Black-hole thermodynamics in Lovelock gravity

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The thermodynamic properties of black holes in Lovelock gravity are examined. In particular, the case of the Einstein Lagrangian plus the four-dimensional Euler density is discussed in detail. In five dimensions, one finds that the specific heat of a black hole becomes positive at small mass, allowing the black hole to achieve stable equilibrium with its environment and giving it an infinite lifetime. This behavior is not universal, however, but it always occurs in $2k+1$ dimensions for a Lovelock theory including the $2k$-dimensional Euler density. For theories including six-derivative or higher-order interactions, black holes with degenerate zero-temperature horizons are also possible.

I. INTRODUCTION

The Hawking effect[1] was a remarkable discovery in the study of quantum field theory in curved space-times. A black hole appears to emit thermal radiation with a temperature of

$$T=\frac{\kappa}{2\pi}=\frac{1}{8\pi GM} , \tag{1}$$

where κ is the surface gravity. The second equality holds for a spherically symmetric black hole in four dimensions, where G is Newton's constant and M is the mass of the black hole. (We have set $\hbar=c=k_B=1$.) The emitted radiation is a thermal ensemble which must be described by a density matrix from the point of view of an observer located outside of the black hole. Naively a dilemma may appear to arise since, to the observer, the incoming pure quantum state appears to evolve into a mixed state. However, for each external radiation state there is a corresponding internal state describing the quantum fields inside of the event horizon, and so the joint system is still described by a well-defined wave function. Unfortunately quantum gravity faces a true paradox if the Hawking radiation continues until the black hole finally vanishes, having radiated away its entire mass. In this case the incoming pure state has actually become a mixed state, since the potential reservoir of information about internal states is lost. This scenario violates a basic principle of quantum field theory: the time evolution of a physical system should be described by a *unitary* operator, the Hamiltonian. It is reasonable though to expect that the relation between the temperature and mass of a black hole given in Eq. (1) will be modified at small masses. The paradox might be avoided if in fact the black-hole temperature begins decreasing and approaches zero for small mass. The black-hole evolution might then end with a zero-temperature soliton, where some of the incoming quantum-mechanical information would continue to reside.

The relation between the temperature and the mass given in Eq. (1) is derived for an explicit solution of Einstein's equations. One should expect that the Einstein action is only an effective gravitational action valid for small curvatures or low energies. One sees that the action will be modified by higher-derivative interactions in any attempt to perturbatively quantize gravity as a field theory.[2] Such terms also arise in the effective low-energy actions of string theories.[3] Higher-derivative gravity theories have also been studied in their own right since in many cases they display the attractive features of renormalizability and asymptotic freedom.[4] One higher-derivative theory which has attracted a great deal of attention recently is Lovelock gravity.[5] A Lovelock Lagrangian is the sum of dimensionally extended Euler densities

$$\mathcal{L}=\sum_{m=0}^{k} c_m\mathcal{L}_m , \tag{2a}$$

where c_m is an arbitrary constant, and $\mathcal{L}_m$ is the Euler density of a $2m$-dimensional manifold:

$$\mathcal{L}_m=2^{-m}\delta^{a_1b_1\cdots a_mb_m}_{c_1d_1\cdots c_md_m}R^{c_1d_1}{}_{a_1b_1}\cdots R^{c_md_m}{}_{a_mb_m} . \tag{2b}$$

Here, the generalized δ function $\delta^{a_1\cdots b_m}_{c_1\cdots d_m}$ is totally antisymmetric in both sets of indices. We set $\mathcal{L}_0=1$ and, hence, c_0 is proportional to the cosmological constant. The Einstein Lagrangian is a special case of Eq. (2) for which only c_1 is nonvanishing. These Lagrangians are exceptional in that the resulting equations of motion contain no more than second derivatives of the metric.[5] They have also been shown to be free of ghosts when expanding about flat space,[6] evading any problems with unitarity. Exact solutions describing black holes have also been found for these theories.[7–9] In this paper we will study the thermodynamics of these black-hole solutions as a model for the possible effects of higher-derivative interactions in strong gravitational fields. References 10 and 11 present related material in the context of string theory. In four dimensions, $\mathcal{L}_2$ is a total derivative while the higher-order interactions (i.e., $\mathcal{L}_m$ with $m>2$) are simply zero. Therefore, we must concern ourselves here with space-times for which $D\geq 5$. This is not a drawback since most recent attempts to construct a quantum theory of gravity have involved space-time dimensions greater than four.

The paper is organized as follows. In Sec. II we begin by examining the simplest nontrivial example of Lovelock gravity, the Einstein action plus the four-dimensional Euler character. Spherically symmetric vacuum solutions have been found independently by Boulware and Deser,[7] and Wheeler.[8] A careful examination of the global topology of these solutions is made, and some new cases without naked singularities are found. The thermodynamic properties of all of the nonsingular cases are explored. In Sec. III a discussion covering more general Lovelock actions is presented, and Sec. IV gives the concluding discussion.

II. THE FOUR-DERIVATIVE THEORY

The simplest higher-derivative Lovelock Lagrangian consists of the Einstein term plus the four-dimensional Gauss-Bonnet density. The recent interest in this theory arose because it appears as the low-energy effective action of some string theories.[3] The action is

$$I=\frac{1}{16\pi G}\int d^Dx\,\sqrt{-g}\left[R+\frac{\lambda}{2}(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)\right]+\frac{1}{8\pi G}\int d^{D-1}x\,\sqrt{h}\,[K+\lambda(\cdots)]\;,$$

where λ is a coupling constant of dimension (length)^2. Presumably λ is small compared to present empirical scales, so that one recovers Einstein gravity in the low-energy limit (i.e., for small curvature). The higher-order surface terms indicated by the ellipses are given in Ref. 12, but will be of no consequence in the following.

Spherically symmetric vacuum solutions for this theory are given by[7,8]

$$ds^2=-f^2dt^2+f^{-2}dr^2+r^2d\Omega_{D-2}^2\;, \tag{3}$$

where $d\Omega_N^2$ is a line element on the unit N-sphere, and

$$f^2=1+\frac{r^2}{\hat{\lambda}}\left[1+\epsilon\left[1+\frac{2\omega\hat{\lambda}}{r^{D-1}}\right]^{1/2}\right]\,.$$

Here, $\hat{\lambda}=\lambda(D-3)(D-4)$ and ω is an integration constant with dimensions of (length)$^{D-3}$, related to the mass of the black hole. The constant $\epsilon=\pm1$ appears because f^2 is determined by solving a quadratic equation. *A priori*, one may consider $\hat{\lambda}$ and ω with either positive or negative values, but for many values of ϵ, $\hat{\lambda}$, and ω, the solutions have naked singularities. Examining the components of the Riemann tensor in an orthonormal frame reveals curvature singularities at $r=0$, but also at $r^{D-1}=-2\omega\hat{\lambda}$ if $\omega\hat{\lambda}<0$. We will determine the ranges of parameters for which these singularities are surrounded by a horizon.

As expected $f^2=0$ is only a coordinate singularity, signaling the presence of an event horizon. One easily extends the (t,r) coordinate patch with the null coordinates

$$dv_{\pm}=dt\pm f^{-2}dr\;. \tag{4}$$

Radial lines of constant $v_+(v_-)$ correspond to infalling (outgoing) null geodesics. Given in terms of (v_+,r), the metric is singularity-free on the future horizon(s), while the (v_-,r) coordinates provide a regular extension across the past horizon(s). (Appendix A of the first paper in Ref. 8 provides an alternative regular extension.) The positions of the horizons may be determined as the real roots of the polynomial $q(r=r_h)=0$, where

$$q(r)=r^{D-3}+\frac{\hat{\lambda}}{2}r^{D-5}-\omega\;. \tag{5}$$

Deriving $q(r)=0$ from $f^2=0$ involved squaring an intermediate equation, and therefore the roots of $q(r)$ must also satisfy

$$\epsilon(r_h^2+\hat{\lambda})\leq 0 \tag{6}$$

for a horizon to actually be present.

If $\epsilon=-1$, the metric is asymptotically flat. This is a case which has already been explored in some detail.[7,8] For large r, f^2 becomes

$$f^2\approx 1-\frac{\omega}{r^{D-3}}\;,$$

and Eq. (3) takes the form of the Schwarzschild metric with mass[13]

$$M=\frac{(D-2)A_{D-2}}{16\pi G}\omega\;, \tag{7}$$

where $A_{N-1}=2\pi^{N/2}/\Gamma(N/2)$ is the area of a unit $(N-1)$-sphere. For $\epsilon=-1$, Eq. (6) becomes

$$r_h^2\geq-\hat{\lambda}\;. \tag{8}$$

This is trivially satisfied if $\hat{\lambda}>0$. In this case, the polynomial $q(r)$ given in Eq. (5) increases monotonically for positive r beginning from $-\omega$ (for $D>5$) at $r=0$. Therefore, q has one and only one zero with $r_h>0$ if and only if $\omega>0$. For the case $D=5$, however, $q(r=0)=(\hat{\lambda}/2)-\omega$, and, hence, one must have $\omega>\hat{\lambda}/2$ for a horizon to exist. For $\hat{\lambda}<0$, examining q shows that one and only one horizon occurs if $\omega>|\hat{\lambda}|^{(D-3)/2}/2$. In this case, r_h satisfies Eq. (8) as well as occurring outside of the singularity at $r^{D-1}>2\omega|\hat{\lambda}|$. Therefore for $\epsilon=-1$ and both signs of $\hat{\lambda}$, there is the possibility that a single horizon occurs enclosing a spacelike singularity. The global topology of these manifolds is then identical to that of a Schwarzschild black hole as illustrated by the Penrose diagram in Fig. 1.

The temperature of the horizon of a black hole is calculated using the periodicity in imaginary time of the metric. To define quantum-field propagators in these black-hole backgrounds, one rotates to Euclidean time $t \to i\tau$. In order to produce a smooth Euclidean manifold, the Euclidean time must be identified with a certain periodicity β. This periodicity in imaginary time will then appear in the propagator so defined and may be interpreted as indicating the fields are in a canonical ensemble in equilibrium with a heat bath of temperature $T=1/\beta$ (Refs. 14 and 15). As in the usual Einstein case then, one finds that the periodicity is

$$\beta=\frac{2\pi}{\kappa} ,$$

where κ is the surface gravity of the horizon.[16] For the present metric, one finds

$$T=\frac{D-3}{4\pi r_h}\left[\frac{r_h^2+\frac{D-5}{D-3}\frac{\hat\lambda}{2}}{r_h^2+\hat\lambda}\right] , \tag{9}$$

where r_h is implicitly given by $q(r_h)=0$. Using Eq. (5) [as well as Eq. (8) for $\hat\lambda<0$] to show that $\partial r_h/\partial\omega$ is always positive in the cases of interest, one can show that $\partial T/\partial\omega$ is strictly negative for $D>5$ (just as it is for a Schwarzschild black hole). For $D=5$, Eq. (9) simplifies to

$$T=\frac{1}{2\pi}\frac{r_h}{r_h^2+\hat\lambda}=\frac{1}{2\pi}\frac{\left[\omega-\frac{\hat\lambda}{2}\right]^{1/2}}{\omega+\frac{\hat\lambda}{2}} . \tag{10}$$

In this case with $\hat\lambda<0$, $|\hat\lambda|$ is a lower bound on the position of the horizon, and one finds then that $\partial T/\partial\omega$ is always negative. On the other hand, for $\hat\lambda>0$, $\partial T/\partial\omega$ is zero at $\omega=3\hat\lambda/2$ and is positive (negative) for smaller (larger) values of ω.

To construct our candidate for the entropy of these black holes, we use the same thermodynamic identities which are applied in Einstein gravity.[17] The action of the Euclideanized manifold $\hat I$ is identified with the free energy multiplied by β, and then the entropy follows as

$$S=\beta\langle E\rangle-\hat I=\beta\frac{\partial\hat I}{\partial\beta}-\hat I . \tag{11}$$

Here $\langle E\rangle\equiv\partial\hat I/\partial\beta$ is the thermodynamic energy of the system. We stress that this is a tentative definition since we have not proven a second law of black-hole mechanics (i.e., $\delta S\geq 0$) for this theory. It is a reasonable definition though, since entropy is automatically conjugate to temperature. For $\epsilon=-1$, the Euclideanized action is

$$\hat I=-\frac{A_{D-2}r_h^{D-5}}{16(D-4)\pi G}\left[(D-2)\beta\left[r_h^2-\frac{\hat\lambda}{2}\right]-8\pi r_h^3\right] , \tag{12}$$

and one finds

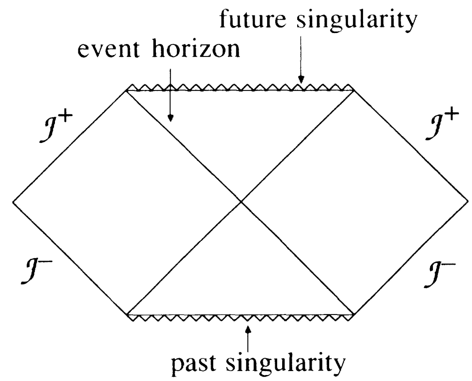

FIG. 1. The Penrose diagram for a black hole with a single nondegenerate horizon. This diagram is similar to that for the Schwarzschild solution in four dimensions (for example, see Ref. 19), but now each point represents a $(D-2)$-sphere for a black hole in D dimensions.

$$\langle E\rangle=\frac{(D-2)A_{D-2}}{16\pi G}\omega=M .$$

Thus $\langle E\rangle$ is precisely equal to the mass (7) determined from the asymptotic behavior of the metric. This result should not be surprising in light of Sorkin's expression for the mass[18] where the variation of the action (12) can be related to an asymptotic integral involving a timelike Killing vector. The entropy (11) is then

$$S=\frac{A_{D-2}}{4G}r_h^{D-2}\left[1+\frac{D-2}{D-4}\frac{\hat\lambda}{r_h^2}\right] . \tag{13}$$

Note that the last factor modifies the usual Einstein gravity result of $S=\mathcal{A}/4G$. For $\hat\lambda>0$ this expression is manifestly non-negative, and it always increases with ω (using $\partial r_h/\partial\omega>0$). For $\hat\lambda<0$, the entropy again always increases with ω but can be negative for small r_h. The minimum entropy occurs as $r_h^2\to|\hat\lambda|$, at which point the horizon vanishes exposing a naked singularity. (Further discussion of the entropy is presented in Sec. IV.)

We now consider the solutions with $\epsilon=+1$. In this case, the metric (3) is no longer asymptotically flat, as can be seen by examining f^2 for $r^{D-1}\gg 2|\omega\hat\lambda|$:

$$f^2\approx 1+\frac{2r^2}{\hat\lambda}+\frac{\omega}{r^{D-3}} . \tag{14}$$

The metric approaches a Schwarzschild–(anti-)de Sitter metric with a cosmological constant proportional to $-\hat\lambda^{-1}$ and a gravitational mass proportional to $-\omega$. In the present theory, the inertial mass is actually still proportional to ω because the graviton becomes a ghost for $\epsilon=+1$ as discussed in Ref. 7.

The restriction imposed by Eq. (6) is now $r_h^2<-\hat\lambda$, and so only the theory with $\hat\lambda<0$ will have horizons. The background cosmological constant in Eq. (14) is positive yielding a de Sitter space. The positions of the horizons are bounded above, $r_h^2<|\hat\lambda|$. They are still determined by $q(r_h)=0$ [see Eq. (5)], and so we begin by giving a qualitative description of the behavior of this function for

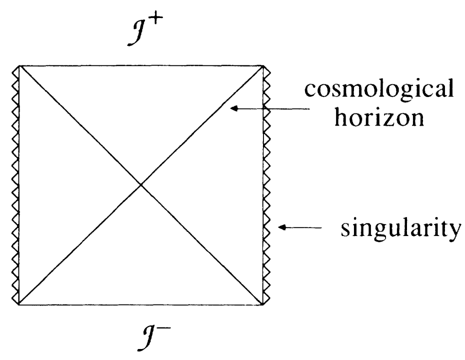

FIG. 2. The Penrose diagram for a solution with a single nondegenerate cosmological horizon and a curved de Sitter background. This diagram is like that of the Schwarzschild–de Sitter solution with negative mass in four dimensions.

$r>0$ and $D>5$. At $r=0$, $q=-\omega$ and its slope is zero (or negative for $D=6$). As r begins to increase, q decreases to a minimum at

$$r_{\min}^2=\frac{D-5}{D-3}\frac{|\hat{\lambda}|}{2} .$$

As r increases beyond $r_{\min}$, q increases monotonically. Depending on the value of ω, there are several regimes to the solutions each with different topologies.

If ω is positive, the horizon must occur at $r_h^{D-1}>2\omega|\hat{\lambda}|$ in order to surround the singularity. This in turn requires that $\omega<|\hat{\lambda}|^{(D-3)/2}/2$. For $0<\omega<|\hat{\lambda}|^{(D-3)/2}/2$, there is a single cosmological horizon enclosing the timelike singularity. The global topology is like that of a Schwarzschild–de Sitter space with negative mass as illustrated in Fig. 2. We identify the singularity as being naked because the space does not have a global Cauchy surface.[19] For $\omega=0$, the solution is no longer a black hole but simply de Sitter space,[7] with cosmological horizon at $r_c^2=|\hat{\lambda}|/2$. The temperature of this horizon is still given by Eq. (9) except that the sign must be reversed:

$$T_c=(2\pi^2|\hat{\lambda}|)^{-1/2} .$$

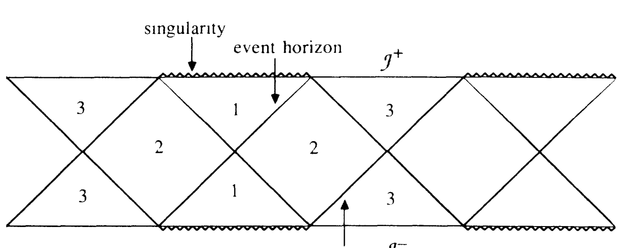

FIG. 3. The Penrose diagram for a black-hole solution in a de Sitter background with two nondegenerate horizons: an event horizon separating the regions labeled 1 and 2, and a cosmological horizon between the coordinate patches labeled 2 and 3. This diagram is like that of the Schwarzschild–de Sitter space in four dimensions with $0<M<\sqrt{\Lambda}/3$ (for example, see Ref. 20). This infinite chain of patches may be reduced to a finite loop by identifying two of the regions labeled 2.

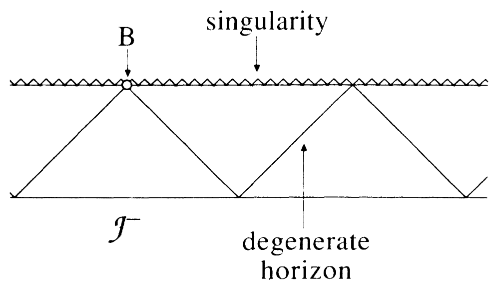

FIG. 4. The Penrose diagram for a solution with a single degenerate cosmological horizon and a curved de Sitter background. This diagram is like that of the Schwarzschild–de Sitter solution with $M=\sqrt{\Lambda}/3$ in four dimensions. The point labeled B on the horizon is an infinite proper distance from any point a finite coordinate distance away in the adjacent exterior region. Note that there is a spacelike infinity for timelike and null lines in either the past (as illustrated) or the future.

Let ω_c correspond to the value of ω for which $q(r_{\min})=0$. One finds that

$$\omega_c=-\frac{2}{D-5}\left[\frac{D-5}{D-3}\frac{|\hat{\lambda}|}{2}\right]^{(D-3)/2} .$$

For $\omega_c<\omega<0$, there are two horizons and the global topology is like that of the Schwarzschild–de Sitter space as illustrated by Fig. 3 (Ref. 20). Equation (9) gives the temperature of the horizons except that the sign must be reversed for the outer or cosmological horizon. For $\omega=\omega_c$ there is a single degenerate horizon with zero temperature. This space (see Fig. 4) should be regarded as singular, since the spacelike singularity spans either the future or past boundary of the manifold. Finally for $\omega<\omega_c$, there are no horizons and there is a naked timelike singularity at $r=0$.

For the special case of $D=5$, one sees from Eq. (5) that $q(0)=-\omega-|\hat{\lambda}|/2$ and q increases monotonically for increasing positive r. If ω is positive, a single horizon will occur outside of the singularity at $r^4=2\omega|\hat{\lambda}|$ for $0<\omega<|\hat{\lambda}|/2$. These solutions contain naked singularities having the global topology illustrated in Fig. 2. Setting $\omega=0$, yields de Sitter space as described above. A single horizon will also occur for $-|\hat{\lambda}|/2<\omega<0$, and these spaces have the topology shown in Fig. 2 again. For $\omega<-|\hat{\lambda}|/2$, there are no horizons and the spaces contain naked singularities once again.

III. THE GENERAL CASE

Recall the general Lovelock Lagrangian given in Eq. (2). In the following we will consider an action which includes up to $\mathcal{L}_k$, and so the dimension of space-time is $D\geq 2k+1$. We will assume that $c_1>0$, so that Einstein gravity is recovered as a low-energy limit. Wheeler[8] considered the general Lagrangian (2) and found solutions which take the form

$$ds^2=-f^2dt^2+f^{-2}dr^2+r^2d\Omega_{D-2}^2 , \tag{15}$$

where $d\Omega_N^2$ is a line element on the unit N-sphere, and

$$f^2=1-r^2F(r) \ . \tag{16}$$

F is determined by solving for the real roots of the following kth-order polynomial equation:

$$P(F)=\sum_{m=0}^{k}\hat{c}_m F^m=\frac{\omega}{r^{D-1}} \ , \tag{17}$$

where the coefficients are defined in terms of those appearing in the Lagrangian:

$$\hat{c}_0=\frac{c_0}{c_1}\frac{1}{(D-1)(D-2)},\quad \hat{c}_1=1 \ ,$$
$$\hat{c}_m=\frac{c_m}{c_1}\prod_{n=3}^{2m}(D-n) \quad \text{for } m>1 \ .$$

The product of factors $(D-n)$ expresses the fact that $\mathcal{L}_m$ would not affect the field equations for $D\le 2m$. In Eq. (17) a positive ω corresponds to a source with a positive identical mass,[7] given for an asymptotically flat solution by Eq. (7).

Considering constant solutions F_0 of $P(F_0)=0$, yields the constant-curvature vacua of the theory.[7,8] The solution is de Sitter space for $F_0>0$, while $F_0<0$ yields an anti–de Sitter background. In principle one might find up to k real roots. If $\hat{c}_0\neq 0$ and k is even, it is not clear that the latter equation has any real roots, in which case there would be no maximally symmetric vacuum solutions. In this case, the theory may simply be undefined, but there is also the tantalizing possibility that the vacuum is a compactified space-time.[21] If $\hat{c}_0=0$, $F_0=0$ is always a solution and so flat space is an allowed vacuum. The sign of the derivative of P with respect to F at F_0 determines whether or not the graviton is a ghost particle when perturbing about this background. If $P'(F_0)>0$, the graviton is a positive-energy particle while $P'(F_0)<0$ indicates that it is a ghost.[7] If one is searching for black-hole solutions in a specific vacuum, one may define $F=F_0+\tilde{F}$, where $\tilde{F}$ now satisfies

$$\tilde{P}(\tilde{F})=P(F_0+\tilde{F})=\sum_{m=1}^{k}\tilde{c}_m\tilde{F}^m=\frac{\omega}{r^{D-1}} \ , \tag{18}$$

where $\tilde{P}$ is simply a new kth-order polynomial in $\tilde{F}$ for which $\tilde{F}=0$ must be a root.

It is useful to examine the nonvanishing components of the Riemann tensor in an *orthonormal* frame

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})F \ ,$$
$$R_{i\hat{r}j\hat{r}}=\frac{1}{2r}\delta_{ij}(r^2F)_{,r}=-R_{i\hat{t}j\hat{t}} \ , \tag{19}$$
$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}=-\tfrac{1}{2}(r^2F)_{,rr} \ .$$

Above i, j, k, and l indicate frame indices corresponding to directions on the $(D-2)$-sphere, while $\hat{t}$ and $\hat{r}$ indicate those parallel to dt and dr, respectively. The subscript comma indicates ordinary differentiation. Clearly a curvature singularity arises when $\tilde{F}$ diverges, but also possibly from divergences in $\tilde{F}_{,r}$ or $\tilde{F}_{,rr}$ (recall that F and $\tilde{F}$ only differ by a constant F_0).

Now we wish to consider the form of the solution of Eq. (18). For simplicity we will assume that $\tilde{P}'(0)\neq 0$. The analysis is only slightly more complicated if $\tilde{c}_1=0$ (see Ref. 8). For large r, the right-hand side of Eq. (18) and, hence, $\tilde{F}$ are small, and one finds

$$\tilde{F}\approx\frac{\omega}{\tilde{c}_1 r^{D-1}} \ . \tag{20}$$

One may expand $\tilde{F}$ as a power series in $y=r^{-D+1}$. Now one wishes to extend this solution in towards $r=0$. As r decreases, the right-hand side of Eq. (18) varies monotonically towards $\mathrm{sgn}(\omega)\times\infty$ at $r=0$. Now on the left-hand side, one has a kth-order polynomial in $\tilde{F}$ with nonvanishing slope at $\tilde{F}=0$. Therefore as r decreases, $\tilde{F}$ will begin varying monotonically with the sign appearing in Eq. (20). Since $\tilde{P}$ is a finite polynomial, $\tilde{F}$ will continue growing with $\tilde{F}_{,r}$ finite as long as $\tilde{P}'$ is nonvanishing. If the latter derivative is always finite, $\tilde{F}$ will only finally diverge at the origin which indicates the presence of a curvature singularity as exhibited in Eq. (19). The other possibility is that at finite radius, one encounters $\tilde{P}'=0$, in which case $\tilde{F}_{,r}$ will diverge which also results in a curvature singularity. Such a singularity arose in the four-derivative theory at $r^{D-1}=-2\omega\hat{\lambda}$ when $\omega\hat{\lambda}<0$.

Given a solution for a specific theory, one must consider whether it contains any horizons to conceal the singularity at the minimum radius. In the metric (15), horizons occur where $f^2=0$. A regular extension of the metric across such a singularity is easily constructed with the null coordinates given in Eq. (4). Consider the case where one is able to extend the solution to the origin. Near that point, one has

$$\tilde{F}\approx\left[\frac{\omega}{\tilde{c}_k}\right]^{1/k} r^{-(D-1)/k} \ .$$

From Eq. (16) for $F_0=0$, $f^2\approx 1$ for large r. Near the origin

$$f^2\to 1-\left[\frac{\omega}{\tilde{c}_k}\right]^{1/k} r^{(2k+1-D)/2} \ ,$$

which diverges at $r=0$ for $D>2k+1$. Therefore the solution will have at least one horizon if $\tilde{F}$ is positive (i.e., $\omega\tilde{c}_1>0$). In fact for odd (even) k, it is possible to have an odd number of horizons up to k $(k-1)$. Solutions with a single horizon have the topology of a Schwarzschild black hole (see Fig. 1), but with more horizons the topology becomes arbitrarily complicated unless one identifies various patches (for example, see Fig. 5). In all cases, the singularities are spacelike because there are an odd number of horizons. If $\tilde{F}$ is negative, no horizons occur and there is a naked timelike singularity at the origin (including $D=2k+1$). For the special case of $D=2k+1$, f^2 has a finite limit at the origin, $f^2\to 1-(\omega/\tilde{c}_k)^{1/k}$. In this case $\omega/\tilde{c}_k>1$ yields an odd number of horizons up to $k(k-1)$ for odd (even) k, and the topology is similar to that described above for positive $\tilde{F}$. If $0<\omega/\tilde{c}_k<1$, an even number (including possibly 0) of horizons occur, and the maximum possible number of horizons given above is decreased by 1. In this case the topology may again be

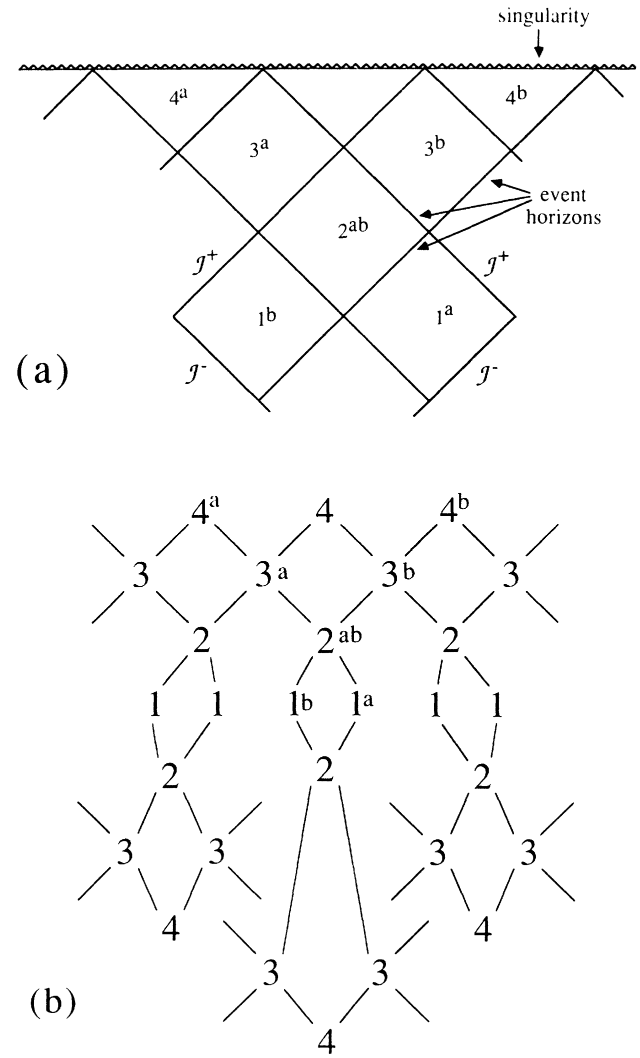

FIG. 5. (a) A Penrose diagram illustrating the overlap of two infalling null coordinate patches (denoted by superscripts a and b) for an asymptotically flat black hole with three nondegenerate horizons, as might occur for Lovelock theories with six-derivative (or higher) interactions. Each such patch covers four distinct types of (t,r) coordinate patches labeled 1, 2, 3, and 4. (b) A schematic illustration of the most general extension of the above space. The two null coordinate patches shown in the previous diagram are again labeled with superscripts a and b. This extension may be reduced in various ways by identifying regions which have the same (numerical) label, and have no neighboring regions in common.

complicated, but the singularities are timelike. In specifying an odd or even number of horizons above, we have assumed that they are all nondegenerate. Of course degenerate horizons will also be possible in which case the parity of the number of horizons may change, but the nature of the singularities remains unchanged.

One should note at this point that our analysis disagrees with that of Ref. 8. There it is implied that the monotonic variation of $\tilde F$ with r results in f^2 varying monotonically as well. In fact, this statement is false and f^2 may have a number of local extrema at finite r. In the asymptotically flat case (i.e., $F_0=0$),

$$(f^2)_{,r} = -2r\tilde F - r^2\tilde F_{,r} = \frac{2r}{\tilde P'}\left[\frac{D-1}{2}\tilde P - \tilde F\,\tilde P'\right], \tag{21}$$

where Eq. (18) and its derivative were used to derive the second equality. Since in the range of interest $\tilde P'$ is monotonic, the sign of $(f^2)_{,r}$ is controlled by the factor in the large parentheses above. Since this factor is a kth-order polynomial in $\tilde F$, one sees that in principle f^2 may have k local extrema [i.e., $(f^2)_{,r}=0$]. A closer examination taking into account the asymptotic behavior of $\tilde F$ discussed above reveals that the preceding is true for odd k, but only $k-1$ extrema are possible if k is even. By adjusting ω, which controls the overall scale of $\tilde F$, one may in principle produce the numbers of horizons given in the previous paragraph. In practice though each theory requires individual consideration. For example, with $k=3$ if $\tilde c_2<0$ and $3\tilde c_3>\tilde c_2^{\,2}$, a new extremum (other than that at $r\to\infty$ which is always present) occurs at finite r in $D=7$. Two new extrema arise in $D=8$ if the above constraints as well as $\tilde c_3<9\tilde c_2^{\,2}/20$ are satisfied. Although for $D\geq 9$ the only extremum in f^2 occurs at $r\to\infty$ (Ref. 22). For larger k, a general analysis becomes intractable, although numerical analysis would be straightforward given a theory with a specific set of coefficients c_m.

For the case in which the solution extends to the origin but with an asymptotically curved background, the conclusions about horizons are similar. In fact for asymptotically anti–de Sitter space ($F_0<0$), it is easy to show that the results are the same as for the asymptotically flat solutions discussed above. If $\tilde F<0$, no horizons and a timelike naked singularity occur, while if $\tilde F>0$, an odd number of nondegenerate horizons and spacelike singularities result except for the special case $D=2k+1$ and $0<\omega/\tilde c_k<1$. The conclusions are modified for asymptotic de Sitter space ($F_0>0$) because f^2 is now negative at large r. If $\tilde F<0$, an odd number of nondegenerate horizons are possible up to k or $k-1$ for odd or even k, respectively. A space with a single horizon has the topology illustrated in Fig. 2, and hence contains a naked singularity. A space with three horizons has a topology similar to that of a Kerr–de Sitter black hole with no naked singularities. The topology becomes more complicated with more horizons, but in all cases the singularities are timelike. If $\tilde F>0$, an even number (including 0) of nondegenerate horizons are possible up to $k+1$ (k) for k odd (even), and the singularities are spacelike. Of course $D=2k+1$ and $0<\omega/\tilde c_k<1$ is again a special case with timelike singularities and up to k ($k-1$) horizons for k odd (even). The topology of the nonsingular spaces here is the obvious extension of the asymptotically flat black holes discussed above by the addition of a cosmological horizon. The above properties are summarized in Table I.

The other possibility is that the solution is singular at finite r where $\tilde F$ approaches a finite limit $\hat F$ but $\tilde F_{,r}$ diverges because $\tilde P'(\hat F)$ vanishes. The maximum number of nondegenerate horizons is identical to the cases enumerated above. Here though for a given F_0 and $\mathrm{sgn}(\tilde F)$, one cannot *a priori* determine the parity of the

TABLE I. Properties of black holes in Lovelock gravity. The results presented here are for theories with $\tilde{P}'\neq 0$ and $\tilde{c}_1\neq 0$, and for which $\tilde{P}$ is a kth-order polynomial in $\tilde{F}$. Case A: $D>2k+1$, or $D=2k+1$ and $\omega/\tilde{c}_c>1$. Case B: $D=2k+1$ and $0<\omega/\tilde{c}_k<1$.

Asymptotic geometry	Dimension	Sign of $\tilde{F}$(or $\omega\tilde{c}_1$)	Number of (nondegenerate) horizons	Maximum number of horizons Even k	Maximum number of horizons Odd k	Nature of singularly
Flat ($F_0=0$)	Case A	Positive	Odd	$k-1$	k	Spacelike
or		Negative	None			Timelike
Anti–de Sitter ($F_0<0$)	Case B	Positive	Even[a]	$k-2$	$k-1$	Timelike
		Negative	None			Timelike
de Sitter ($F_0>0$)	Case A	Positive	Even[a]	k	$k+1$	Spacelike
		Negative	Odd	$k-1$	k	Timelike
	Case B	Positive	Odd	$k-1$	k	Timelike
		Negative	Odd	$k-1$	k	Timelike

[a]Including zero.

number of horizons nor the nature of the singularities.

Finally we wish to briefly consider the temperature of the horizons. As described in Sec. II the temperature is given by

$$T=\frac{\kappa}{2\pi}=\frac{1}{4\pi}\left|\frac{\partial f^2}{\partial r}\right|_{r=r_h} . \tag{22}$$

Of particular interest is the possibility that the temperature approach zero in the final stages of the evaporation of the black hole. There are two distinct ways in which this prospect may be achieved. The first occurs in the special case $D=2k+1$ for which the metric is finite at the origin with $f^2(0)=1-(\omega/\tilde{c}_k)^{1/k}$. Now as $\omega/\tilde{c}_k$ approaches one (from the appropriate side), there will be a horizon at some small r_h which approaches the origin. Now since $\tilde{F}$ is a function of $r^{-D+1}=r^{-2k}$, f^2 is an even function. Combined with the fact that $f^2(0)$ is finite, this shows that $\partial f^2/\partial r$ vanishes at the origin. Therefore the black-hole temperature given in Eq. (22) approaches zero as $r_h\to 0$. Unfortunately the final state is no longer a black hole since the hypersurface $r=0$ is singular. One might say that the horizon vanishes leaving a lightlike singularity (i.e., the tangent vectors to the singularity are spacelike except for one lightlike direction). We will show in the next section though that the lifetimes of these black holes are infinite.

The second case in which zero temperature can be produced is when two horizons coalesce to produce a degenerate horizon. This is an obvious possibility for the theories where a number of horizons are possible. In this case, the final state is a black hole with an actual (degenerate) horizon at a finite radius, and so may be regarded as a zero-temperature soliton rather like an extremal Reissner-Nordström or an extremal Kerr black hole.

IV. DISCUSSION

Our discussion of black-hole solutions for the general Lovelock theory slightly expands that of Ref. 8 by including curved backgrounds (i.e., $F_0\neq 0$ which may even occur when $c_0=0$). It was also pointed out that the statement made there that solutions in a flat background can have at most one horizon is incorrect. In principle it is possible to construct theories in which the solutions have any number of horizons by adding more higher-derivative interactions and increasing the space-time dimension. Hence much more complicated global topologies are possible for these theories than appear for solutions of Einstein's equations.

In the four-derivative theory examined in Sec. II, there is a class of solutions without naked singularities, which were previously overlooked—the asymptotically Schwarzschild—de Sitter solutions, where $\epsilon=+1$, $\hat{\lambda}<0$, and $\omega_c<\omega<0$. This range of parameters was probably not explored in the past because the solutions suffer from two defects. First and most obvious is that there is a large background cosmological constant, proportional to $-1/\hat{\lambda}$. The second is that gravitons in this background are ghosts (i.e., negative-energy particles).[7] This leads to the unusual result that the black holes above have positive gravitational mass but negative inertial mass. If the associated Hawking radiation consisted only of gravitons, the radiation would carry away negative inertial mass and ω would evolve towards zero. (This assumes that the temperature of the event horizon is higher than that of the cosmological horizon. This is always true for the solutions in the four-derivative theory, but it need not be the case for de Sitter background solutions in some of the higher-derivative theories.) This makes a self-consistent picture if gravity is the only field in the theory. If matter fields satisfying a positive-energy condition[19] are included, their contribution to the Hawking radiation would tend to drive ω down to ω_c. In this case, these solutions and the de Sitter vacuum are unstable against the production of gravitational radiation and hence are not suitable backgrounds for reliable field-theory calculations. They may still play a role as extrema in a quantum path integral over geometries.

With regard to black-hole thermodynamics, the most interesting result was that in some cases the black-hole temperature vanishes at a finite mass. This behavior always occurs for asymptotically flat solutions in $D=2k+1$ for the Lagrangians given in Eq. (2), which include $\mathcal{L}_k$ and have coefficients c_m which allow the solutions to be extended to the origin. In the four-derivative theory discussed in Sec. II, this corresponds to the case of $D=5$ and $\lambda>0$. Then from Eq. (10), one finds that

$T \sim (\omega - \hat{\lambda}/2)^{1/2}$ and hence vanishes as $\omega \to \omega_0 = \hat{\lambda}/2$. Unfortunately it is in precisely this limit that the horizon vanishes revealing the naked singularity which occurs for $\omega \le \hat{\lambda}/2$. One may ask how much time is required to reach ω_0 by black-hole evaporation. The rate of mass decrease is given by the luminosity $L \sim \mathcal{A} T^D$, where $\mathcal{A}$ is the area of the horizon. [Note that $\mathcal{A}$ has dimensions $(\text{length})^{D-2}$.] One then finds

$$\frac{d\omega}{dt} \sim -r_h^3 T^5 \sim -(\omega - \omega_0)^4 ,$$

which may be integrated to yield

$$\Delta t \sim (\omega - \omega_0)^{-3} \Big|_{\omega_1}^{\omega_0} \to \infty .$$

Therefore it requires an infinite amount of time for the black hole to evaporate down to ω_0 beginning at any mass ω_1 for which the temperature is finite. In this theory, the black holes would always exist as potential reservoirs of information for external observers, evading any violations of unitary time evolution.

Similar results are found in the asymptotically flat solutions of the higher-derivative theories with $\mathcal{L}_k$ in $D = 2k+1$. The temperature vanishes for $\omega_0 \simeq \hat{c}_k$, but the horizon also vanishes in this limit leaving a naked singularity. Further investigation reveals that if $\hat{c}_n / \hat{c}_k > 0$,

$$\frac{d\omega}{dt} \sim |\omega - \omega_0|^{\beta} \tag{23}$$

with $\beta = 2k+1-1/(k-n)$ where $\hat{c}_n$ is the next nonvanishing coefficient in $P(F)$. Integrating Eq. (23), then shows that the black holes have an infinite lifetime. Unfortunately we have no physical insight into the mechanism by which the temperature of these black holes vanishes. It appears as simply an accident of numerology that yields finite f^2 at $r=0$ for $D = 2k+1$, and the interesting thermodynamics follows as a result of this fact. In the theories with $k > 2$, the complete analysis is complicated by the fact that more than one horizon may occur for $\omega \simeq \omega_0$ (as is the case for $\hat{c}_n / \hat{c}_k < 0$). Therefore one must determine whether the horizon with $r_h \to 0$ is relevant for external observers. Similar complications arise for nonasymptotically flat solutions with $F_0 \ne 0$.

The second possibility for vanishing temperatures was by the occurrence of a degenerate event horizon. In the four-derivative theory discussed in Sec. II, this possibility was only realized by a singular solution with a single cosmological horizon in a de Sitter background. For some of the more general theories though, it is possible to find asymptotically flat solutions with more than one nondegenerate horizon. An appropriate choice of the mass parameter ω will then yield a degenerate horizon, and hence this solution will be a nonsingular zero-temperature soliton. More general analysis of this case is difficult because finding the solutions becomes increasingly complicated. By Galois's famous result the roots of a generic kth-order polynomial are soluble in terms of radial expressions only for $k \le 4$. Therefore, analytic solutions for the metric coming from Eq. (17) only occur for generic cases of the four-, six-, and eight-derivative theories. We have studied the six-derivative theory and found examples of degenerates horizons.[22]

It should be pointed out that these solutions with degenerate horizons actually occur as the final states of the black-hole evaporation process. One can argue this fact by first noting from Eq. (21) that if $(f^2)_{,r}$ has a zero it is determined entirely by the coefficients c_m appearing in the Lagrangian. Equation (17) then shows that the mass parameter ω sets the scale for $\tilde{F}$, and hence, with an appropriate choice $\omega \equiv \omega_0$, the zeros of f^2 and $(f^2)_{,r}$ can be made to coincide. Now in most cases the existence of horizon requires $\tilde{F} > 0$ which in turn requires $P'(0) > 0$ and $\omega > 0$ or $P'(0) < 0$ and $\omega < 0$. In either case Hawking radiation in the form of gravitons drives $\omega \to 0$; recall that in the latter case, both the black hole and the gravitons possess negative inertial mass. So if one begins with a solution with a large value of $|\omega| > |\omega_0|$, where large is defined as yielding $f^2 < 0$ where $(f^2)_{,r} = 0$. Then the black-hole evaporation process will end with a degenerate horizon at $\omega = \omega_0$. For $F_0 > 0$ there also exists the possibility that a number of horizons occur when $\tilde{F} < 0$. This case requires $P'(0) > 0$ and $\omega < 0$ or $P'(0) < 0$ and $\omega > 0$, and so Hawking radiation due to gravitons drives $|\omega|$ to larger values and the evaporation process ends when the cosmological and event horizons coalesce. For the previous case, as the black hole approaches the soliton solution, one may show that

$$T \sim (\omega - \omega_0)^{\alpha} ,$$

where $\alpha \ge \frac{1}{2}$. The time required to reach the final state is then infinite.

These solutions with a zero-temperature limit also display the interesting property that in the final stages of the evaporation process the specific heat is positive. This allows these black holes to come into stable equilibrium with an external thermal bath. For black-hole solutions of Einstein's equations, such stable equilibria have only been found for black holes confined to systems of finite size.[23,24] In the four-derivative theory, the only case with $\partial\omega/\partial T > 0$ was $D=5$ and $\lambda > 0$, which had the zero-temperature limit for $\omega \to \hat{\lambda}/2$. For more general theories there exist cases with $\partial\omega/\partial T > 0$ without a zero-temperature limit. This would be sufficient to avoid complete black-hole evaporation as long as the external universe ended in a thermal state with a sufficiently higher temperature.

Although black holes for many of the Lovelock theories have desirable thermodynamic properties, such behavior is by no means universal. In fact often, the new interactions accelerate the evaporation process. Consider the early stages of the evaporation of an asymptotically flat black hole for the four-derivative theory. Defining $\omega \equiv \varpi^{D-3}$, one is considering $\varpi^2 \gg \hat{\lambda}$. One finds that

$$T \simeq \frac{D-3}{4\pi\varpi} \left[1 - \frac{D-2}{D-3} \frac{\hat{\lambda}}{2\varpi^2} \right]$$

and

$$\frac{d\omega}{dt} \sim -\frac{1}{\varpi^2} \left[1 - \frac{(D+1)(D-2)}{D-3} \frac{\hat{\lambda}}{2\varpi^2} \right] .$$

In these formulas, the factor in parentheses gives the leading correction to the usual Einstein result given by the first factor. Therefore for $\hat{\lambda}>0$, the higher-derivative corrections slow down the early stages of evaporation while for $\hat{\lambda}<0$, they accelerate the process. The same results hold for the general theories since the four-derivative interactions will produce the leading corrections in the regime considered here. One need only substitute $\hat{\lambda}=2\hat{c}_2$ to be consistent with the notation of Sec. III.

Perhaps even more interesting is examining the final stages of the black-hole evaporation when the black hole or the horizon vanishes. Once again begin with asymptotically flat solutions in the four-derivative theory. For $\hat{\lambda}>0$ and $D>5$, ω approaches zero in the final stages, and $d\omega/dt \sim -(\omega/\hat{\lambda})^{-2/(D-5)}$ which diverges faster than for the usual Einstein rate, $d\omega/dt \sim -\omega^{-2/(D-3)}$. For a $(2k)$-derivative theory in $D>2k+1$ in which the solutions are extended to the origin with a single horizon, $d\omega/dt \sim -\omega^{-2/[D-(2k+1)]}$ as ω approaches zero in the final stages of evaporation. Thus the evaporation process is again accelerated by the higher-derivative interactions. In the four-derivative theory with $\hat{\lambda}<0$, $\omega \rightarrow \omega_c = |\hat{\lambda}|^{(D-3)/2}/2$, and $r_h^2 \rightarrow |\hat{\lambda}|$ at which point the horizon vanishes leaving a naked singularity. One finds that $d\omega/dt \sim -(\omega-\omega_c)^{-D}$, and so the naked singularity is produced in a finite amount of time. For the case where a singularity at finite radius and a single horizon occur in the general case, the evaporation process also reaches a naked singularity in a finite amount of time.

Given a theory with a zero-temperature solution at $\omega=\omega_0$, one may wish to consider solutions with $|\omega|<|\omega_0|$. Typically such solutions will have a nonzero temperature, and evaporate down to a singular final state or flat space. These soultions will be small Planck-size black holes which would not arise in a normal stellar collapse, but one might imagine that they could arise in some quantum gravity process. In such a case, it appears that there would still be some quantum incoherence due to the evaporation of these Planckian black holes.

Our definition of the black-hole entropy (11), which may be applied equally well in any of the higher-derivative theories, arises from a thermodynamic definition having identified the Euclidean action with the free energy. Therefore after requiring thermodynamic equilibrium, the first law of black-hole mechanics follows:

$$\delta S = \beta\, \delta M \ .$$

This result makes our choice for the entropy credible. At this point though, we must stress again that the definition (11) is only tentative. We have not proven the second law of black-hole mechanics in the context of these new theories (i.e., $\delta S \geq 0$). One of the basic assumptions used to prove this relation in Einstein gravity,[25] the weak-energy condition,[19] is violated by the effective stress-energy tensor in Lovelock gravity.[7] Unfortunately proving the zeroth law, which states that the surface gravity is constant over the horizon, relies on an even stronger condition, the dominant energy condition.[19] In the solutions considered here, this constancy is enforced by spherical symmetry. *A priori* though, there is no apparent reason that it should be true for a spinning black hole in these theories, in which case one could not interpret the black hole as a thermal bath.

In the four-derivative theory, one finds that the entropy (13) always increases with ω or the mass, just as for a black hole in Einstein gravity. Unlike Einstein's theory though, the entropy in Eq. (13) or for any of the higher-derivative theories is *not* one-quarter of the horizon area except approximately for $r_h^2 \gg |\hat{\lambda}|$ or equivalently for large mass. Therefore the usual intuitive notion that the entropy is a property of the horizon of the black hole fails for these theories. The large deviation of the entropy from the usual Einstein result for $r_h^2 \lesssim |\hat{\lambda}|$ will be important in examining the final stages of black-hole evaporation.[26]

For $\hat{\lambda}<0$, the result in Eq. (13) has the problem that it becomes negative for small ω. This is undesirable from the point of view of statistical mechanics, where $S=-\sum_i P_i \ln P_i$ is a positive-definite quantity. More generally though, one should regard entropy as a relative quantity. When a black hole in Einstein's theory is assigned $S=\mathcal{A}/4G$, implicitly this is relative to $S=0$ for flat empty space. The source of the problem for the four-derivative theory with $\hat{\lambda}<0$ may be that there is not a continuous family of nonsingular solutions which connect a black hole of mass ω to flat space. The solutions considered here become singular when $r_h^2 \rightarrow |\hat{\lambda}|$. Hence one should only apply Eq. (13) to determine the entropy of one black hole relative to another. Perhaps by a careful examination of a series of systems containing a gas of gravitons in flat and black-hole backgrounds, one may be able to deduce the entropy of the black holes relative to flat space.

It would be of interest to consider modifications of our results by the addition of matter fields to Lovelock gravity. Including an electromagnetic field, one may find charged black-hole solutions by simply replacing Eq. (18) by[9]

$$\tilde{P}(\tilde{F}) = \frac{\omega}{r^{D-1}} - \frac{q^2}{r^{2D-4}} \ ,$$

where the charge is proportional to q and $F_{tr} \sim q/r^{D-2}$. For $q \neq 0$ and the special case of theories in $D=2k+1$ including interactions up to $\mathcal{L}_k$, one easily sees that the temperature no longer vanishes as described above. One still expects that zero-temperature degenerate horizons will occur for $q \simeq \omega$ as in an extreme Reissner-Nordström black hole at least for large mass.[9] Also one may construct zero-temperature black holes similar to the cases with degenerate horizons at finite radius described above by choosing q sufficiently small. It would be useful to explore black-hole thermodynamics in these regimes in more detail as well as for $q \gtrsim \omega$.

Many of the theories studied here lead to desirable black-hole thermodynamics and may evade any violations of unitarity time evolution. [One should note that the simple considerations presented in the Introduction seem to be in contradiction with the idea that the remnant black holes have little internal structure as implied by their small entropy (which remains true in the four-derivative theory at least). It seems implausible then that there could be a sufficient number of internal states to totally correlate the emitted radiation.[27]] At the very least then, this demonstrates concretely how black-hole evaporation might qualitatively differ from the naive predictions derived from Einstein's theory. Might we infer any further lessons from the present studies for more serious candidates for the theory of quantum gravity? We expect the theory to have two particular properties of relevance. First no new scales are envisaged to appear, and so any higher-derivative interactions will appear with coefficients on the order of the Planck scale. We should then consider any nonvanishing c_m in Eq. (2) to be of the order (Planck length)$^{2m-D}$. It is encouraging that the zero-temperature solutions of the six-derivative theory in eight dimensions occur precisely in this regime. The second expected property is that the theory will include an infinite number of higher-derivative interactions, as appear in a perturbative-field-theory approach or an effective low-energy string action. This would suggest that black-hole solutions would have a large number of horizons, and yield a degenerate event horizon for the appropriate mass, as we saw occurred for the Lovelock theories with large numbers of higher-derivative interactions. For a practical calculation, one would attempt to solve including only a finite number of these interactions. If the temperature is found to vanish for some solution, the contributions of the higher-derivative terms must equal that of the Einstein tensor at the horizon. Despite the fact that the relevant physics may only involve finite curvature, one should expect that the neglected higher-derivative interactions will make equally important corrections, rendering the calculation invalid. It would require a new insight into the theory to determine a cutoff in the number of interactions beyond which the higher-order terms would add only small corrections. Therefore it will be difficult to produce reliable results for a reasonable candidate theory, but it may still be possible that the problems quantum gravity faces due to the black-hole evaporation are solved by the higher-derivative interactions appearing in the full effective action.

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[1]S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[2]See, for example, G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré **20**, 69 (1974); P. van Nieuwenhuizen, Ann. Phys. (N.Y.) **104**, 197 (1977); S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974); **10**, 411 (1974); S. Deser, H. T. Tsao, and P. van Nieuwenhuizen, *ibid.* **10**, 3337 (1974); M. T. Grisaru, P. van Nieuwenhuizen, and J. A. M. Vermaseren, Phys. Rev. Lett. **37**, 1662 (1976); **37**, 1669 (1976); S. Deser, J. Kay, and K. Stelle, *ibid.* **38**, 527 (1977); M. H. Goroff and A. Sagnotti, Nucl. Phys. **B266**, 709 (1986).

[3]See, for example, R. I. Nepomechie, Phys. Rev. D **32**, 3201 (1985); D. J. Gross, J. Harvey, E. Martinec, and R. Rohm, Nucl. Phys. **B267**, 75 (1986); D. J. Gross and E. Witten, *ibid.* **B277**, 1 (1986); C. G. Callan, E. J. Martinec, M. J. Perry, and D. Friedan, *ibid.* **B262**, 593 (1985); M. T. Grisaru, A. E. M. van de Ven, and D. Zanon, Phys. Lett. B **173**, 423 (1986); Nucl. Phys. **B277**, 388,409 (1986); M. T. Grisaru and D. Zanon, Phys. Lett. B **177**, 347 (1986); J. H. Sloan and D. J. Gross, Nucl. Phys. **B291**, 41 (1987).

[4]See, for example, R. Uityama and B. S. DeWitt, J. Math. Phys. **3**, 608 (1962); S. Deser, in *Gauge Theory and Modern Field Theory,* proceedings of the Boston Conference, 1975, edited by R. Arnowitt and P. Nath (MIT Press, Cambridge, MA, 1976); K. S. Stelle, Phys. Rev. D **16**, 953 (1977); A. Salam and J. Strathdee, *ibid.* **18**, 4480 (1978); E. T. Tomboulis, Phys. Lett. **70B**, 361 (1977); **97B**, 77 (1980); B. Hasslacher and E. Mottola, *ibid.* **99B**, 221 (1981); F. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B201**, 469 (1982); I. Antoniadis and E. T. Tomboulis, Phys. Rev. D **33**, 2756 (1986); D. A. Johnston, Nucl. Phys. **B297**, 721 (1988).

[5]D. Lovelock, J. Math. Phys. **12**, 498 (1971); **13**, 874 (1972).

[6]B. Zwiebach, Phys. Lett. **156B**, 315 (1985); B. Zumino, Phys. Rep. **137**, 109 (1986).

[7]D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).

[8]J. T. Wheeler, Nucl. Phys. **B268**, 737 (1986); **B273**, 732 (1986).

[9]D. L. Wiltshire, Phys. Lett. **169B**, 36 (1986).

[10]C. G. Callan, R. C. Myers, and M. J. Perry (unpublished).

[11]R. C. Myers, Nucl. Phys. **B289**, 701 (1987).

[12]R. C. Myers, Phys. Rev. D **36**, 392 (1987).

[13]R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).

[14]J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976).

[15]G. W. Gibbons and M. J. Perry, Proc. R. Soc. London **A358**, 476 (1978); Phys. Rev. Lett. **36**, 985 (1976).

[16]B. Carter, in *General Relativity,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1985).

[17]Hawking, in *General Relativity* (Ref. 16).

[18]R. D. Sorkin, in Proceedings of the American Mathematical Society Conference, Santa Cruz, 1986.

[19]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1980).

[20]G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).

[21]F. Müller-Hoissen, Phys. Lett. **163B**, 106 (1985).

[22]R. C. Myers and J. Z. Simon, ITP report, 1988 (unpublished).

[23]S. W. Hawking, Phys. Rev. D **13**, 191 (1976).

[24]J. W. York, Phys. Rev. D **33**, 2092 (1986).

[25]S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).

[26]M. Bowick, L. Smolin, and L. C. R. Wijewardhana, Phys. Rev. Lett. **56**, 424 (1986); Gen. Relativ. Gravit. **19**, 113 (1987).

[27]See, for example, R. M. Wald, in *Quantum Gravity 2*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981).

[28]B. Whitt, Phys. Rev. D (to be published).

[29]D. L. Wiltshire, following paper, Phys. Rev. D **38**, 2445 (1988).