Why Be Backward?
Forward Evolution Equations for Barrier Options

Ali Hirsa
Morgan Stanley

with Peter Carr of the Courant Institute, NYU
We thank

Participants of the 2003 Risk Paris conference

Participants of the 2003 Cornell Theory Center conference seminar

Bob Kohn and Pedro Judice of Courant

Students at the Mathematics of Finance Practitioner Seminar, Spring 2003, at Columbia University

for their suggestions and comments.
A European (vanilla) call option is an option that gives the holder the right to buy stock at a specified price (the strike price) at the expiration date (maturity).

A knock-out call option is the same as vanilla if it has not knocked out (the stock has not reached a specified level, barrier, throughout the life of the option.)

In the case of knocking from below, it is called Up-and-Out.

In the case of knocking from above, it is called Down-and-Out.
Motivation

Propagating option prices in the maturity and/or strike directions

Enhances computational efficiency of calibration

Promotes computational efficiency in marking
Outline

Review of the Dupire PDE

Markovian Stock Price Process

Backward PIDE for **Up-and-Out** and **Down-and-Out** Calls

Forward PIDE for **Down-and-Out** Calls

Forward PIDE for **Up-and-Out** Calls
Outline (Cont’d)

Numerical Examples

Future Work
Backward PDE for European Calls

Assuming that the stock price process follows the following

\[ ds_t = [r(t) - q(t)] s_t dt + \sigma(s_t, t) s_t dW_t \]

Applying Ito’s Lemma to show that \( c(S, t) \) solves:

\[
\frac{\partial c(S, t)}{\partial t} + \frac{\sigma^2(S, t)}{2} S^2 \frac{\partial^2 c(S, t)}{\partial S^2} + [r(t) - q(t)] S \frac{\partial c(S, t)}{\partial S} = r(t) c(S, t)
\]

\[ c(S, T_0) = (S - K_0)^+, \quad S \in [0, \infty) \]

\[
\lim_{S \downarrow 0} \frac{\partial^2}{\partial S^2} c(S, t) = 0, \quad t \in [0, T_0]
\]

\[
\lim_{S \uparrow \infty} \frac{\partial^2}{\partial S^2} c(S, t) = 0, \quad t \in [0, T_0].
\]
As a starting point, we look at the Dupire PDE

$$\frac{\partial c}{\partial T} = \frac{\sigma^2(K, T)}{2} K^2 \frac{\partial^2 c}{\partial K^2} - [r(T) - q(T)] K \frac{\partial c}{\partial K} - q(T)c(K, T)$$

By having the market call prices, $c(K, T)$, the local volatility surface, $\sigma(K, T)$, can be calculated.

Or by having the local volatility surface, $\sigma(K, T)$, one can compute call prices, $c(K, T)$, for all strikes and maturities subject to:

$$c(K, 0) = (S_0 - K)^+, \quad K \in [0, \infty)$$

$$\lim_{K \downarrow 0} \frac{\partial^2}{\partial K^2} c(K, T) = 0, \quad T \in [0, \bar{T}],$$

$$\lim_{K \uparrow \infty} \frac{\partial^2}{\partial K^2} c(K, T) = 0, \quad T \in [0, \bar{T}].$$
Markovian Stock Price Process

We assume that under a risk neutral measure $\mathbb{Q}$, the stock price $s_t$ satisfies the following stochastic differential equation:

$$
\text{d}s_t = [r(t) - q(t)] s_t \text{d}t + \sigma(s_t, t) s_t \text{d}W_t + \int_{-\infty}^{\infty} s_t (e^x - 1) \left[ \mu(dx, dt) - \nu(x, t) dx dt \right],
$$

for all $t \in [0, \bar{T}]$. Thus, the change in the stock price decomposes into three parts: The risk-neutral drift, the diffusion part, the jump part.

The random measure $\mu(dx, dt)$ counts the number of jumps of size $x$ in the log price at time $t$. 
The Lévy density \{\nu(x, t), x \in \mathbb{R}, t \in [0, \bar{T}]\} is used to compensate the jump process

\[ J_t \equiv \int_0^t \int_{-\infty}^{\infty} s_t (e^x - 1) \mu(dx, dt) \]

so that the last term is the increment of a \( \mathbb{Q} \) jump martingale. Thus

\[ \mathbb{E}_{\mathbb{Q}}[s_t | s_0] = s_0 e^{\int_0^t [r(u) - q(u)] du} . \]
Backward PIDE for European Calls

\[
\frac{a^2(S,t)}{2} \frac{\partial^2 c(S,t)}{\partial S^2} + \int_{-\infty}^{\infty} \left[ c(Se^x,t) - c(S,t) - \frac{\partial}{\partial S} c(S,t)(e^x - 1) \right] \nu(x,t) dx \\
+ [r(t) - q(t)] S \frac{\partial c(S,t)}{\partial S} - r(t) c(S,t) + \frac{\partial c(S,t)}{\partial t} = 0
\]

A fortiori, the European call value function \( c(S,t) \) solves a backward boundary value problem (BVP), consisting of the backward PIDE subject to the following boundary conditions:

\[
c(S,T_0) = (S - K_0)^+, \quad S \in [0, \infty) \\
\lim_{S \to 0} c(S,t) = 0, \quad t \in [0, T_0] \\
\lim_{S \to \infty} \frac{\partial^2}{\partial S^2} c(S,t) = 0, \quad t \in [0, T_0].
\]
Backward PIDE for Down-and-Out Calls

\[
\frac{a^2(S,t)}{2} \frac{\partial^2 D(S,t)}{\partial S^2} + \int_{-\infty}^{\infty} \left[ D(Se^x,t) - D(S,t) - \frac{\partial}{\partial S} D(S,t) S(e^x - 1) \right] \nu(x,t) dx \\
+ [r(t) - q(t)] S \frac{\partial D(S,t)}{\partial S} - r(t) D(S,t) + \frac{\partial D(S,t)}{\partial t} = 0
\]

A fortiori, the down-and-out call value function \( D(S,t) \) solves a backward boundary value problem (BVP), consisting of the backward PIDE subject to the following boundary conditions:

\[
D(S,T_0) = (S - K_0)^+, \quad S \in [H, \infty) \\
\lim_{S \downarrow H} D(S,t) = 0, \quad t \in [0, T_0] \\
\lim_{S \uparrow \infty} \frac{\partial^2}{\partial S^2} D(S,t) = 0, \quad t \in [0, T_0].
\]
Backward PIDE for Up-and-Out Calls

\[
\frac{a^2(S,t)}{2} \frac{\partial^2 U(S,t)}{\partial S^2} + \int_{-\infty}^{\infty} \left[ U(Se^x,t) - U(S,t) - \frac{\partial}{\partial S} U(S,t)Se^x - 1 \right] \nu(x,t) \, dx \\
+[r(t) - q(t)]S \frac{\partial U(S,t)}{\partial S} - r(t)U(S,t) + \frac{\partial U(S,t)}{\partial t} = 0
\]

A fortiori, the up-and-out call value function $U(S,t)$ solves a backward boundary value problem (BVP), consisting of the backward PIDE subject to the following boundary conditions:

\[
U(S,T_0) = (S - K_0)^+, \quad S \in [0, H] \\
\lim_{S \uparrow 0} U(S,t) = 0, \quad t \in [0, T_0] \\
\lim_{S \downarrow H} U(S,t) = 0, \quad t \in [0, T_0].
\]
Forward PIDE for European Calls

\[
\frac{\partial}{\partial T} c(K, T) = \frac{a^2(K, T)}{2} \frac{\partial^2}{\partial K^2} c(K, T) - [r(T) - q(T)] K \frac{\partial}{\partial K} c(K, T) - q(T) c(K, T) \\
+ \int_{-\infty}^{\infty} \left[ c(Ke^{-x}, T) - c(K, T) - \frac{\partial}{\partial K} c(K, T) K(e^{-x} - 1) \right] e^{x \nu(x, T)} dy.
\]

Boundary conditions are:

\[
c(K, 0) = (S_0 - K)^+, \quad K \in [0, \infty),
\]
\[
\lim_{K \downarrow 0} \frac{\partial^2}{\partial K^2} c(K, T) = 0, \quad T \in [0, \bar{T}],
\]
\[
\lim_{K \uparrow \infty} \frac{\partial^2}{\partial K^2} c(K, T) = 0, \quad T \in [0, \bar{T}].
\]
Forward PIDE for Down-and-Out Calls

\[ \frac{\partial}{\partial T} D^c_0(K, T) = \frac{a^2(K, T)}{2} \frac{\partial^2}{\partial K^2} D^c_0(K, T) - [r(T) - q(T)] K \frac{\partial}{\partial K} U^c_0(K, T) - q(T) D^c_0(K, T) \]
\[ + \int_{-\infty}^{\infty} \left[ D^c_0(Ke^{-x}, T) - D^c_0(K, T) - \frac{\partial}{\partial K} D^c_0(K, T) K(e^{-x} - 1) \right] e^x \nu(x, T) dy. \]

Boundary conditions are:

\[ D^c_0(K, 0) = (S_0 - K)^+, \quad K \in [H, \infty) \quad \text{and} \quad H < S_0, \]
\[ \frac{\partial^2}{\partial K^2} D^c_0(H, T) = 0, \quad T \in [0, \bar{T}], \]
\[ \lim_{K \uparrow \infty} \frac{\partial^2}{\partial K^2} D^c_0(K, T) = 0, \quad T \in [0, \bar{T}]. \]
Forward PIDE for Up-and-Out Calls

\[
\frac{\partial}{\partial T} U_0^c(K,T) = \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} U_0^c(K,T) - [r(T) - q(T)]K \frac{\partial}{\partial K} U_0^c(K,T) - q(T)U_0^c(K,T)
\]
\[
+ \int_{-\infty}^{\infty} \left[ U_0^c(K e^{-x}, T) - U_0^c(K, T) - \frac{\partial}{\partial K} U_0^c(K, T) K(e^{-x} - 1) \right] e^x \nu(x, T) dy
\]
\[
+ (H - K) \frac{a^2(H,T)}{2} \frac{\partial^3}{\partial K^3} U_0^c(H,T)
\]
\[
+ (H - K) \int_{0^+}^{\infty} \frac{\partial}{\partial K} U_0^c(He^{-x}, T) \nu(x, T) dy - \int_{0^+}^{\infty} U_0^c(He^{-x}, T) e^x \nu(x, T) dy
\]

Boundary conditions are:

\[
U_0^c(K, 0) = (S_0 - K)^+, \quad K \in [0, H) \quad \text{and} \quad H > S_0,
\]
\[
\lim_{K \to 0} \frac{\partial^2}{\partial K^2} U_0^c(K, T) = 0, \quad T \in [0, \bar{T}],
\]
\[
\frac{\partial^2}{\partial K^2} U_0^c(H, T) = 0, \quad T \in [0, \bar{T}].
\]
In our numerical examples, we consider the following local volatility surface

$$\sigma(K, T) = 0.3e^{-T}(100/K)^{0.2}$$
The Variance Gamma Process

In our numerical examples, $\nu(x)dx$ is the Lévy density for the VG process in the following form

$$\nu(x) = \frac{e^{-\lambda_p x}}{\nu x} \text{ for } x > 0 \text{ and } \nu(x) = \frac{e^{-\lambda_n |x|}}{\nu |x|} \text{ for } x < 0$$

and

$$\lambda_p = \left( \frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2} \quad \lambda_n = \left( \frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 \nu} \right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}.$$

where $\sigma$, $\nu$, and $\theta$ are VG parameters.
The variables are: spot $S_0=100$, Up-Barrier $H = 140$, risk-free rate $r = 0.06$, dividend rate $q = .02$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = -0.3$. 
Illustration 1(b): Up-and-Out Call Prices (Backward)
Illustration 1(c): Up-and-Out Call Prices (Backward)

Maturity = 1 year and Strike = 90

Maturity = 1 year and Strike = 110
Illustration 2: Up-and-Out Call Prices (Forward)

The variables are: spot $S_0 = 100$, Up-Barrier $H = 140$, risk-free rate $r = 0.06$, dividend rate $q = 0.02$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = -0.3$. 
Up-and-Out Call Prices (Forward vs. Backward)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>( T_1 = 0.25 )</th>
<th>( T_2 = 0.5 )</th>
<th>( T_3 = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bwd</td>
<td>Fwd</td>
<td>Bwd</td>
</tr>
<tr>
<td>Barrier</td>
<td>Strike</td>
<td>Bwd</td>
<td>Fwd</td>
</tr>
<tr>
<td>140</td>
<td>90</td>
<td>110</td>
<td>11.9869</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.38287</td>
<td>2.38951</td>
</tr>
</tbody>
</table>
Illustration 3(a): Down-and-Out Call Prices (Backward)

The variables are: spot $S_0 = 100$, Down-Barrier $H = 60$, risk-free rate $r = 0.06$, dividend rate $q = .02$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = -0.3$. 
Illustration 3(b): Down-and-Out Call Prices (Backward)

Maturity = 6 months and Strike = 90

Maturity = 6 months and Strike = 110
Illustration 3(c): Down-and-Out Call Prices (Backward)
Illustration 4: Down-and-Out Call Prices (Forward)

The variables are: Spot $S_0=100$, Down-Barrier $H = 60$, risk-free rate $r = 0.06$, dividend rate $q = .02$, and VG parameters $\sigma = 0.3$, $\nu = 0.25$, $\theta = −0.3$. 
Down-and-Out Call Prices (Forward vs. Backward)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>$T_1 = 0.25$</th>
<th>$T_2 = 0.5$</th>
<th>$T_3 = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barrier</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Bwd</td>
<td>Fwd</td>
<td>Bwd</td>
</tr>
</tbody>
</table>
Future work

Bermudan, Compound, and American options in the Markov setting

Derivatives with stochastic volatility
Assuming the Markovian stock price process, the Tanaka Meyer formula implies:

\[
(S_T - K)^+ = e^{\int_0^T r(u) du} (S_0 - K)^+ + \int_0^T e^{\int_0^T r(u) du} 1(S_t > K) dS_t \\
+ \int_0^T e^{\int_0^T r(u) du} \left\{ a^2(S_t, t) \frac{\delta(S_t - K) - r(t)(S_t - K)^+}{2} \right\} dt \\
+ \int_0^T e^{\int_0^T r(u) du} \int_{-\infty}^\infty \left[ (S_t e^x - K)^+ - (S_t - K)^+ - 1(S_t > K)S_t(e^x - 1) \right] \mu(dx, dt),
\]
Derivation of Forward PIDE for Up-and-Out Calls

Multiplying by $e^{-\int_0^T r(u) du} 1(\tau_H > T)$ and taking expectations on both sides under an equivalent martingale measure $\mathbb{Q}$, we have:

$$U_0^c(K,T) = (S_0 - K)^+ E_0^Q 1(\tau_H > T) + \int_0^T e^{-\int_0^t r(u) du} E_0^Q \{ 1(\tau_H > T) 1(S_t > K) [r(t) - q(t)] S_t^- \} dt$$

$$+ \int_0^t e^{-\int_0^u r(u) du} \left\{ \frac{a^2(K,t)}{2} E_0^Q [1(\tau_H > T) \delta(S_t - K)] - r(t) E_0^Q [1(\tau_H > T) (S_t - K)^+] \right\} dt$$

$$+ \int_0^T e^{-\int_0^u r(u) du} E_0^Q 1(\tau_H > T) \int_{-\infty}^{\infty} \left[ (S_t e^x - K)^+ - (S_t - K)^+ - 1(S_t > K) S_t (e^x - 1) \right] \nu(x,t) dx dt.$$
Derivation of Forward PIDE for Up-and-Out Calls

Differentiating w.r.t. \( T \) implies:

\[
\frac{\partial}{\partial T} U_0^c(K, T) = -e^{-\int_0^T r(u) du} E_0^Q \{ \delta(\tau_H - T)(S_T - K)^+ \} + e^{-\int_0^T r(u) du} E_0^Q \{ 1(\tau_H > T)1(S_T^- > K)[r(T) - q(T)]S_T^- \} + \frac{a^2(K, T)}{2} e^{-\int_0^T r(u) du} E_0^Q [1(\tau_H > T)\delta(S_T^- - K)] - r(T) e^{-\int_0^T r(u) du} E_0^Q [1(\tau_H > T)(S_T^- - K)^+] + e^0 E_0^Q \left\{ 1(\tau_H > T) \int_{-\infty}^{\infty} \left[ (S_T e^x - K)^+ - (S_T - K)^+ - 1(S_T > K)S_T(e^x - 1) \right] \nu(x, T) dx \right\}.
\]
Derivation of Forward PIDE for Up-and-Out Calls

Subtracting and adding \( e^{-\int_0^T r(u)du} E_0^Q \{1(\tau_H > T)[r(T) - q(T)]K \ 1(S_T > K)\} \) to the second term on the RHS gives:

\[
\begin{align*}
\frac{\partial}{\partial T} U_0^\xi(K,T) &= -e^{-\int_0^T r(u)du} E_0^Q \{1(\tau_H < T)1(S_T \geq H)(S_T - K)\} \\
+ &\ e^{-\int_0^T r(u)du} E_0^Q \{1(\tau_H > T)1(S_T^- > K)[r(T) - q(T)](S_T - K)\} \\
+ &\ e^{-\int_0^T r(u)du} E_0^Q \{1(\tau_H > T)[r(T) - q(T)]K1(S_T > K)\} + \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} U_0^\xi(K,T) \\
- &\ r(T)U_0^\xi(K,T) \\
- &\ \int_T^\tau \ e^{-\int_0^u r(v)dv} E_0^Q \left\{1(\tau_H > T) \int_{-\infty}^\infty \left[e^x(S_T - Ke^{-x})^+ - 1(S_T > K)(S_T - K + S_T e^x - S_T)\right] \nu(x,T)dx \right\} \\
\end{align*}
\]
Derivation of Forward PIDE for Up-and-Out Calls (Cont’d)

\[
\frac{\partial}{\partial T} U_0^c(K,T) = -e^{-\int_0^T r(u)du} E_0^Q \left\{ 1(\tau_H < T)1(S_T \geq H)(H - K) + 1(\tau_H < T)1(S_T \geq H)(S_T - H) \right\} + [r(T) - q(T)]U_0^c(K,T) - [r(T) - q(T)]K \frac{\partial}{\partial K} U_0^c(K,T) + \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} U_0^c(K,T) - r(T)U_0^c(K,T) + e^{-\int_0^T r(u)du} E_0^Q \left\{ 1(\tau_H > T) \int_{-\infty}^{\infty} e^x \left[ (S_T - Ke^{-x})^+ - 1(S_T > K)(S_T - Ke^{-x} + K - K) \right] \nu(x,T)dx \right\}.
\]

The first term on the RHS is the sum of the payoffs from \( H - K \) partial barrier up-and-out binary calls with barrier \( H \) and one partial barrier up-and-out call, with barrier and strike \( H \). For both options, the end of the barrier monitoring period is \( T - \), while the options mature at \( T \).
Derivation of Forward PIDE for Up-and-Out Calls (Cont’d)

For the partial barrier up-and-out call with strike and barrier $H$, we have:

$$e^{-\int_0^T r(u)du} E_0^Q \{ 1(\tau_H < T) 1(S_T \geq H)(S_T - H) \} = \frac{a^2(H,T)}{2} \frac{\partial^2}{\partial K^2} U_0^c(H,T).$$

However, $\frac{\partial^2}{\partial K^2} U_0^c(H,T) = 0$, since, loosely speaking, this is just the discounted probability of surviving beyond $T$ and that $S_T = H$. For the partial barrier up-and-out binary call with barrier $H$, we have:

$$e^{-\int_0^T r(u)du} E_0^Q \{ 1(\tau_H < T) 1(S_T \geq H) \} = -\frac{a^2(H,T)}{2} \frac{\partial^3}{\partial K^3} U_0^c(H,T).$$

The third derivative does not vanish.
If we now account for jumps when valuing the first term, we obtain:

\[
\frac{\partial}{\partial T} U_0^c(K,T) = (H - K) \left[ \frac{a^2(H,T)}{2} \frac{\partial^3}{\partial K^3} U_0^c(H,T) - e^{-\int_0^T r(u)du} E_0^Q \{ 1(\tau_H < T) \int_{0^+}^{\infty} 1(S_T e^x \geq H) \nu(x,T)dx \} \right]
\]

\[
- e^{-\int_0^T r(u)du} E_0^Q \{ 1(\tau_H < T) \int_{0^+}^{\infty} (S_T e^x - H)^+ \nu(x,T)dx \} - q(T) U_0^c(K,T)
\]

\[
+ [r(T) - q(T)]K \frac{\partial}{\partial K} U_0^c(K,T) + \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} U_0^c(K,T)
\]

\[
+ e^{-\int_0^T r(u)du} E_0^Q \left\{ 1(\tau_H > T) \int_{-\infty}^{\infty} \left[ (S_T - Ke^{-x})^+ - (S_T - K)^+ - \frac{\partial}{\partial K}(S_T - K)^+ K(e^{-x} - 1) \right] e^x \nu(x,T)dx \right\}
\]
Derivation of Forward PIDE for Up-and-Out Calls (Cont’d)

\[
\frac{\partial U_0^c(K,T)}{\partial T} = (H - K) \left[ \frac{a^2(H,T)}{2} \frac{\partial^3}{\partial K^3} U_0^c(H,T) - e^{-\int_0^T r(u)du} E_0^Q \left\{ 1(\tau_H < T) \int_0^\infty 1(S_{T^-} \geq He^{-x}) \nu(x,T)dx \right\} \right] \\
- e^{-\int_0^T r(u)du} E_0^Q \left\{ 1(\tau_H < T) \int_0^{\infty} (S_{T^-} - He^{-x})^+ e^x \nu(x,T)dx \right\} \\
- q(T) U_0^c(K,T) - [r(T) - q(T)]K \frac{\partial}{\partial K} U_0^c(K,T) + \frac{a^2(K,T)}{2} \frac{\partial^2}{\partial K^2} U_0^c(K,T) \\
+ \int_{-\infty}^{\infty} \left[ U_0^c(Ke^{-x},T) - U_0^c(K,T) - \frac{\partial}{\partial K} U_0^c(K,T) K(e^{-x} - 1) \right] e^x \nu(x,T) dx
\]