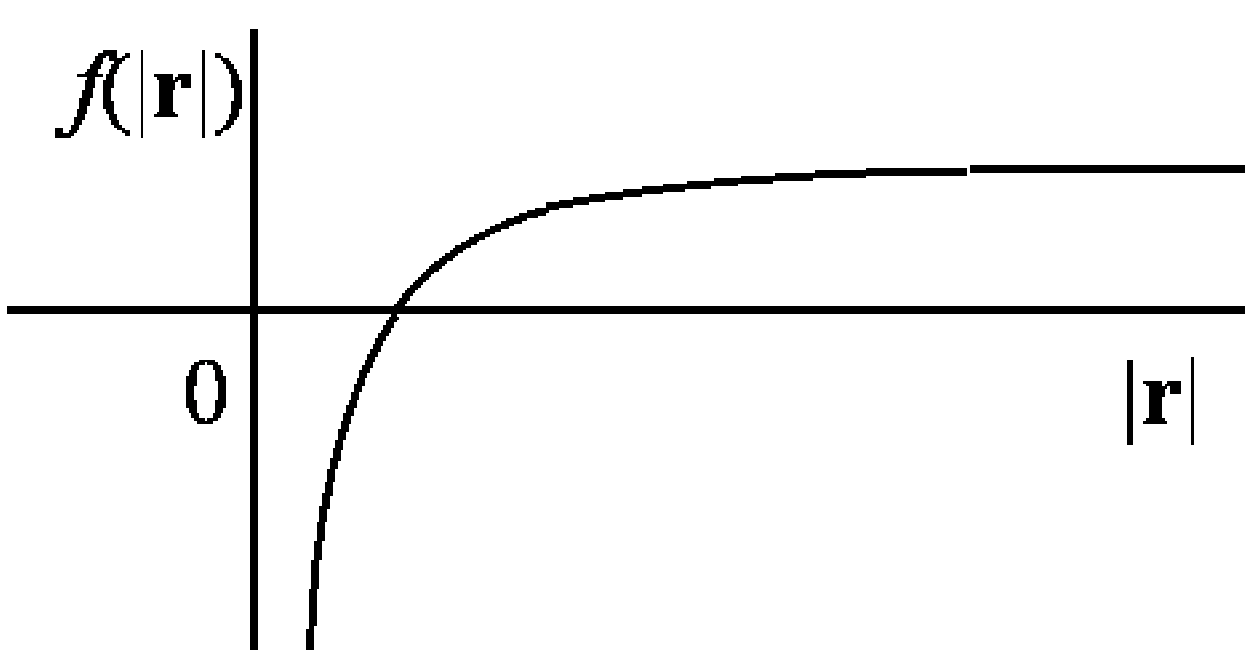
- L.-S. Wang and P.S. Krishnaprasad, *J. Nonlin. Sci.*, 2, pp. 367- 415,1992.
- J.E. Marsden and T.S. Ratiu, *Intro.to Mechanics and Symmetry*, 2nd ed, Springer, 1999.

Two-Vehicle Law: Key Ideas

- Unit-speed motion with steering control.
  - **Gyroscopic forces** preserve kinetic energy of each particle.
  - In mechanics, gyroscopic forces are associated with **vector potentials**.
- Shape variables: relative distances and angles.
- Lyapunov function $\Rightarrow$ convergence result for the shape dynamics.
- Equilibria of the shape dynamics = relative equilibria of the vehicle dynamics.
- Vehicle re-labeling symmetry.
- Lie group formulation:
  - The dynamics of each particle can be expressed as a left-invariant system evolving on $SE(2)$, the group of rigid motions in the plane.
  - $G=SE(2)$ is a symmetry group for the dynamics: the control law is invariant under rigid motions of the entire formation.
  - V_{pair} is also invariant under G.
  - Therefore, we can consider the reduced system evolving on shape space = $(G \times G)/G = G$.

Planar Formation Laws for Two Vehicles





$$\dot{\mathbf{r}}_1 = \mathbf{x}_1 \qquad \dot{\mathbf{r}}_2 = \mathbf{x}_2 \qquad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1 \qquad \dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$

$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1 \qquad \dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

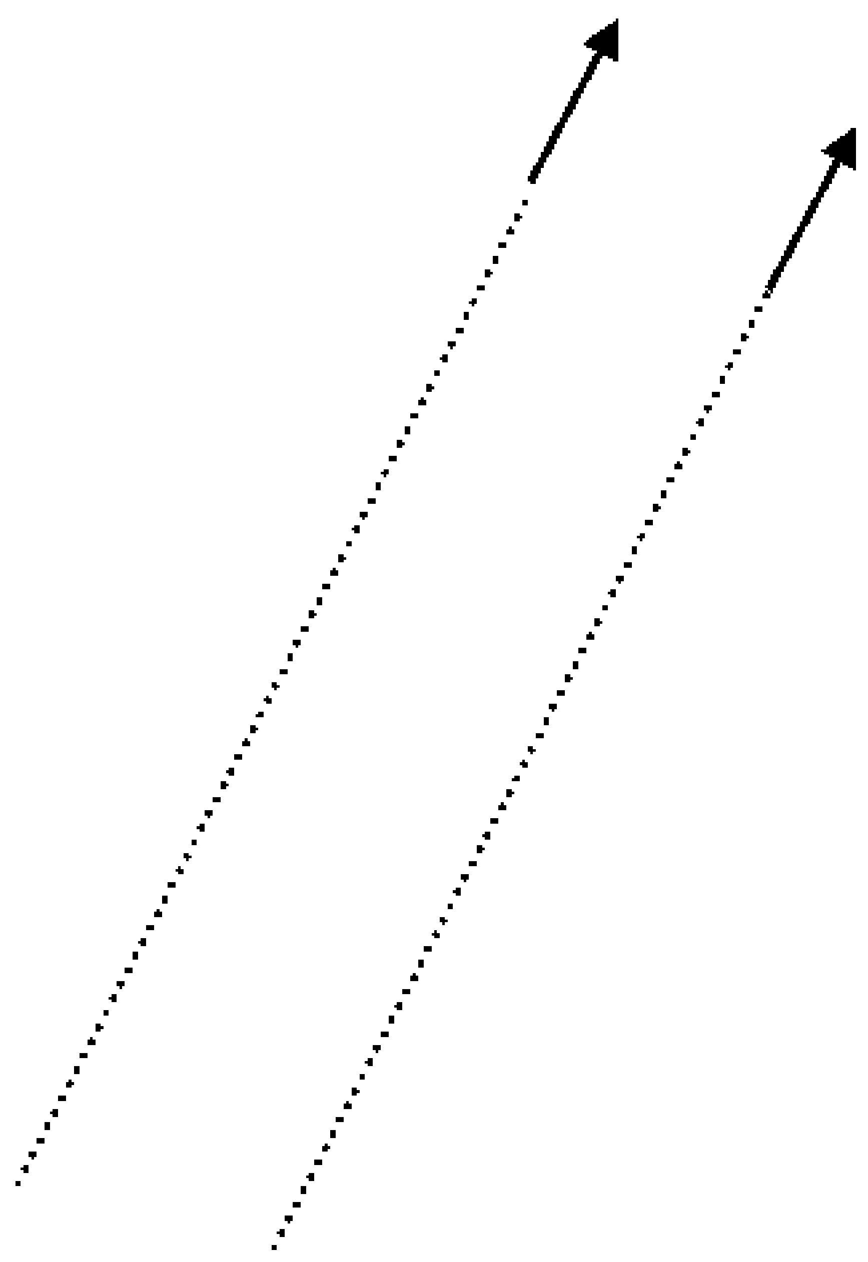
$$u_1 = F(-\mathbf{r}, \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_1\right)$$

$$u_2 = F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1, \mathbf{y}_1) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_2\right)$$

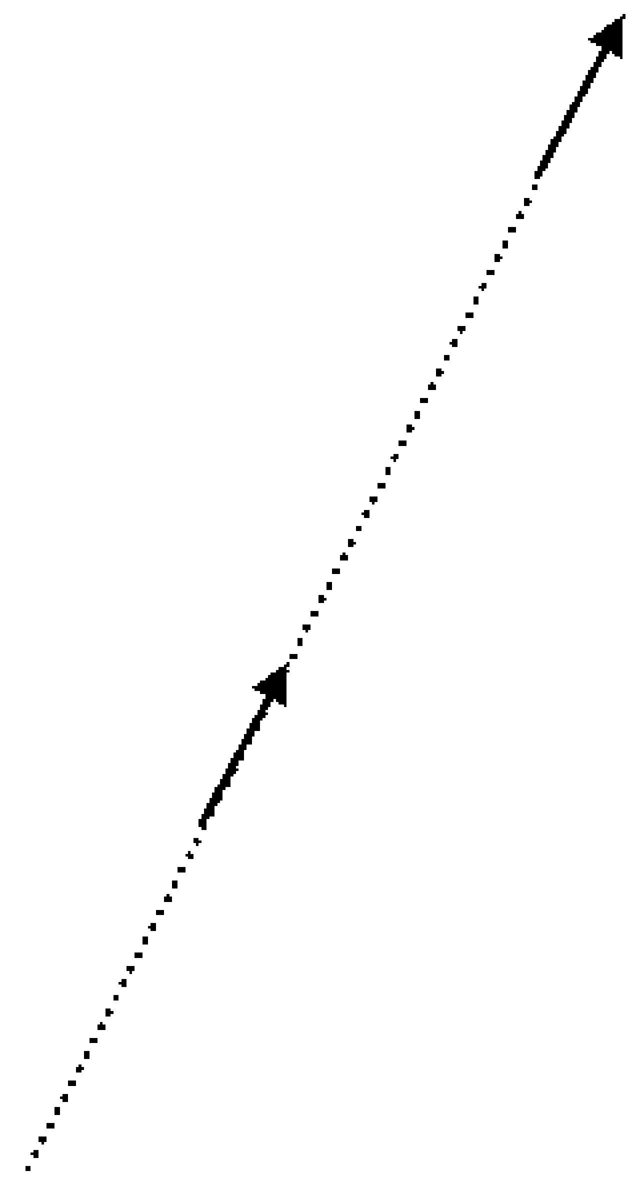
$$F(\mathbf{r}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_1, \mathbf{y}_1) = \sum_{j=1}^{2} \sum_{k=1}^{2} \nu_{jk}(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{x}_j\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|} \cdot \mathbf{y}_k\right)$$

$$f(|\mathbf{r}|) = \alpha\left[1 - \left(\frac{r_o}{|\mathbf{r}|}\right)^2\right]$$

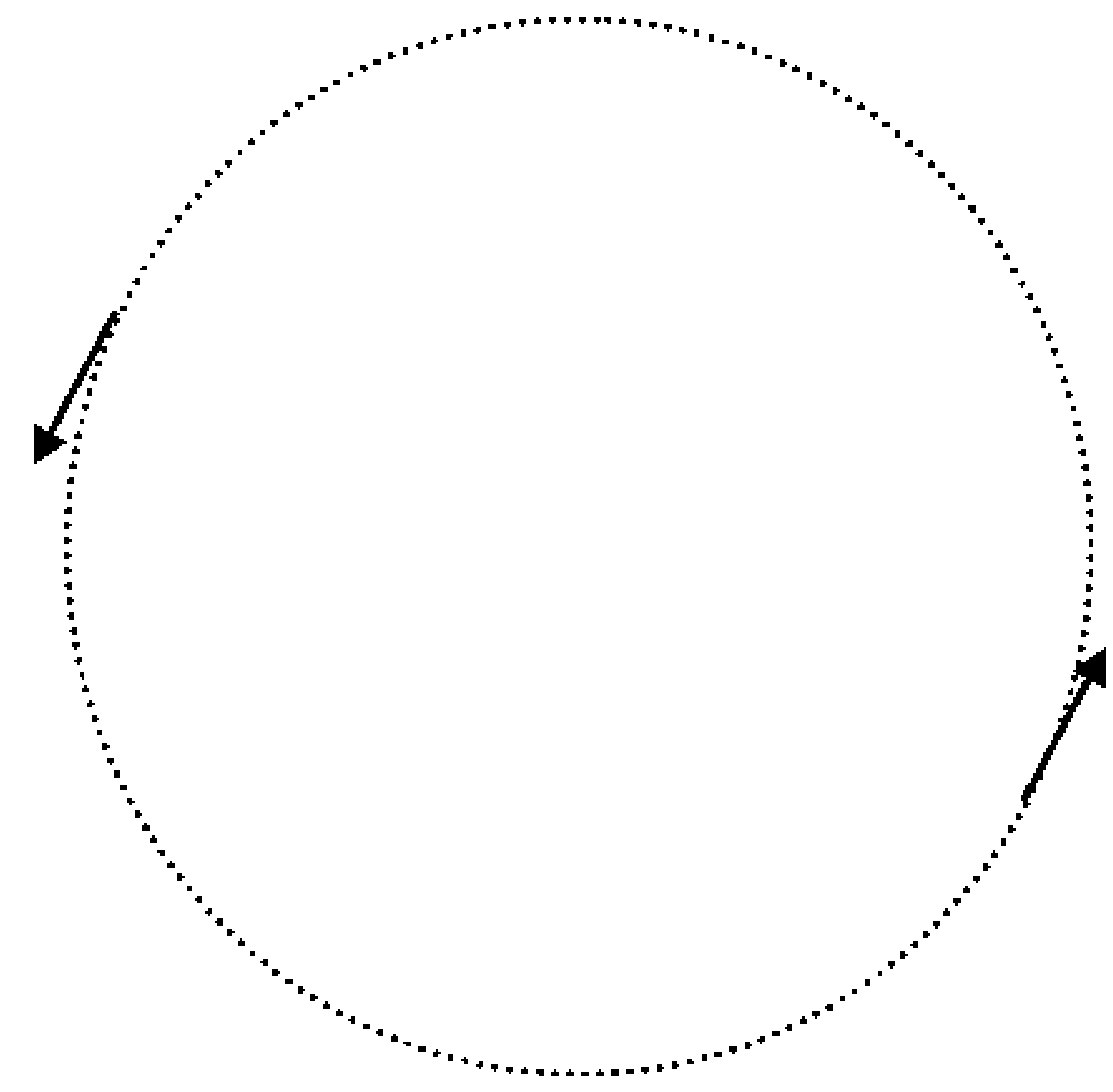
Equilibrium Formations of Two Vehicles



Rectilinear formation (motion perpendicular to the baseline)



Collinear formation



Circling formation (vehicle separation equals the diameter of the orbit)

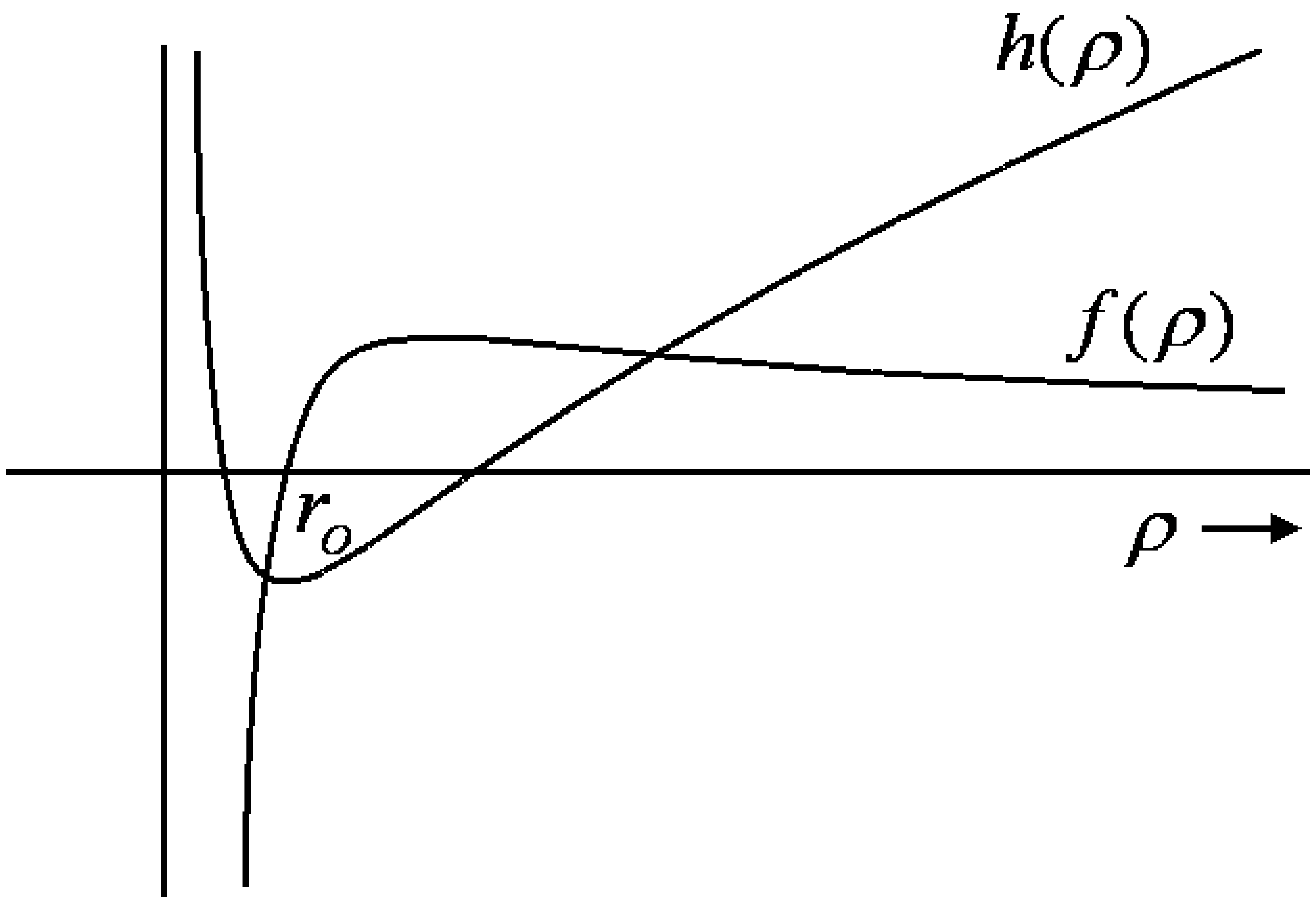
Lyapunov Functions for Two Vehicles

- Rectilinear formation (perpendicular to baseline) or collinear formation:

$$V_{pair} = -\ln(1+\cos(\phi_2-\phi_1)) + h(\rho)$$

$$v_{12}(\rho) - v_{21}(\rho) > |v_{11}(\rho) - v_{22}(\rho)|, \quad \forall \rho > 0$$

$$v_{11}(\rho) + v_{12}(\rho) + v_{21}(\rho) + v_{22}(\rho) \neq 0, \quad \forall \rho > 0$$



Circling formation or collinear formation:

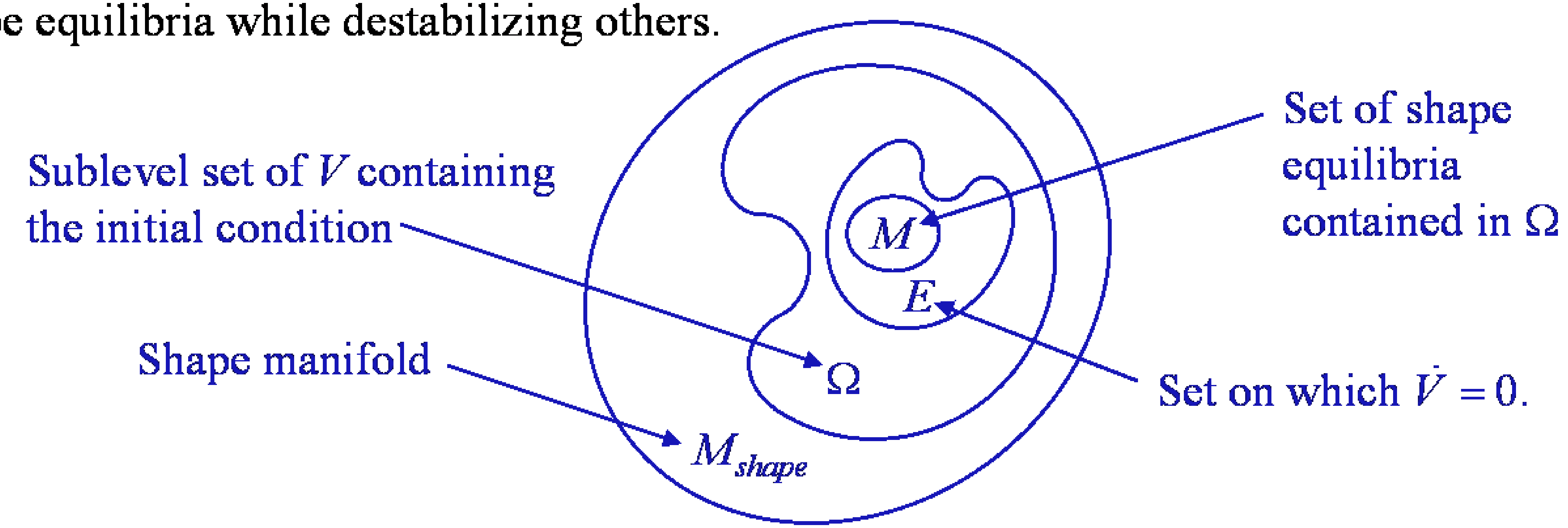
$$V_{circ} = -\ln(1-\cos(\phi_2+\phi_1)) + h(\rho)$$

$$v_{12}(\rho) + v_{21}(\rho) > |v_{11}(\rho) + v_{22}(\rho)|, \quad \forall \rho > 0$$

$$v_{11}(\rho) - v_{12}(\rho) + v_{21}(\rho) - v_{22}(\rho) \neq 0, \quad \forall \rho > 0$$

Global Convergence for Two Vehicles

- We consider the (reduced) dynamics on shape space.
- Conditions on the $\nu_{\mathrm{jk}}(\rho)$ determine whether $V = V_{\mathrm{pair}}$ or $V = V_{\mathrm{circ}}$, and ensure that $\dot{V} \leq 0$.
- For any initial condition such that V is finite, the trajectory remains in a sublevel set Ω of V, which is compact because V is radially unbounded.
- By LaSalle's Invariance Principle, the trajectory converges to the largest invariant set M of the set E of all points in Ω such that $\dot{V} = 0$.
- The invariant set M is the set of shape equilibria contained in Ω .
- Further conditions can be imposed on the $\nu_{\mathrm{jk}}(\rho)$ to (linearly) stabilize particular shape equilibria while destabilizing others.



Formation Control for *n* vehicles

Generalization of the two-vehicle formation control law to *n* vehicles:

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$

$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$

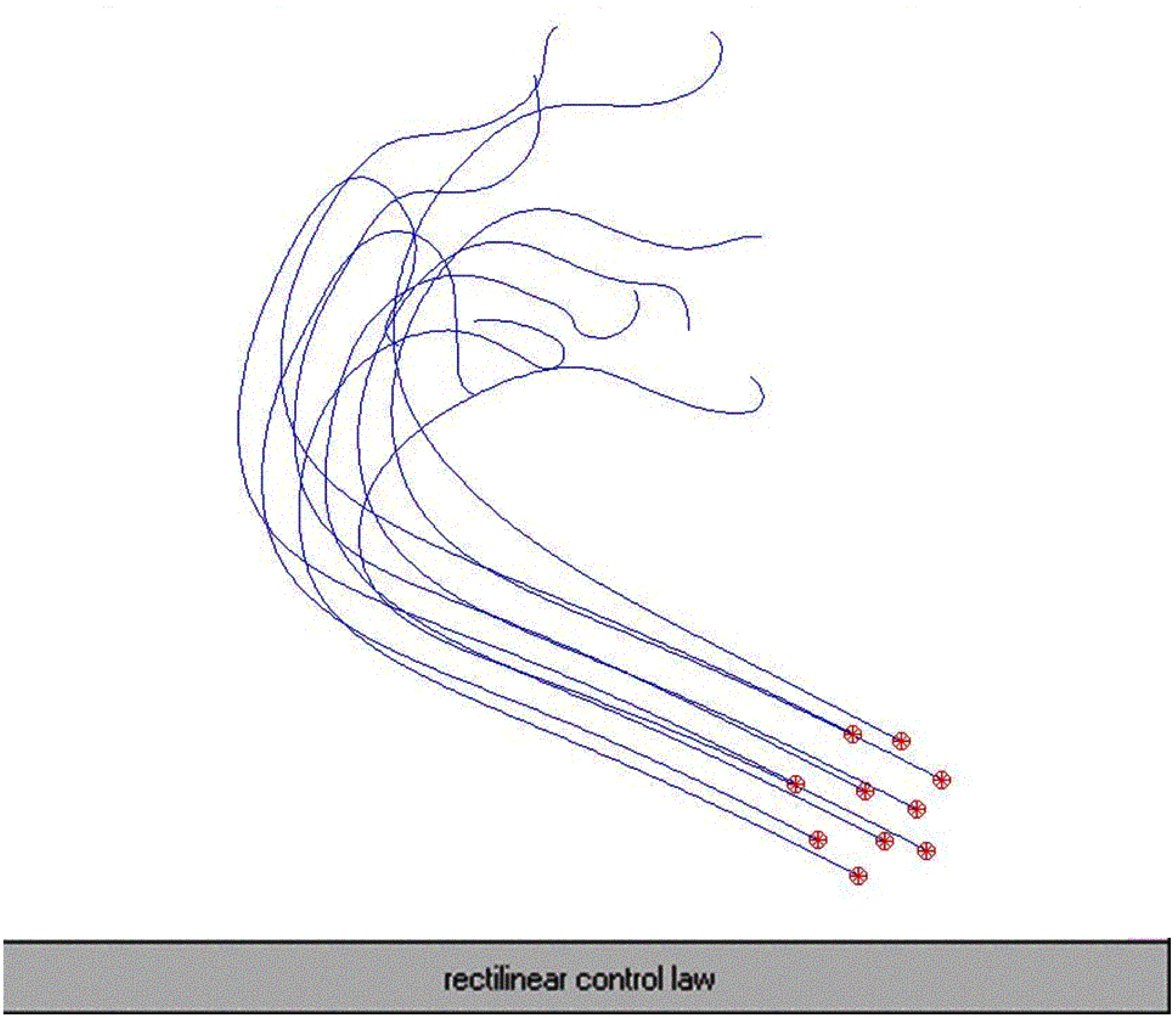
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$

$$u_j = \frac{1}{n} \sum_{k \neq j} \left[F(\mathbf{r}_j - \mathbf{r}_k, \mathbf{x}_j, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_k) - f(|\mathbf{r}_j - \mathbf{r}_k|) \left(\frac{\mathbf{r}_j - \mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|} \cdot \mathbf{y}_j \right) \right]$$

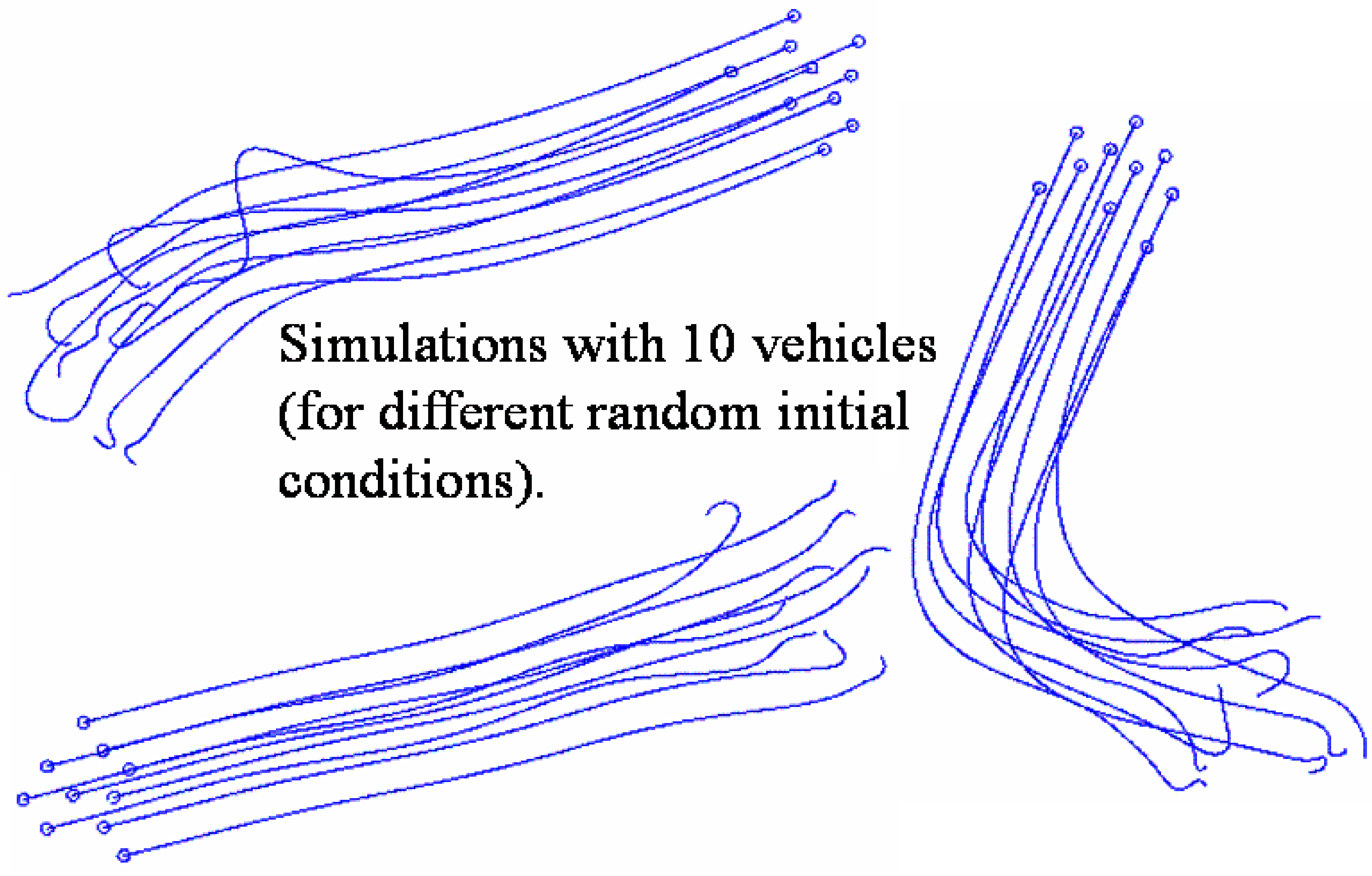
$$j = 1, 2, \ldots, n$$

At present, it is **conjectured** (based on simulation results) that such control laws stabilize certain formations. However, analytical work is ongoing.

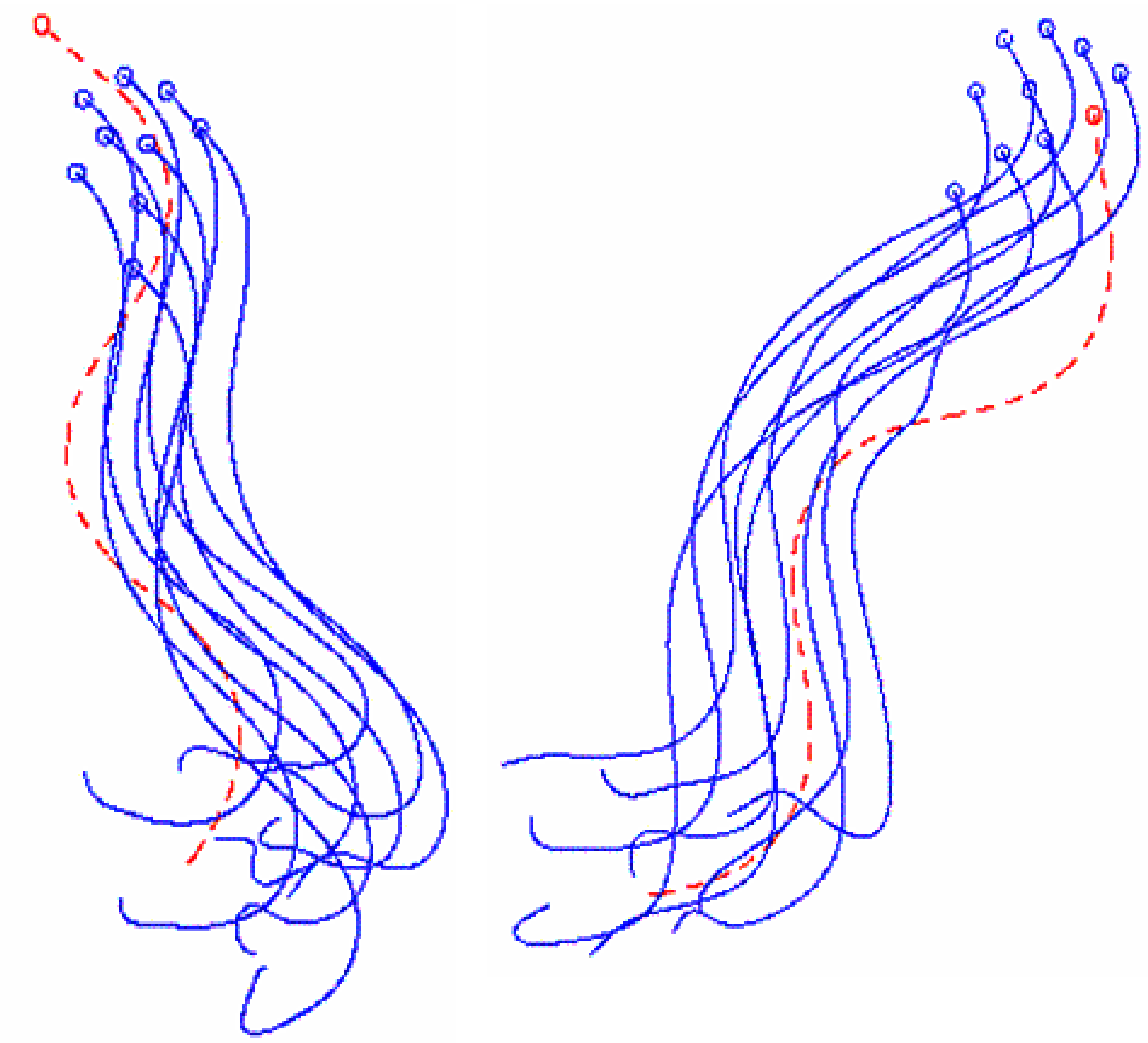
Rectilinear Control Law Simulations



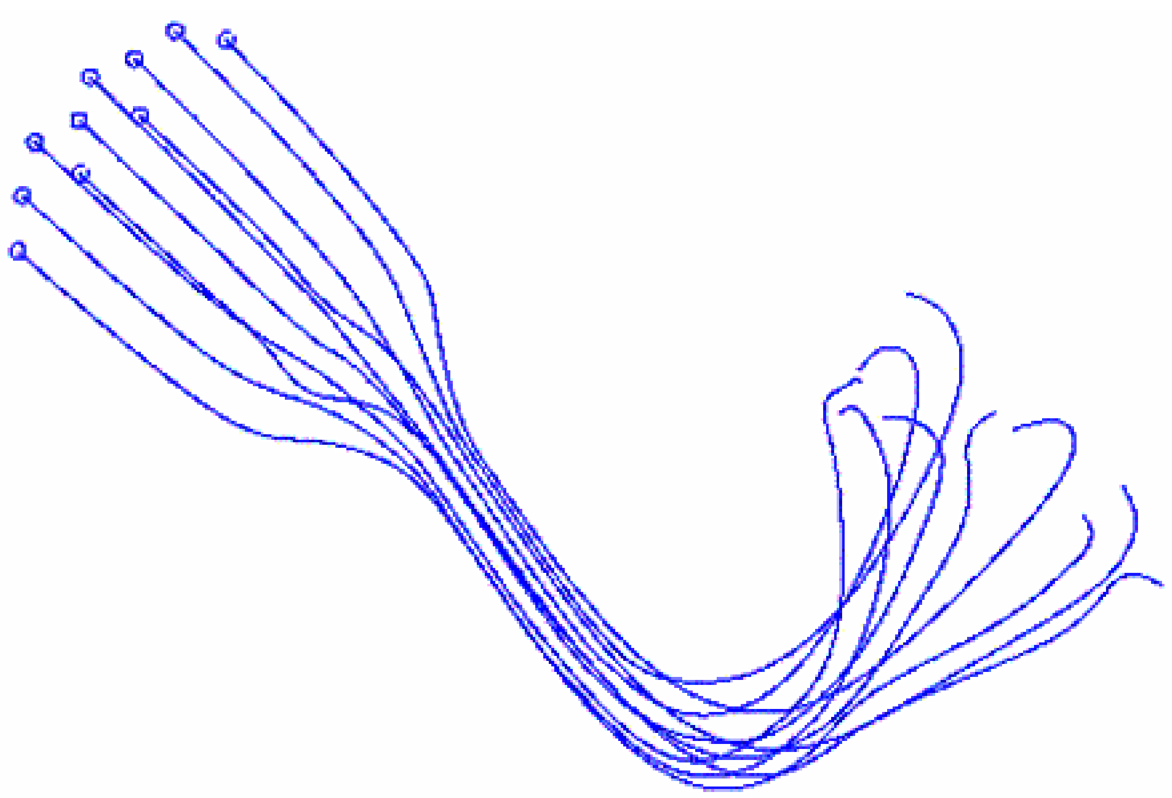
Rectilinear Control Law Simulations

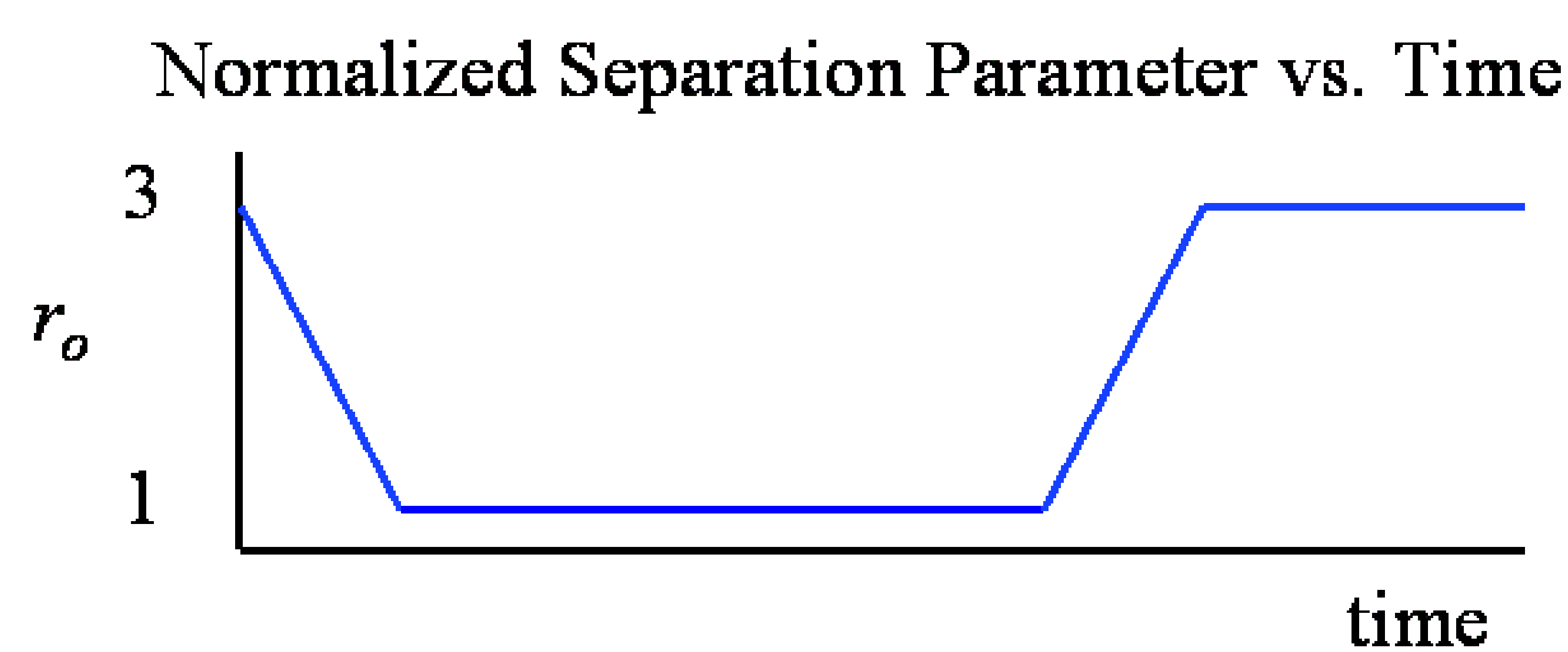


Simulations with 10 vehicles (for different random initial conditions).



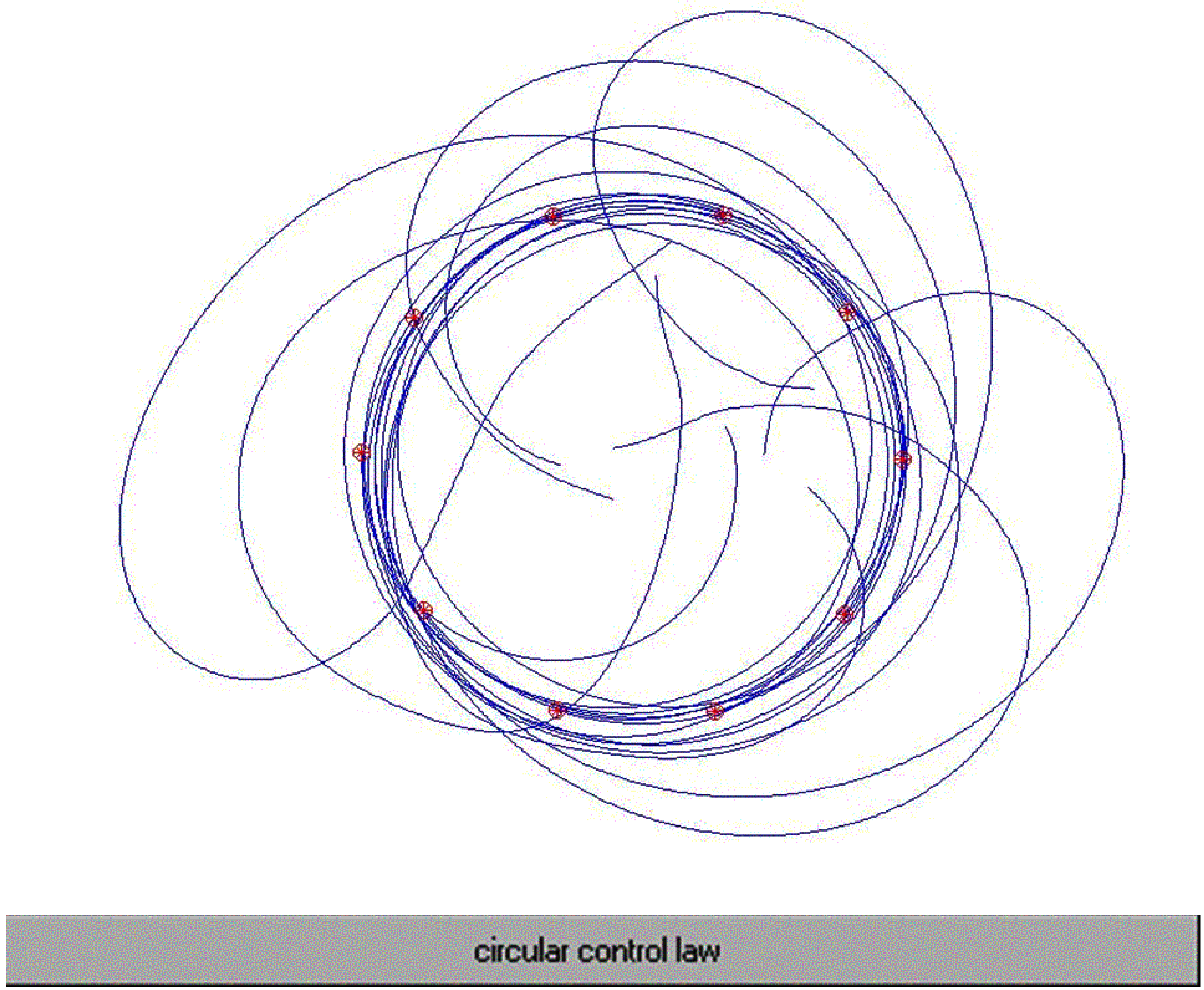
Leader-following behavior: the red vehicle follows a prescribed path (dashed line).





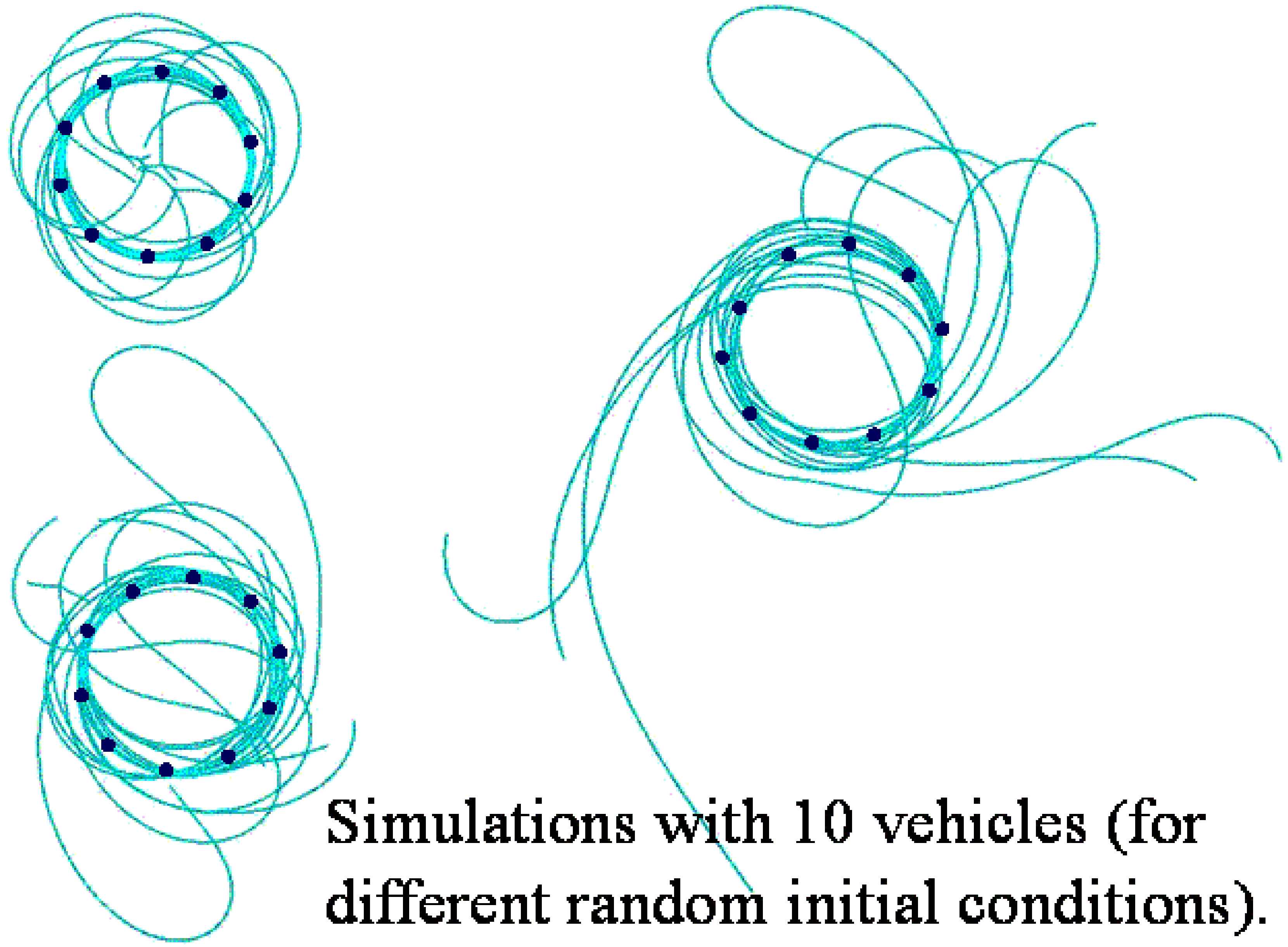
On-the-fly modification of the separation parameter.

Circling Control Law Simulations



Circling Control Law

$$u_j = \frac{1}{n}\sum_{k\neq j}\left[\pm\eta\left(\frac{\mathbf{r}_j-\mathbf{r}_k}{|\mathbf{r}_j-\mathbf{r}_k|}\cdot\mathbf{x}_j\right)+f(|\mathbf{r}_j-\mathbf{r}_k|)\left(\frac{\mathbf{r}_j-\mathbf{r}_k}{|\mathbf{r}_j-\mathbf{r}_k|}\cdot\mathbf{y}_j\right)\right]$$



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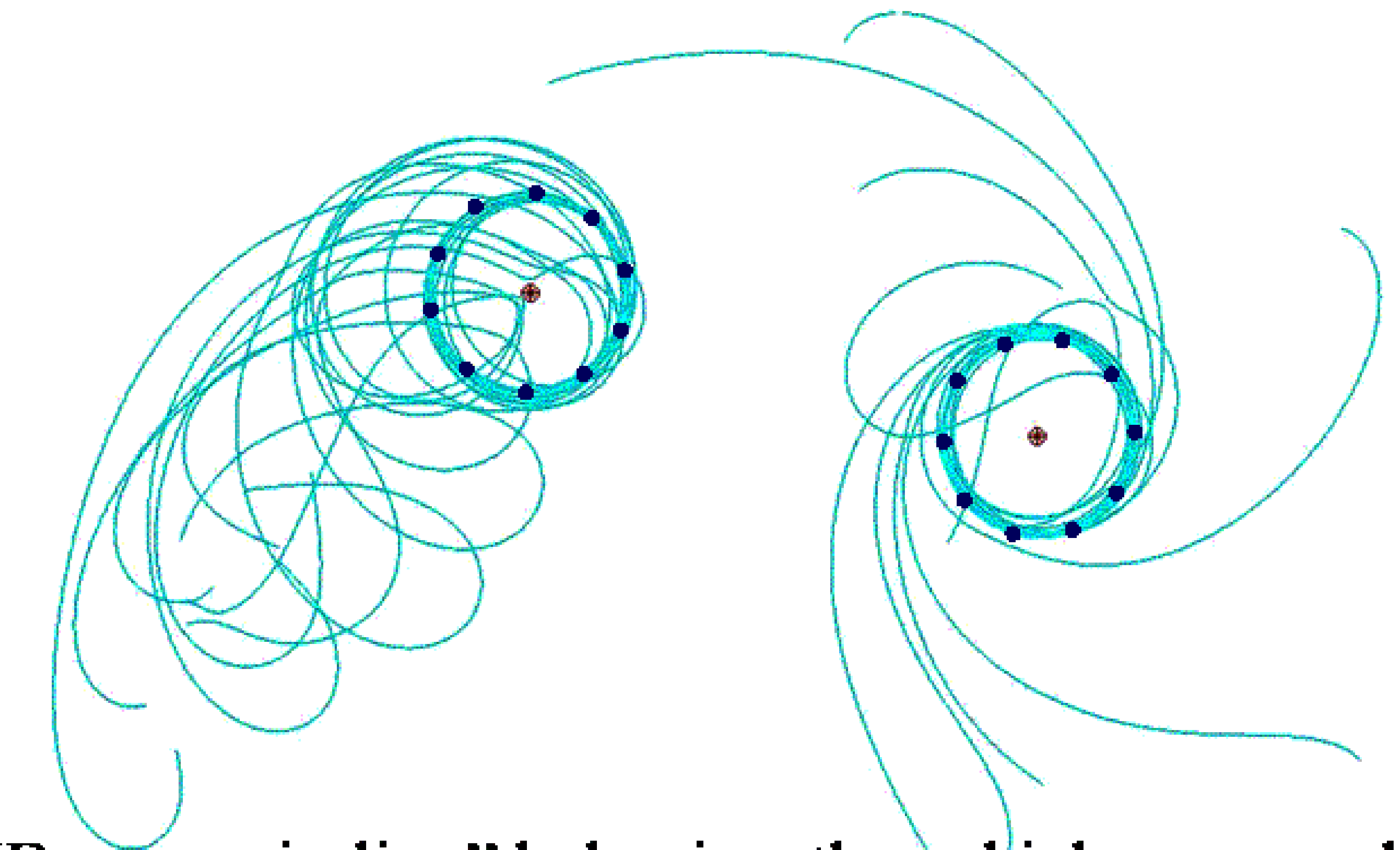
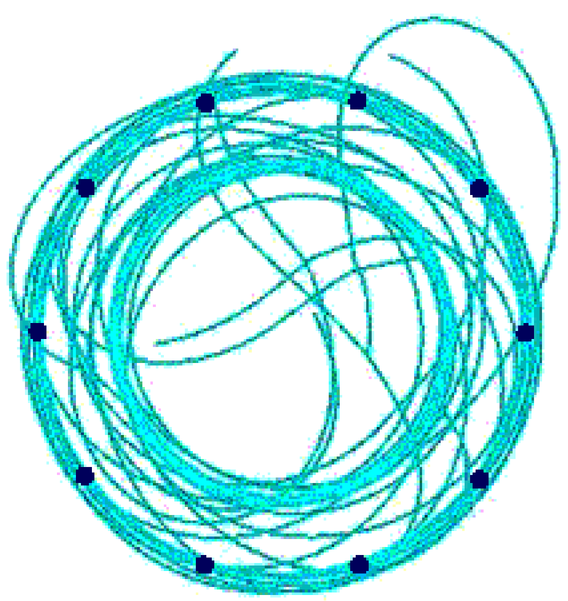
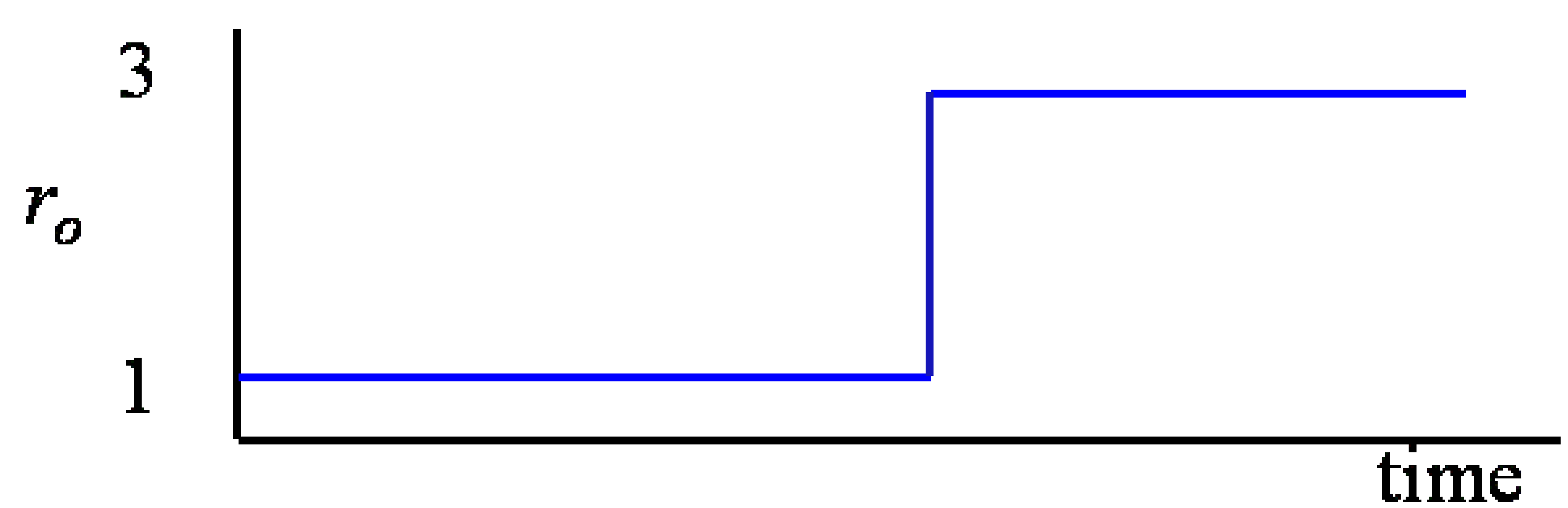


“Beacon-circling” behavior: the vehicles respond to a beacon, as well as to each other.



On-the-fly modification of the separation parameter.

Normalized Separation Parameter vs. Time



Convergence Result for $n > 2$

- We consider rectilinear relative equilibria, and the Lyapunov function

$$V = \sum_{j=1}^{n} \sum_{k<j} \left[-\ln\left(1 + \cos(\theta_j - \theta_k)\right) + h(|\mathbf{r}_j - \mathbf{r}_k|) \right]$$

- **Convergence Result** (Justh, Krishnaprasad): There exists a sublevel set Ω of V and a control law (depending only on shape variables) such that $\dot{V} \leq 0$ on Ω.

- With this Lyapunov function, we cannot prove global convergence for $n > 2$.

- Although we obtain an explicit formula for the controls u_j, $j=1,\ldots,n$, there is no guarantee that this particular choice of controls will result in convergence to a particular desired equilibrium shape in Ω.

Shape Space for n Vehicles

Frenet-Serret Equations

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$
$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$
$$j = 1, 2, \ldots, n.$$

Group variables

$$g_j = \left[\begin{array}{cc|c} \mathbf{x}_j & \mathbf{y}_j & \mathbf{r}_j \\ \hline 0 & 0 & 1 \end{array}\right]$$

$j = 1, 2, \ldots, n.$

$g_1, g_2, \ldots, g_n \in G = SE(2)$, the group of rigid motions in the plane.

Dynamics

$$\dot{g}_j = g_j \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_j \right)$$

$$= g_j \xi_j, \quad \xi_j \in se(2), \quad j = 1, 2, \ldots, n.$$

Configuration space

$$S = \overbrace{G \times G \times \cdots \times G}^{n \text{ copies}}$$

Assume the controls $u_1, u_2, \ldots, u_n$ are functions of shape variables only.

Shape variables capture **relative** vehicle positions and orientations.

Shape variables

$$\tilde{g}_j = g_1^{-1} g_j, \quad j = 2, \ldots, n.$$

Shape space

$$R = \overbrace{G \times G \times \cdots \times G}^{n-1 \text{ copies}}$$

Two-Vehicle Law: Lie Group Setting

- Dynamics on configuration space S=G×G, where G=$SE(2)$:

$$g_1 = \left[\begin{array}{cc|c} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{r}_1 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_1 & -\sin\theta_1 & \mathbf{r}_1 \\ \sin\theta_1 & \cos\theta_1 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_1 = g_1\xi_1 = g_1(A_0 + A_1 u_1).$$

$$\xi_1, \xi_2 \in se(2)$$

$$g_2 = \left[\begin{array}{cc|c} \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{r}_2 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_2 & -\sin\theta_2 & \mathbf{r}_2 \\ \sin\theta_2 & \cos\theta_2 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_2 = g_2\xi_2 = g_2(A_0 + A_1 u_2).$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Shape variable: $g = g_1^{-1} g_2$
- Dynamics on shape space R=G: $\dot{g} = g\xi$,

$$\xi = \xi_2 - g^{-1}\xi_1 g = \xi_2 - Ad_{g^{-1}}\xi_1 \in se(2).$$

- Controls as functions of the shape variable g:

$$u_1(g) = F(r, g_{13}, g_{23}, g^{13}, g^{23}) + f(r)\left(\frac{g_{23}}{r}\right), \quad g = [g_{ij}], \quad g^{-1} = [g^{ij}], \quad r = \sqrt{g_{13}^2 + g_{23}^2},$$

$$u_2(g) = F(r, g^{13}, g^{23}, g_{13}, g_{23}) + f(r)\left(\frac{g^{23}}{r}\right).$$

Lyapunov Function:

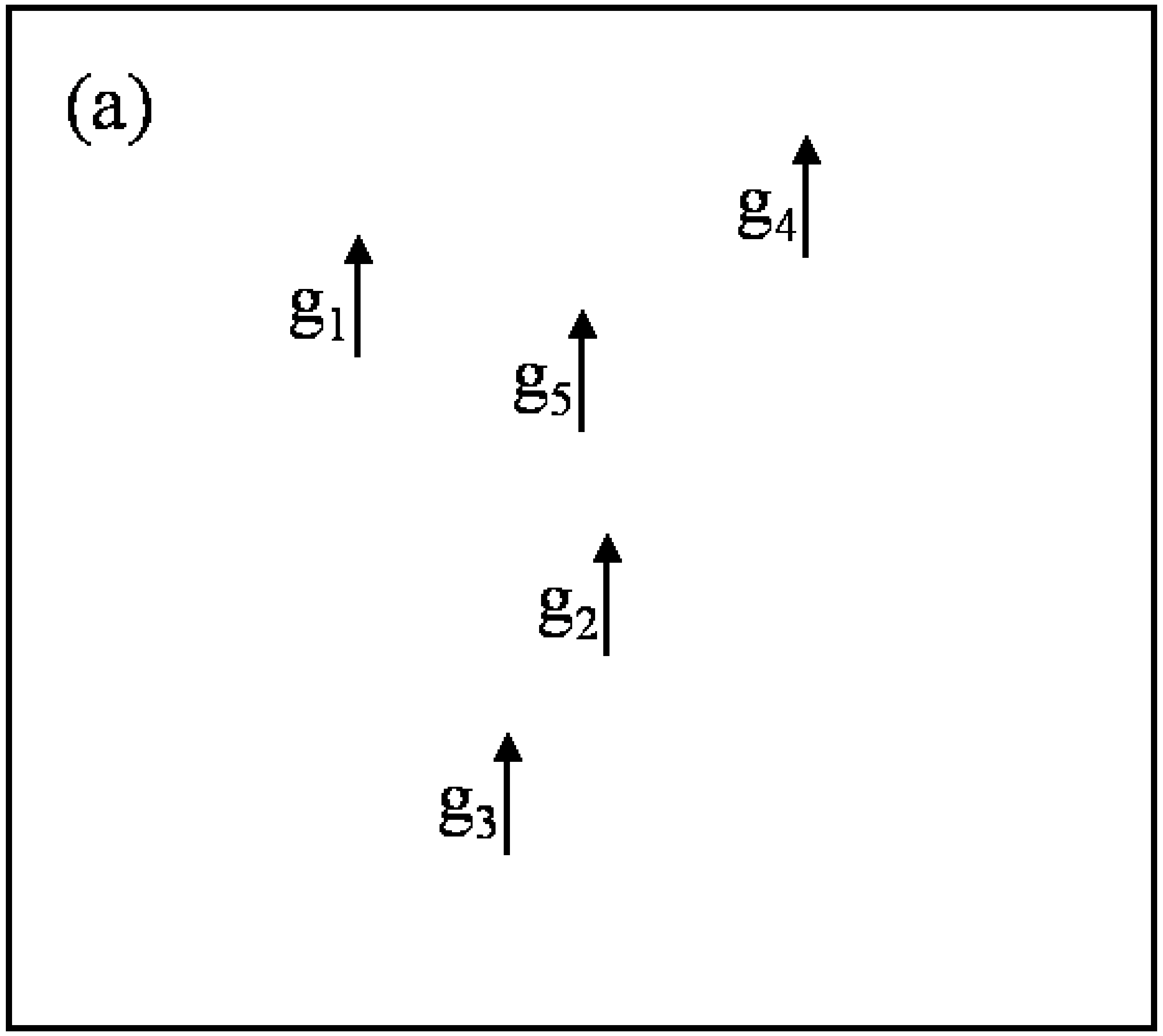
$$V_{pair} = V_{pair}(g).$$

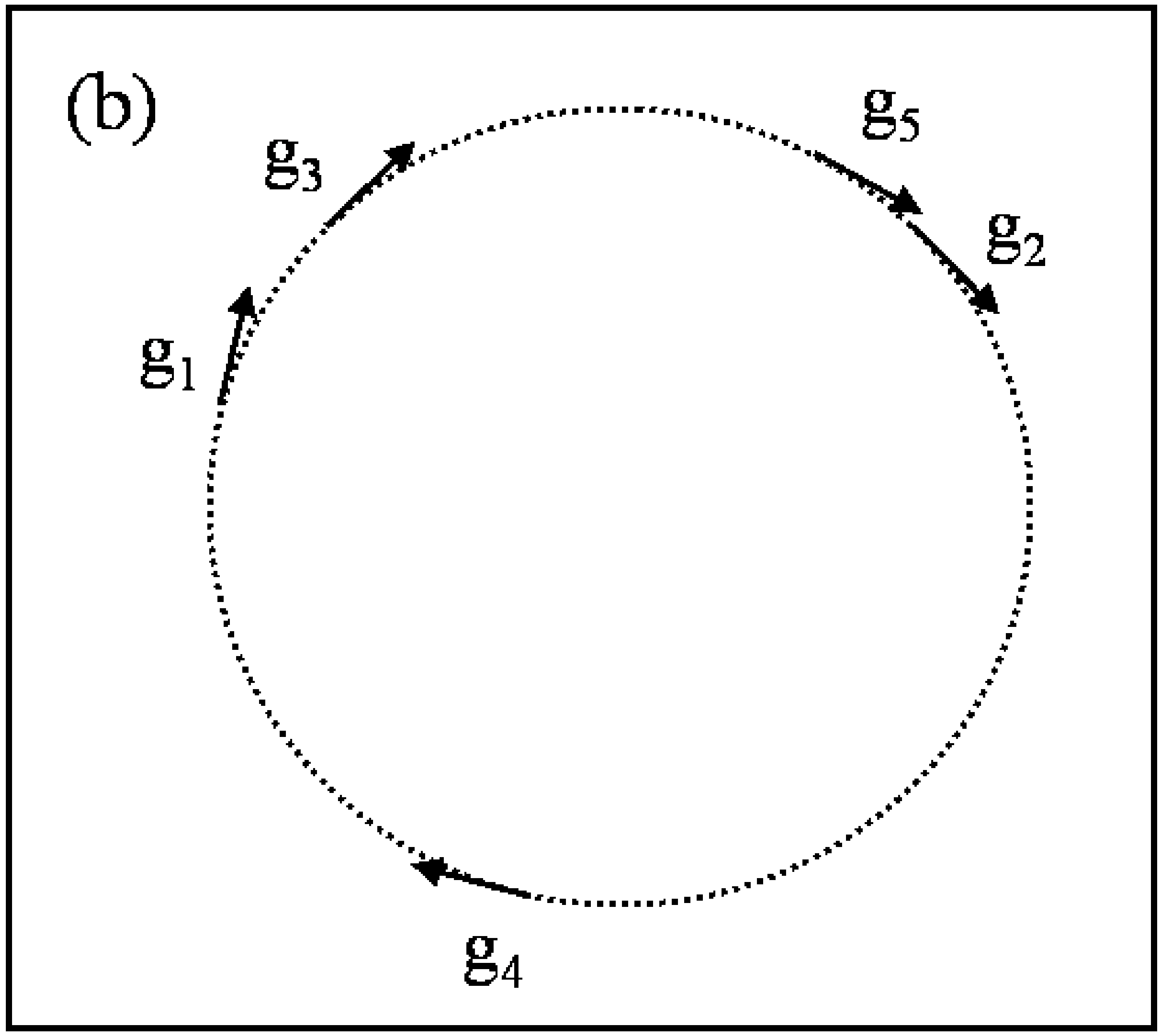
Characterization of Equilibrium Shapes

Proposition (Justh, Krishnaprasad): For equilibrium shapes (i.e., relative equilibria of the dynamics on configuration space), $u_1 = u_2 = \ldots = u_n$, and there are only two possibilities:

(a) $u_1 = u_2 = \ldots = u_n = 0$: all vehicles head in the same direction (with arbitrary relative positions), or

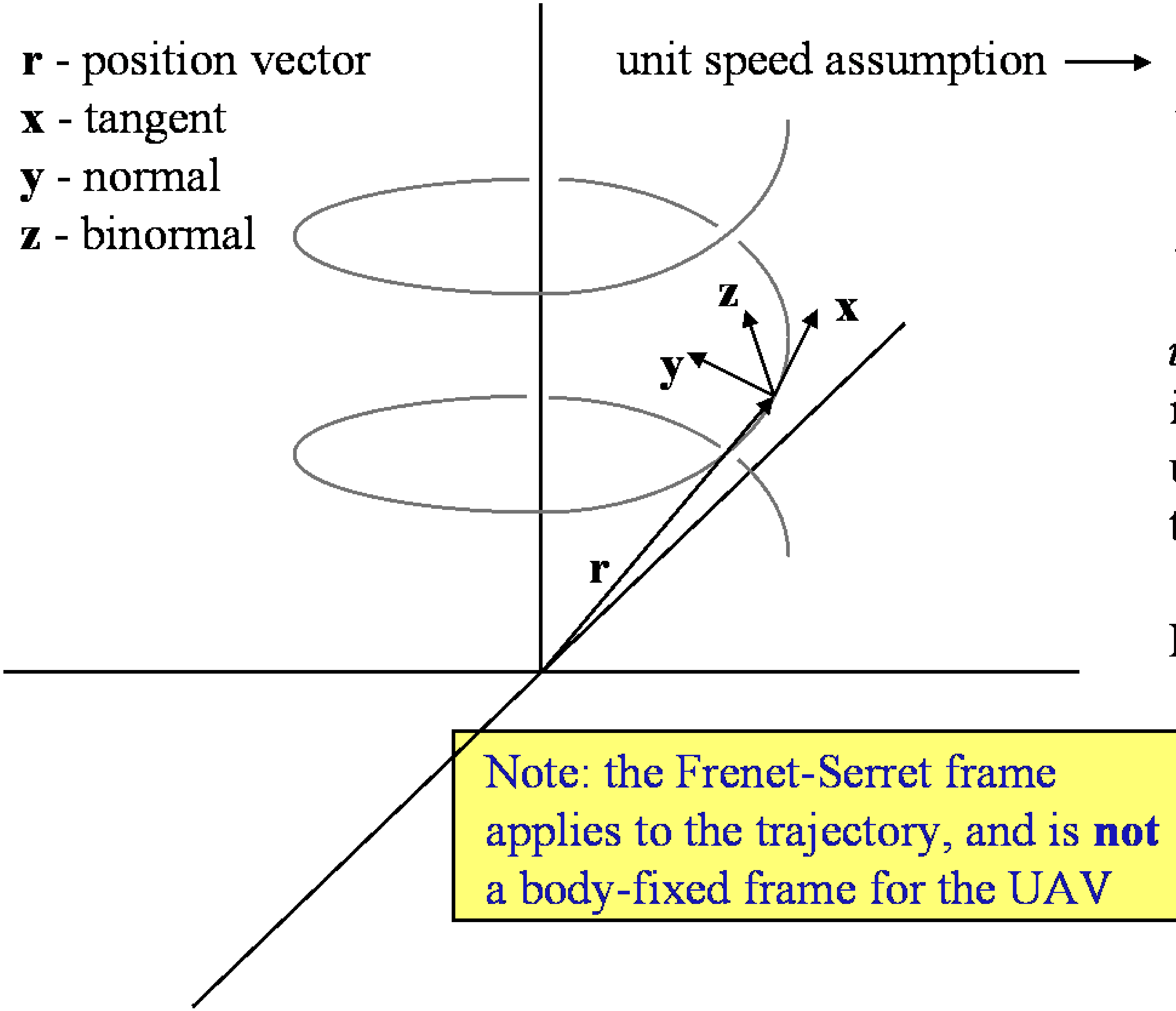
(b) $u_1 = u_2 = \ldots = u_n \neq 0$: all vehicles move on the same circular orbit (with arbitrary chordal distances between them).





3-Dimensional Frenet-Serret Equations

r - position vector
x - tangent
y - normal
z - binormal



unit speed assumption → $\dot{\mathbf{r}} = \mathbf{x}$

$$\dot{\mathbf{x}} = \mathbf{y}u - \mathbf{z}v$$
$$\dot{\mathbf{y}} = -\mathbf{x}u + \mathbf{z}w$$
$$\dot{\mathbf{z}} = \mathbf{x}v - \mathbf{y}w$$

u, v, w are control inputs (two of which uniquely specify the trajectory)

Frenet-Serret:

$v = 0$

u = curvature

w = torsion

Note: the Frenet-Serret frame applies to the trajectory, and is **not** a body-fixed frame for the UAV

Continuum Model

- Vector field (in polar coordinates):

$$\begin{pmatrix} d\mathbf{r}/dt \\ \\ d\theta/dt \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ u \end{pmatrix}.$$

- Continuity equation (Liouville equation):

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial (u\rho)}{\partial \theta} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \nabla_{\mathbf{r}} \rho \right].$$

- Conservation of matter:

$$\int_G \rho(t, \mathbf{r}, \theta) d\mathbf{r} d\theta = 1, \quad \forall t.$$

- Energy functional:

$$V_c(t) = \frac{1}{2} \int_G \int_G \left[-\ln\left(1 + \cos(\theta - \tilde{\theta})\right) + h(|\mathbf{r} - \tilde{\mathbf{r}}|) \right] \rho(t, \mathbf{r}, \theta) \rho(t, \tilde{\mathbf{r}}, \tilde{\theta}) d\mathbf{r} d\theta d\tilde{\mathbf{r}} d\tilde{\theta}.$$

- This continuum formulation only involves two scalar fields: the density $\rho(t, \mathbf{r}, \theta)$ and the steering control $u(t, \mathbf{r}, \theta)$.
- However, the underlying space is 3-dimensional (for planar formations).
- Incorporating time and/or spatial derivatives in the equation for u yields a coupled system of PDEs for ρ and u.

References

E. W. Justh and P. S. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research Technical Report TR 2002-38, 2002 (see http://www.isr.umd.edu). (Abridged to appear in *Systems and Control Letters*.)

E. W. Justh and P. S. Krishnaprasad, "Steering laws and continuum models for planar formations," *Proc. 42nd IEEE Conference on Decision and Control*, to appear, December 2003.

E. W. Justh and P. S. Krishnaprasad, "Steering laws and convergence for planar formations", to appear in, V. Kumar, N. E. Leonard and A. S. Morse (eds.), *Proceedings of the Block Island Workshop on Cooperative Control*, Springer-Verlag, New York, 2003.

See http://www.isr.umd.edu/~justh for downloadable preprints and movies.