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Implicit Energy Cost of Feedback in Noisy Channels

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Abstract

We investigate the energy cost of negative feedback in first and second order linear systems. Previous work demonstrated that, in noisy physical channels, tradeoffs between information and power lead to an energy cost for information. In this paper we show that, for a fixed signal power and output noise, the introduction of negative feedback reduces capacity for first and second order lowpass linear systems. This capacity reduction constitutes a fundamental energy cost associated with the use of feedback in linear systems.

1 Introduction

Noise is an omnipresent feature of the real world, guaranteed to exist in any dissipative technology by fluctuation-dissipation theorems [6], also known as generalized Nyquist relations [9]. Many interesting and important technologies are inherently noisy – including biological transducers and electronic circuits. While invertible transformations such as linear filtering operations produce no change in information capacity, irreversible transformations such as corruption by noise fundamentally limit the information that a channel is able to represent. The presence of noise ultimately sets a minimum energy for even trivial channel uses.

In previous work [1, 2, 3, 4] we examined the energy cost of information in biological and silicon photoreceptors. The capacity and energy cost of information in the channel was determined from basic physical principles. While biological and silicon photoreceptors are nonlinear systems, for many visual tasks their behavior is well modelled as linearized for small signals about an operating point. That operating point is determined by the background light level; the feedback mechanism which implements the adaptive gain control and determines the operating point is the subject of this paper.

Feedback is a simple and common example of both control and adaptation in a physical system. Adaptation allows sensory systems to: represent a large range of input signals while maintaining a moderate dynamic range internally; exhibit high performance despite uncertainty and component mismatch; and adjust to regularities in the input so that output signals are maximally informative. Traditional measures of performance such as stability, sensitivity, and accuracy of systems employing feedback have been studied extensively and can easily be found in basic textbooks. In recent years several authors have addressed control in the context of communication constraints [11, 8]. Here we study another aspect of the interaction between control and communication, examining the energy cost of simple feedback control in a physical channel.

1.1 Energy Cost of information

We consider linear time invariant (LTI) systems with signal and noise inputs normally distributed.

1.1.1 Channel capacity: We quantify system performance using the Shannon channel capacity, defined as the maximimum of the mutual information $I(X;Y)$ between the input and the output of the channel. For the additive white Gaussian noise channel with an average signal power constraint P this becomes $C = F\log_2(1+P/N)$ where N is the noise power, with the capacity C in units of bits per second. For colored Gaussian noise is colored, the capacity is given by [10]:

$$C = \max_{X(f):\sigma_X^2 \le P} \int_0^\infty \log_2\left(1+\frac{X(f)}{N(f)}\right) df \qquad (1)$$

where P is the input power constraint, $X(f)$ is the power spectral density of the signal (given by the Fourier transform of the autocorrelation), and $N(f)$ is the power spectral density of the noise. The signal that maximizes the capacity can be found by "water-filling"; typically the input-referred noise power spectral density is cup-shaped, and finding the total output power spectral density corresponds to putting a fixed amount of water in the cup. The basic idea is that signal energy is concentrated at frequencies where noise is low [7].

1.1.2 Power: We quantify system resources, or costs, using the power P_{sys} required by the system for signal transduction. Usually this is not the same as the signal power P, which can be substantially smaller depending on the system implementation. In particular, for linearized systems the signal is variation about an operating point. For many systems the power cost of signal transduction is related to the total signal rather than to the small increments which encode information.

Power constraints can arise in a number of ways, from material properties of the system to properties of the environment to statistical properties of the task. Ex-

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amples include: battery-powered operation, which imposes the constraints of finite voltage rails, a maximum current which can be delivered, and finite overall power available; and operation from a renewable power source such as wind, solar, or kinetic energy that provides a power source that can be delivered at a fixed rate, perhaps with seasonal variation in the power available. Power is also constrained by the ability of a material to handle high energy densities; often a material suffers irreparable damage if the current density or temperature exceeds some critical bound which is a function of the material properties. Power is also constrained by the ability of a system to dissipate away the energy that is produced as heat, for example by conduction, convection, or radiation.

1.1.3 Bit-Energy: Our measure of performance, the capacity, and our measure of cost, the power, allow us to define a measure for efficiency, the bit-energy, which is simply the ratio between the power dissipated and the information capacity. The bit-energy gives the minimum power required to transmit a single bit of information through the system: it provides a standard for comparing the efficiency of communication among different technologies [5].

$$BE = \frac{P_{sys}}{C} \tag{2}$$

1.2 Feedback cost

The energy cost of feedback control comprises explicit costs associated with the physical implementation of negative feedback in addition to implicit costs incurred by reduction of the channel capacity of the linear system. The physical implementation of negative feedback necessitates additional resources if additional circuitry is used to implement the feedback. In a trivial sense feedback requires additional power to bias the circuits as well as additional space and weight. These trivial costs associated with feedback depend on the particular implementation of the system under study.

Feedback incurs cost in a subtler sense as well. Our previous work has explored the connection between information and power in several physical systems, with the general result that there are tradeoffs between information and power – higher information rates and higher channel capacities are possible when a system is allocated more power. Limitations in power imply limitations in information, and the challenge facing any low power system designer is to get maximum performance given limited resources. As defined above, the bit-energy is the minimum power required to transmit a bit of energy through the channel. When feedback reduces the capacity of a channel, this increases the bit-energy of that channel – the increment in bit-energy is the implicit cost of feedback. This basic result pertains regardless of the specific implementation of feedback in the system, whether it requires additional circuitry and power consumption or not.

We neglect the explicit costs associated with specific feedback implementations in order to focus on the implicit costs related to channel capacity of the linear system.

2 Main Results

Consider the negative feedback system depicted in Figure 1, with input signal S and additive output noise N. In general, physical systems will also have noise at the input, but we do not consider that additional contribution here. Input noise contributions limit capacity for any system regardless of system characteristics such as feedback, gain, or time constants; the capacity of a system with signal power constraint P and white Gaussian input noise N is $P/(\ln 2N$ bits/sec.

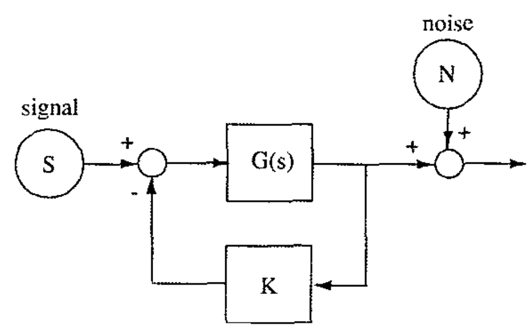

Figure 1: Linear system with negative feedback and additive output noise

We neglect input noise in order to more clearly demonstrate the influence of feedback in determining channel capacity and ultimately energy cost. We assume that the input signal and additive output noise are Gaussian in distribution, that the power spectral density of the output noise is uniform in frequency, and that the system is linear with feedforward tranfer function $G(s)$ and feedback gain K. For the special case when the feedback gain K is zero, the model shown in Figure 1 also represents a system without feedback; in that case the transfer function from input to output is given by $G(s)$. The transfer function from input to output of the linear system with feedback is given by $G(s)/(1 + KG(s))$.

2.1 Capacity of first order system

Consider a linear system with first order lowpass transfer function $G(s) = A/(\tau s + 1)$ with gain A and time constant τ and with white Gaussian output noise of variance N. In order to determine the capacity, we compute the input-referred noise, then use waterfilling to find the signal spectrum satisfying the signal power constraint P which provides maximal information rate, or capacity, in this channel, and calculate the capacity for that signal.

The input-referred noise spectrum is:

$$N_{in}(f) = \frac{N}{A^2}\left[1 + (2\pi\tau f)^2\right] \tag{3}$$

The channel capacity is attained using a normally distributed signal in which power is concentrated in those spectral regions where the noise is lowest, with the total power of signal plus noise constant in those regions of the spectrum where signal power is allocated [7]. The total signal power is the integral of this allocated signal power spectrum, equivalently described by an integral of the total power minus the noise power:

$$P = \int_0^{f_s} \left[SN - \frac{N}{A^2}\left[1 + (2\pi\tau f)^2\right]\right] df \tag{4}$$

$$= \frac{2}{3}(2\pi\tau)^2 \frac{N}{A^2} f_s^3 \tag{5}$$

where the signal is allocated in the frequency band from 0 to f_s, and $SN = (N/A^2)\left[1 + (2\pi\tau f_s)^2\right]$ is the total signal and noise power, constant over the signal bandwidth. The channel capacity is the integral of the logarithm of the signal to noise ratio, equivalently described by an integral of the logarithm of the total signal plus noise power minus the logarithm of the noise power:

$$C = \int_0^{f_s} \log_2\left[\frac{SN}{N_{in}}\right] df \tag{6}$$

$$= \log_2\left[1 + (2\pi\tau f_s)^2\right] f_s - \int_0^{f_s} \log_2\left[1 + (2\pi\tau f)^2\right] df \tag{7}$$

$$= \frac{2}{\ln 2} f_s \left[1 - \frac{\arctan 2\pi\tau f_s}{2\pi\tau f_s}\right] \tag{8}$$

$$= \frac{\sqrt[3]{\frac{3A^2P}{(\pi\tau)^2N}}}{\ln 2}\left[1 - \frac{\arctan\sqrt[3]{\frac{6\pi\tau A^2 P}{N}}}{\sqrt[3]{\frac{6\pi\tau A^2P}{N}}}\right] \tag{9}$$

The capacity is shown in Figure 2 as a function of the time constant τ and the ratio $SN/N(0) = PA^2/N$ of signal power to noise power evaluated at zero frequency. Shading is determined by logarithm of the capacity, with lighter shades indicating larger values; the solid black lines are contour lines at constant capacity.

2.2 Effect of feedback on capacity

The closed loop transfer function of the LTI system with negative feedback shown in Figure 1 is $G_{fb} = G(s)/(1 + KG(s))$. For the first order lowpass transfer function $G(s) = A/(\tau s + 1)$ with scalar feedback K this is again a first order lowpass system:

$$G_{fb}(s) = \frac{\frac{A}{\tau s+1}}{1 + \frac{AK}{\tau s+1}} \tag{10}$$

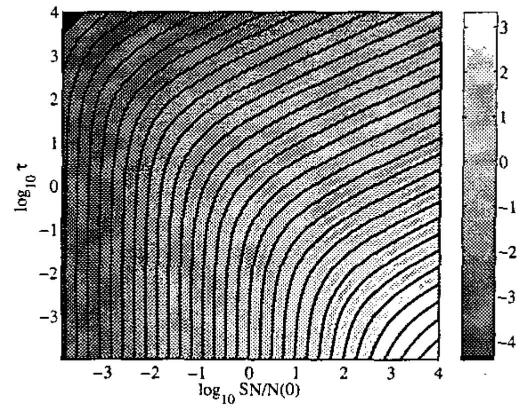

Figure 2: Capacity of first order lowpass system as a function of time constant τ and ratio of signal power to DC noise power PA^2/N

$$= \frac{A}{\tau s + 1 + AK} = \frac{\frac{A}{1+AK}}{\frac{\tau}{1+AK}s + 1} \tag{11}$$

The introduction of feedback can be modelled as a transformation of the gain and time constant of the original feedforward system:

$$A \rightarrow A' = \frac{A}{1+AK} \tag{12}$$

$$\tau \rightarrow \tau' = \frac{\tau}{1+AK} \tag{13}$$

Under this transformation the relationship defining the signal bandwidth f_s remains unchanged, and the capacity C_{fb} of the feedback system becomes:

$$C_{fb} = \frac{\sqrt[3]{\frac{3A'^2P}{(\pi\tau')^2N}}}{\ln 2}\left[1 - \frac{\arctan\sqrt[3]{\frac{6\pi\tau' A'^2P}{N}}}{\sqrt[3]{\frac{6\pi\tau' A'^2P}{N}}}\right] \tag{14}$$

$$= \frac{\sqrt[3]{\frac{3A^2P}{(\pi\tau)^2N}}}{\ln 2}\left[1 - \frac{\arctan\frac{\sqrt[3]{\frac{6\pi\tau A^2P}{N}}}{1+AK}}{\frac{\sqrt[3]{\frac{6\pi\tau A^2P}{N}}}{1+AK}}\right] \tag{15}$$

For $K > 0$ the capacity of the first order lowpass system with feedback is less than the capacity of the feedforward system (with equality for $K = 0$). The difference in capacity between the feedforward and feedback systems is:

$$C - C_{fb} = \frac{(1+AK)\arctan\frac{\sqrt[3]{\frac{6\pi\tau PA^2}{N}}}{1+AK} - \cdots \arctan\sqrt[3]{\frac{6\pi\tau PA^2}{N}}}{\ln 2\pi\tau\sqrt[3]{2}} \tag{16}$$

The product $(C - C_{fb})\tau$ is plotted in Figure 3 as a function of the feedback gain AK and the factor $\sqrt[3]{6\pi\tau PA^2/N}$. The reduction in capacity equates to

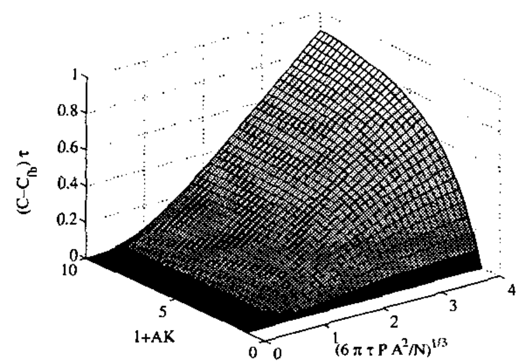

Figure 3: Decrease in capacity $(C - C_{fb})/\tau$ as a function of feedback gain AK and $\sqrt[3]{6\pi\tau PA^2/N}$

an increase in bit energy of the system. This increase in bit energy is the cost of feedback for a first order lowpass system:

$$\Delta BE = BE_{fb} - BE = \frac{P_{sys}}{C_{fb}} - \frac{P_{sys}}{C} \tag{17}$$

$$= P_{sys} \frac{(1+AK)\arctan \frac{\sqrt[3]{\frac{6\pi\tau PA^2}{N}}}{1+AK} - \cdots \arctan \sqrt[3]{\frac{6\pi\tau PA^2}{N}}}{\ln 2\pi\tau\sqrt[3]{2}(CC_{fb})} \tag{18}$$

2.3 Capacity of second order system

Consider an LTI system with second order lowpass transfer function $G(s) = Aw_n^2/(s^2 + 2\xi w_n s + w_n^2) = A/(\tau^2 s^2 + 2\xi\tau s + 1)$ with gain A, time constant τ and damping factor ξ and with white Gaussian output noise of variance N. Given the signal power constraint P, we compute the input-referred noise, then use waterfilling to find the signal spectrum which provides capacity, and finally calculate the capacity for that signal.

The input-referred noise spectrum is:

$$N_{in}(f) = \frac{N}{A^2}\left[(2\pi\tau f)^4 + \cdots 2(2\xi^2-1)(2\pi\tau f)^2 + 1\right] \tag{19}$$

The signal power is an integral of the total power minus the noise power:

$$P = \int_0^{f_s}\left[SN - \frac{N}{A^2}\left[(2\pi\tau f)^4 + \cdots 2(2\xi^2-1)(2\pi\tau f)^2 + 1\right]\right] df \tag{20}$$

$$= \frac{4}{5(2\pi\tau)}\frac{N}{A^2}\left[(2\pi\tau f_s)^5 + \cdots \frac{5}{3}(2\xi^2-1)(2\pi\tau f_s)^3\right] \tag{21}$$

where the signal power is allocated in the frequency band from 0 to f_s, and $SN = (N/A^2)\left[1 + 2(2\xi^2-1)(2\pi\tau f_s)^2 + (2\pi\tau f_s)^4\right]$ is the total signal and noise power, constant over the signal bandwidth. The frequency f_s which specifies the signal bandwidth is determined as a solution of Equation 21.

For underdamped systems with $\xi < 1/\sqrt{2}$, the input-referred noise spectrum is not a monotonically increasing function of frequency; it exhibits a global minimum in the range $[0, \sqrt{2(1-2\xi^2)}/(2\pi\tau)]$. For small values of the signal power, the total signal and noise power in the channel remains smaller than the noise power at low frequencies. In this case, the signal power is allocated between two frequencies, f_1 and f_2, $0 \le f_1 < f_2 \le \sqrt{2(1-2\xi^2)}/(2\pi\tau)$. The frequencies f_1 and f_2 are determined by solving the set of two equations:

$$P = \frac{4}{5(2\pi\tau)}\frac{N}{A^2}\left[(2\pi\tau f_2)^5 - (2\pi\tau f_1)^5 + \cdots \frac{5}{3}(2\xi^2-1)\left[(2\pi\tau f_2)^3 - (2\pi\tau f_1)^3\right]\right] \tag{22}$$

$$0 = (2\pi\tau f_2)^4 - (2\pi\tau f_1)^4 + \cdots 2(2\xi^2-1)\left[(2\pi\tau f_2)^2 - (2\pi\tau f_1)^2\right] \tag{23}$$

When the signal power is large ($P > 8\sqrt{2}N\left(1-2\xi^2\right)^{5/2}/(30\pi\tau A^2)$) or for systems with sufficient damping ($\xi \ge 1/\sqrt{2}$), the signal is allocated in the frequency band from 0 to f_s, with f_s determined by the solution to Equation 21.

The channel capacity takes several functional forms, depending on the value of the damping factor ξ. For overdamped or critically damped systems ($\xi \ge 1$), the capacity is very similar to the first order case:

$$C = \int_0^{f_s}\log_2\left[\frac{SN}{N_{in}}\right] df$$

$$= \frac{2f_s}{\ln 2}\left[1 - \frac{\arctan\frac{2\pi\tau f_s}{\sqrt{2\xi^2-1+\sqrt{(2\xi^2-1)^2-1}}}}{\frac{2\pi\tau f_s}{\sqrt{2\xi^2-1+\sqrt{(2\xi^2-1)^2-1}}}}\right] + \cdots \frac{2f_s}{\ln 2}\left[1 - \frac{\arctan\frac{2\pi\tau f_s}{\sqrt{2\xi^2-1-\sqrt{(2\xi^2-1)^2-1}}}}{\frac{2\pi\tau f_s}{\sqrt{2\xi^2-1-\sqrt{(2\xi^2-1)^2-1}}}}\right] \tag{24}$$

For underdamped systems ($0 \le \xi < 1$), the capacity takes a different form than the first order case:

$$C(f_s) = \frac{2f_s}{\ln 2}\left[1 - \frac{\arctan\frac{2\pi\tau f_s + \sqrt{1-\xi^2}}{\xi}}{\frac{2\pi\tau f_s}{\xi}}\right] + \cdots \frac{2f_s}{\ln 2}\left[1 - \frac{\arctan\frac{2\pi\tau f_s - \sqrt{1-\xi^2}}{\xi}}{\frac{2\pi\tau f_s}{\xi}}\right] - \cdots \frac{\sqrt{1-\xi^2}}{2\pi\tau} \times \cdots$$

$$\log_2\left[\frac{(2\pi\tau f_s)^2+2\sqrt{1-\xi^2}(2\pi\tau f_s)+1}{(2\pi\tau f_s)^2-2\sqrt{1-\xi^2}(2\pi\tau f_s)+1}\right] \quad (25)$$

In the case of small damping factor $0 \leq \xi < 1/\sqrt{2}$ and small channel power $P < 8\sqrt{2}N\left(1-2\xi^2\right)^{5/2}/(30\pi\tau A^2)$, the channel capacity is the difference between Equation 25 evaluated at the upper and lower frequencies.

$$\begin{aligned} C(f_1,f_2) &= C(f_2)-C(f_1) \quad (26)\\ &= \frac{4}{\ln 2}(f_2-f_1)+\frac{\xi}{(\ln 2)\pi\tau}\times\cdots\\ &\left[\arctan\frac{2\xi(2\pi\tau f_1)}{1-(2\pi\tau f_1)^2}-\cdots\right.\\ &\left.\arctan\frac{2\xi(2\pi\tau f_2)}{1-(2\pi\tau f_2)^2}\right]+\cdots\\ &\frac{\sqrt{1-\xi^2}}{2\pi\tau}\times\cdots\\ &\left\{\log_2\left[\frac{(2\pi\tau f_1)^2+2\sqrt{1-\xi^2}(2\pi\tau f_1)+1}{(2\pi\tau f_1)^2-2\sqrt{1-\xi^2}(2\pi\tau f_1)+1}\right]-\cdots\right.\\ &\left.\log_2\left[\frac{(2\pi\tau f_2)^2+2\sqrt{1-\xi^2}(2\pi\tau f_2)+1}{(2\pi\tau f_2)^2-2\sqrt{1-\xi^2}(2\pi\tau f_2)+1}\right]\right\} \quad (27)\end{aligned}$$

2.4 Capacity of feedforward versus feedback second order system

For the second order lowpass system $G(s)=A/(\tau^2s^2+2\xi\tau s+1)$ with scalar feedback K the closed loop transfer function $G_{fb}=G(s)/(1+KG(s))$ is again a second order lowpass system:

$$\begin{aligned} G_{fb}(s) &= \frac{\frac{A}{\tau^2s^2+2\xi\tau s+1}}{1+\frac{AK}{\tau^2s^2+2\xi\tau s+1}} \quad (28)\\ &= \frac{A}{\tau^2s^2+2\xi\tau s+1+AK} \quad (29)\\ &= \frac{\frac{A}{1+AK}}{\frac{\tau^2}{1+AK}s^2+2\frac{\xi}{\sqrt{1+AK}}\frac{\tau}{\sqrt{1+AK}}s+1} \quad (30)\end{aligned}$$

The introduction of feedback can be modelled as a transformation of the gain, time constant, and damping factor of the original feedforward system:

$$A \rightarrow A'=\frac{A}{1+AK} \quad (31)$$

$$\tau \rightarrow \tau'=\frac{\tau}{\sqrt{1+AK}} \quad (32)$$

$$\xi \rightarrow \xi'=\frac{\xi}{\sqrt{1+AK}} \quad (33)$$

Note that systems that were overdamped or critically damped without feedback can become underdamped when feedback is introduced. Under this transformation the relationship defining the signal bandwidth f_s becomes:

$$\begin{aligned} P &= \frac{4}{5(2\pi\tau)}\frac{N}{A^2}\left[(2\pi\tau f_s)^5+\cdots\right.\\ &\left.\frac{5}{3}(2\xi^2-1-AK)(2\pi\tau f_s)^3\right] \quad (34)\end{aligned}$$

For overdamped and critically damped systems ($\xi \geq \sqrt{1+AK}$), the capacity C_{fb} of the feedback system becomes:

$$\begin{aligned} C_{fb} &= \frac{2f_s}{\ln 2}\left[1-\frac{\arctan x_1}{x_1}\right]+\cdots\\ &\frac{2f_s}{\ln 2}\left[1-\frac{\arctan x_2}{x_2}\right] \quad (35)\end{aligned}$$

where

$$x_1 = \frac{2\pi\tau f_s}{\sqrt{2\xi^2-1-AK+\sqrt{(2\xi^2-1-AK)^2-(1+AK)^2}}} \quad (36)$$

$$x_2 = \frac{2\pi\tau f_s}{\sqrt{2\xi^2-1-AK-\sqrt{(2\xi^2-1-AK)^2-(1+AK)^2}}} \quad (37)$$

For underdamped systems ($0 \leq \xi < \sqrt{1+AK}$), the capacity C_{fb} of the feedback system becomes:

$$\begin{aligned} C_{fb}(f_s) = &\frac{2f_s}{\ln 2}\left[1-\frac{\arctan\frac{2\pi\tau f_s+\sqrt{1+AK-\xi^2}}{\xi}}{\frac{2\pi\tau f_s}{\xi}}\right]+\cdots\\ &\frac{2f_s}{\ln 2}\left[1-\frac{\arctan\frac{2\pi\tau f_s-\sqrt{1+AK-\xi^2}}{\xi}}{\frac{2\pi\tau f_s}{\xi}}\right]-\cdots\\ &\frac{\sqrt{1+AK-\xi^2}}{2\pi\tau}\times\cdots\\ &\log_2\left[\frac{(2\pi\tau f_s)^2+2\sqrt{1+AK-\xi^2}(2\pi\tau f_s)+1+AK}{(2\pi\tau f_s)^2-2\sqrt{1+AK-\xi^2}(2\pi\tau f_s)+1+AK}\right] \quad (38)\end{aligned}$$

In the case of small damping factor ($0 \leq \xi < \sqrt{(1+AK)/2}$) and small signal power ($P < 8\sqrt{2}N\left(1+AK-2\xi^2\right)^{5/2}/(30\pi\tau A^2)$), the input-referred noise power spectral density exhibits a global minimum in the range $[0, \sqrt{2(1+AK-2\xi^2)}/(2\pi\tau)]$. Because the total power is smaller than the noise power at low frequencies, the signal power is allocated between two frequencies, f_1 and f_2, $0 \leq f_1 < f_2 \leq \sqrt{2(1+AK-2\xi^2)}/(2\pi\tau)$. The frequencies f_1 and f_2 are determined by solving the set of two equations:

$$\begin{aligned} P &= \frac{4}{5(2\pi\tau)}\frac{N}{A^2}\left[(2\pi\tau f_2)^5-(2\pi\tau f_1)^5+\cdots\right.\\ &\left.\tfrac{5}{3}(2\xi^2-1-AK)\left[(2\pi\tau f_2)^3-(2\pi\tau f_1)^3\right]\right] \quad (39)\\ 0 &= (2\pi\tau f_2)^4-(2\pi\tau f_1)^4+\cdots\\ &2(2\xi^2-1-AK)\left[(2\pi\tau f_2)^2-(2\pi\tau f_1)^2\right] \quad (40)\end{aligned}$$

The channel capacity is the difference between Equation 38 evaluated at the upper and lower frequencies.

$$\begin{aligned} C_{fb}(f_1,f_2) &= C_{fb}(f_2)-C_{fb}(f_1)\\ &= \frac{4}{\ln 2}(f_2-f_1)+\frac{\xi}{(\ln 2)\pi\tau}\times\cdots\\ &\left[\arctan\frac{2\xi(2\pi\tau f_1)}{K_2-(2\pi\tau f_1)^2}-\cdots\right.\\ &\left.\arctan\frac{2\xi(2\pi\tau f_2)}{K_2-(2\pi\tau f_2)^2}\right]+\cdots\end{aligned}$$

$$\frac{K_1}{2\pi\tau} \times \cdots$$
$$\left\{\log_2\left[\frac{(2\pi\tau f_1)^2+2K_1(2\pi\tau f_1)+K_2}{(2\pi\tau f_1)^2-2K_1(2\pi\tau f_1)+K_2}\right] - \cdots \right.$$
$$\left. \log_2\left[\frac{(2\pi\tau f_2)^2+2K_1(2\pi\tau f_2)+K_2}{(2\pi\tau f_2)^2-2K_1(2\pi\tau f_2)+K_2}\right]\right\} \quad (41)$$

where $K_1 = \sqrt{1 + AK - \xi^2}$ and $K_2 = 1 + AK$. Unlike the first order case, it is hard to determine by inspection how the introduction of feedback affects the channel capacity by comparing Equations 24, 25, and 27 with Equations 35, 38, and 41.

Capacity of the feedforward system is a function of the parameters specifying time constant τ, damping factor ξ, and ratio of the signal power to the input-referred DC noise PA^2/N. Capacity of the system with feedback is a function of the same three parameters plus the loop gain AK. Preliminary investigation of this four-dimensional parameter space supports the conjecture that the feedback capacity is less than or equal to the feedforward capacity over the entire parameter space, with equality only for zero feedback $K = 0$.

Therefore for $K \geq 0$ the capacity of the feedback system C_{fb} is less than or equal to the capacity of the feedforward system C. This reduction in capacity equates to an increase in bit energy of the system, which is the cost of feedback for a second order lowpass system:

$$BE_{fb} - BE = P_{sys}\left[\frac{1}{C_{fb}} - \frac{1}{C}\right] \qquad (42)$$

3 Conclusions

Detailed physical models such as those presented in [1, 2, 3, 4] permit the computation of channel capacity from first principles. Such models also predict the power required for communication and computation. The total energy cost of adaptive gain control will comprise both those explicit costs associated with the physical implementation of negative feedback and those implicit costs incurred by reduction of the channel capacity of the linear system.

There is no theoretical lower limit on the power required for communication, computation, or control. Theoretically we can hypothesize a conservative adiabatic process which performs reversible computations, dissipates no energy, and requires no power if we are willing to wait long enough. These results have limited usefulness for many physical systems of interest. This work directly addresses the performance limitations and lower limits of power required for computation and control.

In summary, in this paper we have shown that:

- *Feedback decreases capacity* for linear systems with first and second order lowpass transfer functions, and that
- *Feedback incurs an implicit energy cost* in addition to any explicit costs related to the implementation of the system.

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