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CHAPLYGIN DYNAMICS AND LAGRANGIAN REDUCTION

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ABSTRACT In this paper we present a formulation of reduced dynamic equations for nonholonomic Lagrangian systems with symmetry. Under our hypotheses on constraints and exterior force, we show that the dynamics of a nonholonomic Lagrangian system with non-Abelian symmetry can be reduced to a lower dimensional space determined by a principal fiber bundle. This formulation generalizes the one for classical Chaplygin systems which possess Abelian symmetry, and the one having non-Abelian symmetry but with linear constraints. In addition, if the mechanical connection of Kummer and Smale is considered, our formulation for nonholonomic Lagrangian systems specializes to the one in Lagrangian reduction discovered recently by Marsden and Scheurle.

I. INTRODUCTION

It is known that, in general for a system with nonholonomic constraints, one cannot get reduced dynamics to a lower dimensional space by eliminating constraints, as one can do for a system with holonomic constraints. Instead, one usually has to expand the space by bringing in more variables, i.e., Lagrange multipliers. But Chaplygin showed that one can do such reduction if the system and constraints admit an Abelian symmetry [1]. A natural question one can ask is, if this is also possible for constrained systems which admit a non-Abelian symmetry? This question is answered in this paper. As in [2], the constraints here are constructed on principal fiber bundles with connections. Under our hypotheses, they take the form of affine functions in velocity, instead of linear ones. In using the theory of distribution to interpret constraints, one has to consider distributions on the tangent bundle of the configuration space, rather than on configuration space. Here, the approach of constructing intrinsic constrained dynamics on the second tangent bundle, introduced by Vershik [3], is used. Since we consider constraints arising from principal connections which decompose the velocity phase space, the dynamics of the system can be described to a horizontal subspace and, consequently, on the tangent bundle of the quotient space of the configuration space with respect to the symmetry group. Following this idea, we obtain a reduction theorem. An important application of our reduction theorem is the derivation of Lagrangian reduction, where the principal connection used is the mechanical connection determined by the conserved momentum map. In this application, our result coincides with the one in [4].

II. DYNAMICS ON HORIZONTAL DISTRIBUTION

Consider a simple mechanical system with symmetry given by a four-tuple (Q, K, V, G), where Q is the n-dimensional configuration space; G is a Lie group of dimension p acting on Q on the left freely and properly. This action is denoted by $\Phi : G \times Q \to Q$; K is a Riemannian metric and G acts on Q by isometries; V is a G-invariant potential function. The Lagrangian of this system is given by

$$L(q, v_q) = \frac{1}{2}K(q)(v_q, v_q) - V(q). \tag{2.1}$$

In addition, we let the configuration space Q be a principal G'-bundle:

$$\wp = (Q, B, \pi, G'), \tag{2.2}$$

where G' is a p'-dimensional closed subgroup of G, $B = Q/G'$ is the $m(= n - p')$ dimensional base space and $\pi : Q \to B$ is the bundle projection. On this bundle one can choose a connection, which defines a horizontal distribution on TQ. Associated to such a connection, there is a unique connection form, $\omega \in \varpi^1(Q;\mathcal{G}')$. We now consider a class of constraints which relate to the connection form as follows.

Constraint Hypothesis: We assume that the motion of the system, $(q(\cdot), v_q(\cdot))$, is constrained to a $2n - p'$ dimensional subspace of TQ defined by

$$\mathcal{S} \triangleq \{ (q, v_q) \in TQ \mid \omega(q)(v_q) = \xi(q) \}, \tag{2.3}$$

where the mapping $\xi : Q \to \mathcal{G}'$ is smooth and also G'-equivariant, i.e., $\xi(g \cdot q) = Ad_g \xi(q), \forall g \in G'$.

Since $\mathcal{G}'$ is a p'-dimensional vector space and $\omega(q)(v_q)$ is linear in v_q, the above assumption can be viewed as giving p' affine constraints on TQ. In addition, if $\xi(q) \equiv 0$, the subspace $\mathcal{S}$ is just the horizontal distribution.

With the above setting, according to Lagrange-d'Alembert principle, the dynamics of the system is given by the following equations,

$$\frac{d}{dt} D_2 L(q, v_q) \cdot u_q - D_1 L(q, v_q) \cdot u_q = F \cdot u_q \tag{2.4a}$$

for (q, v_q) satisfying

$$\omega(q)(v_q) = \xi(q) \tag{2.4b}$$

and u_q belonging to the horizontal subspace at q, $\mathbf{H}_q \subset T_q Q$, i.e.,

$$\omega(q)(u_q) = 0. \tag{2.4c}$$

Here, F is an exterior force satisfying the following hypothesis:

Exterior Force Hypothesis: We assume that an exterior force is a mapping $F : TQ \to T^*Q$ such that $\forall (q, v_q)$ and $\forall g \in G$, $F(\Phi_g(q), T_q\Phi_g \cdot v_q) = T_q^*\Phi_g \cdot F(q, v_q)$.

Let $q(\cdot) = \{q(t), t \geq 0\}$ be the solution of (2.4) with initial condition $q(0) = q_0$, and $x(\cdot) = \{x(t) = \pi(q(t)), t \geq 0\}$ the projection of $q(\cdot)$ on B. Our final goal is to explicitly formulate the projected dynamics on TB. The strategy is to determine the unconstrained dynamics on the horizontal subspace for the given connection, and then, project it to the base space B, which will be shown in next section.

Given a connection on Q, let $r(\cdot) = \{r(t), t \geq 0\}$ be the horizontal lift of $x(\cdot)$ to Q for given $r(0) = r_0, \pi(r_0) = x(0) = \pi(q_0)$. From theory of principal connections, we know that this lifted curve is unique. The question we will address is that if $r(\cdot)$ satisfies some Euler-Lagrange equation $\frac{d}{dt} D_2 \hat{L}(r, v_r) \cdot u_r - D_1 \hat{L}(r, v_r) \cdot u_r = \hat{F} \cdot u_r$, for some function $\hat{L}$ and one-form $\hat{F}$ on $\mathbf{H}_r$, where $u_r \in \mathbf{H}_r$ (the horizontal subspace at r), $v_r = \dot{r}(t)$.

From the uniqueness of horizontal lift, we know that for a given $q(\cdot)$ in Q and the horizontal lift $r(\cdot)$, there exists a unique curve $g(\cdot) = \{g(t), t \geq 0\} \in G'$ such that $q(t) = \Phi(g(t), r(t)) \triangleq \Phi_g(r)$. Then,

$$v_q(t) = T_r\Phi_g \cdot v_r + [\eta(t)]_Q(q), \tag{2.5}$$

where $\eta(t) = T_g R_{g^{-1}} \dot{g}(t) \in \mathcal{G}'$ for $\mathcal{G}'$ the Lie algebra of G' and $[\eta(t)]_Q(q)$ infinitesimal generator of the action Φ corresponding to $\eta(t)$. Evaluating $\omega(q)$ on both sides of (2.5) and using Constraint Hypothesis we have $\eta(t) = \omega(q)(v_q) = \xi(q)$. Therefore,

$$v_q(t) = T_r\Phi_g \cdot v_r + [\xi(q)]_Q(q) \quad \text{or} \quad v_r = T_q\Phi_{g^{-1}} \cdot v_q - [\xi(r)]_Q(r) \tag{2.6}$$

Equation (2.6) presents the splitting of a tangent vector on Q according to a choice of connection. From (2.6) and a direct, but tedious, derivation, we get the following theorem.
Theorem: If $q(\cdot)$ is a solution of the constrained dynamics (2.4), then, for given choice of connection on principal G'-bundle (2.2), any horizontal lift, $r(\cdot)$, of $q(\cdot)$'s projection satisfies the unconstrained equation

$$\begin{aligned}\frac{d}{dt}D_2L^{\xi}(r,v_r)\cdot u_r - D_1L^{\xi}(r,v_r)\cdot u_r =& F(r, v_r + [\xi(r)]_Q(r))\cdot u_r - \Xi(r)(u_r,v_r) \\ &+ d\omega_{\xi}(r)(u_r,v_r) + d\omega_{\xi}(r)(u_r,[\xi(r)]_Q(r))\end{aligned} \tag{2.7}$$

for any $u_r \in \mathbf{H}_r \subset T_rQ$, where

$$\begin{aligned} L^{\xi}(r,v_r) &= \frac{1}{2}K(r)(v_r,v_r) - (V(r) + \frac{1}{2}K(r)([\xi(r)]_Q(r),[\xi(r)]_Q(r))), \\ \omega_{\xi}(r) &= K^{\flat}(r)([\xi(r)]_Q(r)), \\ \Xi(r)(u_r,v_r) &= K(r)(v_r,[D_r\xi(r)\cdot u_r]_Q(r)). \end{aligned}$$

Remark: Once a horizontal curve is determined by solving the unconstrained equation (2.7) for an initial condition $r(0)$, the solution for the original constrained equations (2.4) can be determined by first solving the differential equation $\dot{g}(t) = g(t)\cdot\xi(r(t))$ for $g(0)$ satisfying $q(0) = g(0)\cdot r(0)$, and then setting $q(t) = g(t)\cdot r(t)$.

III. NON-ABELIAN CHAPLYGIN SYSTEMS

We now show how to drop the unconstrained dynamics on the horizontal bundle given in (2.7) down to the base space for a given principal fiber bundle. To get explicit expressions, we will consider the formulation on product bundles. Since a principal fiber bundle is locally trivial, the following results will be true locally in general. Also, for simplicity, we will assume the symmetry group G is the same as the structure group G' of the principal bundle.

Let $Q = B \times G$ be the configuration space parametrized by $q = (x,h)$ for $x \in B$ and $h \in G$. Then, the tangent space is $T_qQ = T_xB \times T_hG$. The tangent vector at any point q in Q is given by $v_q = [\ v_x\ ,\ h\cdot\zeta\ ]_{(x,h)}$ for some $\zeta \in \mathcal{G}$. Consider the principal bundle $(B\times G, B, G)$, where G acts on $Q = B\times G$ on the left and a connection given by connection form $\omega \in \varpi^1(Q;\mathcal{G})$. One can show that, on this bundle there exists an unique $\mathcal{G}$-valued one-form, $\widetilde{\omega}$, on B such that, at each point $q=(x,h)$ in Q,

$$\omega(q)\cdot v_q = Ad_h(\zeta + \widetilde{\omega}(x)\cdot v_x). \tag{3.1}$$

With the above connection, the tangent vector v_q on Q at $q=(x,h)$ has its horizontal-vertical splitting,

$$v_q = Ver(v_q) + Hor(v_q), \tag{3.2}$$

where $Ver(v_q) = [\ 0\ ,\ h\cdot(\zeta + \widetilde{\omega}(x)\cdot v_x)\ ]$ and $Hor(v_q) = [v_x, -h\cdot(\widetilde{\omega}(x)\cdot v_x)]$. Using this explicit splitting of v_q and the theorem in the above section, one can get the following result.
Corollary: If $q(\cdot) = (x(\cdot), g(\cdot))$ is a solution of constrained dynamic equations (2.4) on principal fiber bundle $(B\times G, B, \pi, G)$ with connection in (3.1), then its projection $x(\cdot)$ in B satisfies unconstraint equation

$$\frac{d}{dt}D_2\widetilde{L}_{\xi}(x,v_x)\cdot u_x - D_1\widetilde{L}_{\xi}(x,v_x)\cdot u_x = \widetilde{F}(x,v_x)\cdot u_x + d\widetilde{\omega}_{\xi}(x)(u_x,v_x) + \tilde{\Upsilon}(x,v_x)\cdot u_x \tag{3.3}$$

for any $u_x \in T_xB$, where $\widetilde{L}_{\xi}(x,v_x) \triangleq \widetilde{K}(x)(v_x,v_x) - (\widetilde{V}(x) - \frac{1}{2}\langle \mathbb{I}(x)\widetilde{\xi}(x), \widetilde{\xi}(x)\rangle)$ and $\tilde{\Upsilon}(x,v_x)$ is a one-form (therefore, a force) on TB defined by, for any $u \in T_xB$,

$$\tilde{\Upsilon}(x,v_x)\cdot u \triangleq K(x,e)([v_x, \widetilde{\xi}(x) - \widetilde{\omega}(x)v_x], [0, d\widetilde{\omega}(x)(u,v_x) + [\widetilde{\omega}(x)u, \widetilde{\xi}(x) - \widetilde{\omega}(x)v_x] - D_x\widetilde{\xi}(x)u]).$$

(For more details on Eq. (3.3), see [5].)

Remark: If $\widetilde{\xi}(x) \equiv 0$, Equation (3.3) becomes the one derived by Koiller in [2], for linear constrained, nonabelian chaplygin systems.

IV. LAGRANGIAN REDUCTION

In Section II we showed constrained Lagrangian dynamics on horizontal distribution for any principal connection, by which the nonholonomic constraints are constructed. Here, we consider a special connection, namely, *mechanical connection.* Recall that, given a simple mechanical system with symmetry (Q, K, V, G) with its Lagrangian given in (2.1), the lift of the G-action to TQ induces an equivariant momentum map $\mathbf{J} : TQ \to \mathcal{G}^*$ given by $\langle \mathbf{J}(q, v_q), \zeta \rangle = K(q)(v_q, \zeta_Q(q)), \forall \zeta \in \mathcal{G}$. Assume that the motion of the system $q(\cdot) = \{q(t), t \geq 0\}$ satisfies $\mathbf{J}(q, \dot{q}) = \mu$ for a constant $\mu \in \mathcal{G}^*$. If μ is a regular value of $\mathbf{J}$, the question is what the dynamics on subspace $\mathcal{S} = \mathbf{J}^{-1}(\mu) \subset TQ$ is.

Consider a principal fiber bundle given by $\wp = (Q, B, \pi, G_\mu)$, where $G_\mu = \{g \in G \mid Ad^*_{g^{-1}}\mu = \mu\}$, for constant $\mu \in \mathcal{G}^*$ given above, is an isotropy group. Let $\mathcal{G}_\mu$ be the Lie algebra of G_μ. On this bundle, the mechanical connection is given by a $\mathcal{G}_\mu$-valued one-form: $\omega(q) : T_qQ \to \mathcal{G}_\mu : v_q \mapsto \mathbb{I}_\mu^{-1}(q)\mathbf{J}(q, v_q), \forall q \in Q$, where $\mathbb{I}_\mu(q) : \mathcal{G}_\mu \to \mathcal{G}_\mu^*$ is called μ-locked inertia tensor. Now the subspace $\mathcal{S} = \mathbf{J}^{-1}(\mu)$ can be represented as $\mathcal{S} = \{ (q, v_q) \in TQ \mid \omega(q)(v_q) = \mathbb{I}_\mu^{-1}(q)\mu \}$, which is of the same form as (2.3) in Constraint Hypothesis. Now from Theorem in Section II, we have the following result [5], first obtained in [4].

Corollary: If $q(\cdot)$ is the motion of a Lagrangian system with symmetry and preserves the equivariant momentum map $\mathbf{J}(q, v_q) = \mu$, then its restriction, $r(\cdot)$, on the horizontal distribution determined by the mechanical connection satisfies the dynamic equation

$$\frac{d}{dt} D_2 L_\mu(r, v_r) \cdot u_r - D_1 L_\mu(r, v_r) \cdot u_r = F(r, v_r + [\mathbb{I}_\mu^{-1}(r)]_Q(r)) \cdot u_r + \Omega_\mu(r)(u_r, v_r),$$

where $L_\mu(r, v_r) = \frac{1}{2}K(r)(v_r, v_r) - (V(r) + \frac{1}{2}\langle \mu, \mathbb{I}_\mu^{-1}(r)\mu \rangle)$ and Ω_μ is μ-component of the curvature form of the mechanical connection.

V. FINAL REMARK

In the literature of analytical mechanics, the reduction theory for Hamiltonian systems has been well developed [6–8] and has been successfully applied to many problems in engineering. However, since in many physical problems, Lagrangian dynamics is the natural starting point, the construction of a reduction theory directly applicable to Lagrangian systems becomes a challenge. Recent work done by Koiller [2], Marsden and Scheurle [4], and Bloch and Crouch [9] have contributed to this goal. In [5], we apply our approach to problems with exterior forces of great interest in control theory.

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