Sensitivity Analysis for the Problem of Matrix Joint Diagonalization

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Outline

- Introduction and the problem of Exact Joint Diagonalization (EJD)
- A motivating example
- (Approximate) Joint Diagonalization and its cost functions
- A Differential Equation on the $\text{GL}(n)$ Group for Joint Diagonalization
- Sensitivity Analysis
Consider the set \( \{C_i\}_{i=1}^{N} \) of \( N \) symmetric \( n \times n \) matrices which are of this form:

\[
C_i = A\Lambda_i A^T, \quad 1 \leq i \leq N
\]

where \( A \) is a non-singular \( n \times n \) matrix and \( \Lambda_i \) are diagonal. Denote \( j^{th} \) diagonal element of \( \Lambda_i \) by \( \lambda_{ij} \).

Our goal is to find \( A \) just by observing \( C_i \)'s.

Problem: assume that we observe the set \( \{C_i\}_{i=1}^{N} \) and we only know that \( \Lambda_i \)'s are diagonal, under what conditions can we find \( A \) based only on \( C_i \)'s.

Only based on diagonally of \( \Lambda_i \)'s can not find \( A \) beyond permutation and diagonal scaling factors:

\[
C_i = A\Lambda_i A^T = A\Lambda_i^{-1} \Pi T \Pi \Lambda_i \Lambda_i \Lambda_i^T (A\Lambda_i^{-1} \Pi T)^T = \hat{A}\hat{\Lambda}_i \hat{A}^T
\]

where \( \Lambda \) is a non-singular diagonal matrix and \( \Pi \) is a permutation matrix.
Let permutation and scaling be the only indeterminacies, then by finding a non-singular matrix \( B \) such that all \( BC_iB^T \), \( 1 \leq i \leq N \) are diagonal we can essentially find \( A \), i.e. \( A = B^{-1} \) up to scaling and permutation.

This means that restoring diagonality results in identification.

Exact Joint Diagonalization (EJD): Find a non-singular matrix \( B \) such that all \( BC_iB^T \) are diagonal. We say that EJD has a unique solution if scaling and permutation are the only ambiguities in the solution.

How many matrices are needed? Let's count

We are looking for \( A_{n \times n} \) only up to a permutation and scaling by diagonal matrices. So we can assume that the diagonal elements of \( A \) are 1. So we have \( n^2 - n \) unknowns.

A symmetric \( n \times n \) matrix has \( \frac{n(n+1)}{2} \) and a diagonal \( n \times n \) matrix has \( n \) degrees of freedom.

\[
C_i = \begin{bmatrix} A \Lambda_i \end{bmatrix} A^T, \quad 1 \leq i \leq N
\]

\[
\frac{Nn(n+1)}{2} = n^2 - n + nN \Rightarrow N = 2
\]
How many matrices are needed?...

If no degeneracy happens we need \( N = 2 \) matrices independent of \( n \). Degeneracy means that in the matrix:

\[
L = \begin{bmatrix}
\lambda_{11} & \ldots & \lambda_{1n} \\
\lambda_{21} & \ldots & \lambda_{2n} \\
\ldots & \ldots & \ldots \\
\lambda_{N1} & \ldots & \lambda_{Nn}
\end{bmatrix}
\]

at least two columns are multiple of each other. If \( N = 1 \) that would be the case.

Theorem: EJD has a unique solution if and only if no two columns of \( L \) are co-linear.

Sketch of the proof: if \( C_1 \) is non-singular then:

\[
C_iC_1^{-1} = AA_i(A^{-1})^T \Lambda_1^{-1} A^{-1} = AA_i \Lambda_1^{-1} A^{-1}, \quad 2 \leq i \leq N
\]

From these (the eigen decompositions of \( C_iC_1^{-1} \)) \( A \) cannot be found uniquely if and only if there are two columns \( j \) and \( k \) such that \( \frac{\lambda_{ij}}{\lambda_{ik}} = \frac{\lambda_{1j}}{\lambda_{1k}}, \forall 1 < i \leq N \). Which is equivalent to the conditions of the theorem. If all \( C_i \)'s are singular a modified argument works.

Let us measure the co-linearity of the columns of \( L \) by modulus of uniqueness \( |\rho| \) (with \( |\rho| \leq 1 \), and \( |\rho| < 1 \Leftrightarrow \) uniqueness):

\[
|\rho| = \max_{k,l} \left| \frac{\sum_{i=1}^{N} \lambda_{ik} \lambda_{il}}{(\sum_{i=1}^{N} \lambda_{il}^2)^{1/2} (\sum_{i=1}^{N} \lambda_{ik}^2)^{1/2}} \right|, \quad 1 \leq k \neq l \leq N
\]
A motivating example

Let $\bar{s}(t)$ be an unknown $n$-dimensional vector random non-stationary process. Such that each component of the vector $\bar{s}$ are uncorrelated process of zero mean, i.e. its correlation matrix is diagonal through time. Suppose this process is mixed by a matrix $A_{n \times n}$ and we observe the mixture:

$$\bar{x}(t) = A\bar{s}(t)$$

Then for the correlation matrix of this process we have that:

$$C(t) = A\Lambda_s(t)A^T$$

where $\Lambda_s(t)$ is the diagonal correlation of $\bar{s}(t)$ which is unknown.

We can sample the correlation matrix of the observed mixture and if we can find instances of times $t_1, \ldots, t_N$ for which $\{C(t_i)\}_{i=1}^N$ the EJD has a unique solution, then we can find $A$ uniquely.

The non-stationarity assumption is crucial, otherwise $C(t)$ would remain constant and we only have one matrix.

Example: $n$ sources in your brain emitting signals and $n$ sensors on your scalp collect the mixture signals from the sources in the brain. Can we recover the source signals without access to the mixing process?
Joint Diagonalization

(Approximate Joint Diagonalization (JD)) In practice we only have:

\[ C_i \approx A\Lambda_i A^T \]  

hence we approximately jointly diagonalize all of them, i.e. find \( B \) such that all \( BC_i B^T \)s are “as diagonal as possible”.

We hope that after this \( B \approx A^{-1} \). If little error in \( C_i \) result in large deviation in \( B \) from \( A^{-1} \) then the problem is sensitive. The question is what factors make the problem sensitive?

Sensitivity is closely related to uniqueness as measured by \( |\rho| \). In fact if \( |\rho| = 1 \), then sensitivity is infinity, since even in the case no error in the model there are non-unique solutions different from \( A^{-1} \).
Cost functions for Joint Diagonalization

Joint diagonalization as minimization of a cost function defined on a suitable Lie group.

Historically JD was first considered for a case where $A$ (or $B$) are assumed to be orthogonal.

Orthogonal Joint Diagonalization

Orthogonal JD: If assume that $A$ is orthogonal then we have $\frac{n(n-1)}{2}$ unknowns and we need $N = 1$ matrix (say $C_1$) if $C_1$ has distinct eigen values. $C_1 = A \Lambda_1 A^T$ is the eigen decomposition of $C_1$. To combat noise and consider the approximate model we look for $\Theta \in SO(n)$ that minimizes:

$$J_1(\Theta) = \sum_{i=1}^{n} \left\| \Theta C_i \Theta^T - \text{diag}(\Theta C_i \Theta^T) \right\|_F^2$$

where $\text{diag}(X)$ and $\|X\|_F$ are the diagonal part the Frobenius norm of matrix $X$.

$SO(n)$ is a compact manifold so the problem of minimization of $J_1$ over $SO(n)$ is a well defined problem.
An extension of $J_1$ from $\text{SO}(n)$ to bigger group of non-singular matrices $\text{GL}(n)$ is not suitable, since it is not scale invariant:

$$J_1(\Lambda B) \to 0 \text{ as } \|\Lambda\| \to 0, \quad \Lambda = \text{diagonal}$$

Reminder: EJD was a scale-invariant problem, i.e. if $B$ is joint diagonalizer $\Lambda B$ is also another joint diagonalizer.

Equip $\text{GL}(n)$ with a Riemannian metric $\langle \cdot, \cdot \rangle_B: T_B \text{GL}(n) \times T_B \text{GL}(n) \to \mathbb{R}$:

$$\langle \xi_1, \xi_2 \rangle_B = \text{tr}(((\xi_1 B^{-1})^T \xi_2 B^{-1}), \quad \xi_1, \xi_2 \in T_B \text{GL}(n)$$

Reminder: every tangent vector at $B$ is of the form $\xi = \Delta B$, where $\Delta \in \text{gl}(n)$.

The gradient of $J_1 : \text{GL}(n) \to \mathbb{R}$:

$$\nabla J_1(B) = \Delta B = \sum_{i=1}^{N} \left( BC_i B^T - \text{diag}(BC_i B^T) \right) BC_i B^T B$$
We want to reduce $J_1(B)$ without going along directions that correspond to reduction by diagonal matrices.

At each point $B$ the orbit of the action of the group of non-singular matrices is a manifold and the tangent space to this manifold is $T_B \mathcal{O}_B \subset T_B GL(n)$.

A flow for JD:

\[
\dot{B} = -\nabla J_1^\perp(B) = -\Delta^\perp B = -(\Delta - \text{diag}(\Delta))B
\]  

(14)

Define the joint diagonalizer of $\{C_i\}_{i=1}^N$ as the stationary points of this flow:

\[
\left(\sum_{i=1}^N \left(BC_iB^T\right)^\perp BC_iB^T\right)^\perp = 0
\]  

(15)
Let us model the noise or error in $C_i$ as:

\[
C_i(t) = A \Lambda_i A^T + t N_i, \quad 1 \leq i \leq N, \quad t \in [-\delta, \delta]
\]  

where $t$ measures the contribution of noise and $N_i$'s are symmetric noise or error matrices.

Now as $t$ changes $B(t)$ satisfies (16) and if $\delta$ is small (from the Implicit Function Theorem) we have that:

\[
B(t) = (I + t \Delta) A^{-1} + o(t)
\]

where $I$ is the $n \times n$ identity and $\Delta$ is a matrix with zero diagonal and $\frac{\|o(t)\|}{t} \to 0$ as $t \to 0$.

$\|\Delta\|$ measures the sensitivity to noise.
Theorem 1  Let \( C_i = A \Lambda_i A^T + t N_i, 1 \leq i \leq N \) \((t \in [-\delta, \delta])\). Let us define \( B(t) \) the non-orthogonal joint diagonalizer for \( \{C_i\}_{i=1}^{N} \) as (16). Then for small enough \( \delta \) the joint diagonalizer can be written as: \( B(t) = (I + t \Delta) A^{-1} + o(t) \) where \( \Delta \) (with \( \text{diag}(\Delta) = 0 \)) satisfies

\[
\|\Delta\|_F < \frac{\alpha}{(1 - \rho^2)} \|T\|_F
\]

with

\[
T = - \sum_{i=1}^{N} (A^{-1} N_i (A^{-1})^T) \perp \Lambda_i
\]

and

\[
\gamma_{kl} = \left( \sum_{i=1}^{N} \lambda_{ik}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N} \lambda_{il}^2 \right)^{\frac{1}{2}}, \quad \eta_{kl} = \frac{\left( \sum_{i=1}^{N} \lambda_{ik}^2 \right)^{\frac{1}{2}}}{\left( \sum_{i=1}^{N} \lambda_{il}^2 \right)^{\frac{1}{2}}}
\]

and

\[
\alpha = \max_{k \neq l} \frac{\eta_{kl} + \frac{1}{\eta_{kl}}}{\gamma_{kl}}
\]

The derivation of this bound shows that this is not a loose bound

Corollary: Modulus of uniqueness and \( \|A^{-1}\| \) (and consequently the condition number of \( A \)) affect the sensitivity of JD.
On Number of Matrices

- $|\rho|$ is the cosine of the smallest angle between $n$ points in the real projective plane $\mathbb{RP}^N$. These vectors are columns of the matrix $L$ formed from $\Lambda_i$'s.

- If $N$ is small this angle can be very small.

- For $N = 2$, we can think of putting $n$ points on the unit semi-circle. Which means that $\rho \geq \cos \frac{\pi}{n-1}$. Or if $\lambda_{ij}$ are non-negative then $\rho \geq \cos \frac{\pi}{2(n-1)}$. If $n$ is large these bounds can be close to 1 which makes the JD problem very sensitive.

- How to find a similar bound for $N \geq 3$?(I haven’t done it!)

- Also if $N$ is small we will not see much cancelation of the effect of noise in $\mathcal{T}$ (averaging effect) and that also contributes to more sensitivity.

- These results can be confirmed numerically.

Conclusion

- Introduced EJD and JD and the motivation for these problems

- Defined the joint diagonalizer as the stationary points of a flow on the group $\text{GL}(n)$

- Defined the sensitivity of the problem and the factors that affect it: especially the modulus of uniqueness and the condition of $A$