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APPROXIMATION OF DYNAMICAL EFFECTS DUE TO IMPACT ON FLEXIBLE BODIES [1]

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Abstract

This paper presents a nonlinear model to describe the dynamical effects on flexible bodies when they are undergoing impact with each other. The main aim is to develop approximations that can be used to predict the magnitude and duration of impact, and the force profile during the impact period. Two ordinary differential equations are shown to play a central role in this problem.

1. Introduction

The problem of accurate modeling, analysis and control of systems that undergo impact forces has drawn much attention recently [1, 2, 3]. This is because of the needs in control design for high precision robotic manipulators, a topic which has become very important in the recent past. Control of such systems requires taking into consideration such usually ignored features as dynamical effects of impact [4, 5, 6, 7, 8].

Hertz's law of impact is the most popular principle used for modeling impact behavior. Applied to the case of impacting bodies coupled to flexible systems, this method yields a nonlinear integral equation, e.g. [9, 10]. In general this equation does not admit closed form solutions, hence giving rise to a heavy computational burden.

Our objective here is to show that it is possible to take advantage of the fact that the impact period is very short in general in order to simplify matters. Therefore, one may approximate the transfer functions of the systems to which the impacting bodies belong by Taylor polynomials of low order. We explicitly carry out this computation in the cases of first and second order Taylor polynomial approximations. We show that in the case of the first order approximation, there is a universal ordinary differential equation that describes the impact behavior completely. Therefore, one can numerically solve this equation beforehand, save the results, and can use it to predict the impact behavior with only a minimal computational burden. In the case of the second order approximation, there is a two parameters family of ordinary differential equations that govern the impact behavior.

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2. Impact Dynamics

Impact phenomena have interested scientists and engineers for a long time. The fundamental law that describes the impact between two elastic bodies was derived by Hertz [9]. This law, despite of being based on some ideal conditions, has been widely applied to various impact situations [2, 9, 10]. In this paper, we apply this law to model impact dynamics of flexible systems.

In order to motivate the problem, for the sake of simplicity, let us take two flexible bodies in linear motion. This could perhaps be a simplified model for a robot and a work-piece, see Figure 1. Here m_i is the mass of the i_{th} body, c_i and k_i are the associated viscous frictional coefficient and stiffness, and $u(t)$ is control input. Let $f(t)$ denote the impact force between the bodies. The dynamical equations for the system during the impact period are:

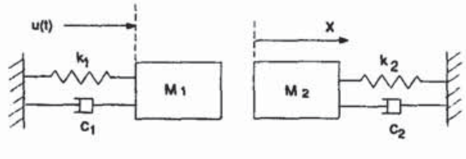

Figure 1: Model of Robot and Workpiece

$$
\begin{aligned}
m_1\ddot{x}_1(t) + c_1\dot{x}_1(t) + k_1x_1(t) &= u(t) - f(t), \\
m_2\ddot{x}_2(t) + c_2\dot{x}_2(t) + k_2x_2(t) &= f(t). \qquad (2.1)
\end{aligned}
$$

2.1. Hertz Law of Impact

Without loss of generality, we assume that the approach velocity of the first body just before contact is $v_0 \geq 0$ and the second body was at rest. The Hertz law of impact is assumed valid, i.e.

$$\alpha(t) = K[f(t)]^{2/3}, \tag{2.2}$$

where $\alpha(t)$ is the relative approach, i.e. the difference between the displacements of the bodies, and K is the Hertz constant [9, 11],

$$K = \frac{4}{3}\{q/(Q_1 + Q_2\sqrt{A+B})\},$$

where q, A and B are constants which depends on the local geometry of the region of contact,and,

$$Q_1 = (1-\mu_1^2)/E_1\pi, \qquad Q_2 = (1-\mu_2^2)/E_2\pi.$$

where, μ_i and E_i are the Poisson ratio and Young's modulus for the two bodies respectively.

Let us assume that the external forces acting on the bodies are negligible in comparison to the impact force. We can write the displacements of the bodies as,

$$\begin{aligned} x_1(t) &= h_0(t) - \int_0^t h_1(t-\tau)f(\tau)d\tau, \\ x_2(t) &= \int_0^t h_2(t-\tau)f(\tau)d\tau. \end{aligned} \tag{2.3}$$

where $h_0(t)$ is a function of initial condition and $h_0(t) = v_0 m_1 h_1(t)$. The Green's functions h_i are given by,

$$h_i(t) = \frac{\omega_i}{k_i\sqrt{1-\zeta_i^2}} e^{-\zeta_i\omega_i t} sin(\omega_i\sqrt{1-\zeta_i^2}\ t), \tag{2.4}$$

where $i = 1, 2$ and ω_i and ζ_i are defined by,

$$\omega_i^2 = \frac{k_i}{m_i}, \qquad \frac{c_i}{m_i} = 2\zeta_i\omega_i.$$

Hence, the relative approach can be written as,

$$\begin{aligned} \alpha(t) &= h_0(t) - \int_0^t [h_1(t-\tau) + h_2(t-\tau)]f(\tau)d\tau, \\ &= h_0(t) - \int_0^t \bar{h}(t-\tau)f(\tau)d\tau. \end{aligned} \tag{2.5}$$

where $\bar{h}(t) := h_1(t) + h_2(t)$.

Equations (2.2) - (2.5) lead to

$$K[f(t)]^{2/3} = h_0(t) - \int_0^t \bar{h}(t-\tau)f(\tau)d\tau. \tag{2.6}$$

The following theorem was established in [12].

Theorem 2.1 *For given $v_0 \geq 0$, the equation (2.6) has a unique solution if the green's function $\bar{h}(t)$ is uniformly bounded $\forall t \geq 0$.*

It was shown in [12] that the boundedness hypothesis is satisfied in the case considered here (and more generally if one of the discrete masses of Figure 1 is replaced by a set of finite masses, a cantilevered or a simply-supported beams).

2.2. Numerical Solution

In general, the nonlinear integral equation (2.6) does not admit a closed form solution. There are several numerical methods available to solve this equation. The small-increment method is the most popular one, which is served as the basis for evaluting other approximation method [10]. We have developed an iterative approximation method in [12].

The following examples are used to show how the impact force is affected by the approach velocity v_0 and material properties, and these results give some "feel" to the problem. The model parameters for the numerical simulations are given in Table 1.

Table 1: Model Parameters

m_1:	2.0 kg
c_1 :	20.0 $N.s/m$
k_1:	10^4 N/m
m_2:	0.5 kg
c_2:	15 $N.s/m$
k_2:	10^4 N/m
v_0:	0.1 m/s

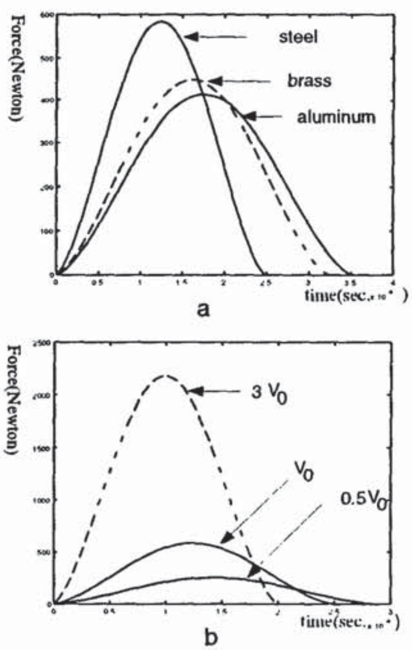

Figure 2: Numerical Solutions Of Impact Forces

Remark 2.1 *The first example illustrates how the im-*

pact force is affected by material properties. We use three different materials typically used in robotics. The approach velocity is fixed and the impact force profiles are plotted. The following conclusions can be drawn from Figure 2(a). For a given approach velocity v_0, the force magnitudes and impact durations are determined by material properties of both objects. The softer material (aluminum) results in smaller impact force and longer impact duration while the harder one (steel) results in greater impact force and shorter impact duration. This relationship leads to the concept of passive compliance. When the approach velocity cannot be controlled to be arbitrarily small, the passive compliance can be introduced by the use of soft contact surfaces (such as soft force sensors), thereby reducing the impact forces greatly [13].

Remark 2.2 *The second numerical example addresses the question of how the impact force is affected by the approach velocity. We use different approach velocities while fixing other parameters. The force profiles are plotted in Figure 2(b). For given impacting objects, the impact forces and impact durations are determined by the approach velocity v_0. Smaller v_0's results in smaller impact forces and longer impact durations, while the larger v_0's result in greater impact forces and shorter impact durations. The approach velocity v_0 is usually controllable in robotics applications. Hence, it can be effectively exploited in the control design.*

Numerical data obtained from the impact model (2.6) provides useful information as illustrated above. The main drawback of this method is that each case, e.g. varying initial velocities, has to be numerically solved separately. Also, a fairly large computational burden has to be incurred for each numerical solution. In the next section we show how to obtain approximate solutions by exploiting the fact that the impact duration is very short in general.

3. Approximation of Impact Dynamics

Since the impact duration is very short, if the Green's function $h(t)$ belongs to $C^\infty[0,T]$ (this condition is satisfied in many applications), $h(t)$ can be expanded in the form of a Taylor series,

$$h(t) = k_1 t + \frac{1}{2!}k_2 t^2 + \frac{1}{3!}k_3 t^3 + \cdots, \qquad (3.1)$$

where the coefficients k_i are determined by $h(t)$ and are finite. Since the impact duration is short, one can frequently approximate $h(t)$ by its first or second order Taylor polynomials very well.

3.1. First-order Approximation

The solution of nonlinear equation (2.6) can be greatly simplified if we use the first-order approximation. Somewhat surprisingly, equation(2.6) renders a universal ordinary differential equation in this case. The first-order approximations of functions $h_0(t)$ and $\bar{h}(t)$ are $v_0 t$ and $k_1 t$ respectively. By plugging these approximations into equation (2.6) we obtain,

$$K[f(t)]^{2/3} = v_0 t - \int_0^t k_1(t-\tau)f(\tau)d\tau. \qquad (3.2)$$

Since the relative approach $\alpha(t)$ is equal to $Kf^{2/3}(t)$, substituting $f(t)$ by $\alpha(t)$ in equation (3.2)

$$\alpha(t) = v_0 t - \int_0^t k_1(\frac{1}{K})^{3/2}(t-\tau)\alpha^{3/2}(\tau)d\tau. \qquad (3.3)$$

Differentiating equation (3.3) twice leads to,

$$\frac{d^2\alpha(t)}{dt^2} = -k_1(\frac{1}{K})^{3/2}\alpha^{3/2}(t). \qquad (3.4)$$

By our assumptions, the initial conditions are $\alpha(0) = 0$, and $\dot{\alpha}(0) = v_0$. Equation (3.4) can be integrated once to obtain,

$$(\frac{d\alpha(t)}{dt})^2 - v_0^2 = -\frac{4}{5}k_1(\frac{1}{K})^{3/2}\alpha^{5/2}(t). \qquad (3.5)$$

From the equation (3.5), the maximum relative approach α_{max} can be immediately obtained by letting $\dot{\alpha}(t) = 0$, and the maximum impact force is easily obtained from (2.2),

$$\alpha_{max} = [\frac{5}{4}\frac{K^{3/2}v_0^2}{k_1}]^{2/5}, \quad f_{max} = [\frac{5}{4}\frac{v_0^2}{k_1 K}]^{3/5}. \qquad (3.6)$$

Next, we shown that equation (3.5) can be put into a universal form by scaling both time and magnitude of $\alpha(t)$. Let us introduce two new variables β and τ by,

$$\beta(\cdot) = \lambda\alpha(\cdot), \qquad t = \mu\tau. \qquad (3.7)$$

where β and τ will play the roles of the dependent and the independent variables, λ and μ are two constants to be determined shortly. It is easy to show that $\frac{d\beta(\tau)}{d\tau} = \lambda\mu\frac{d\alpha(t)}{dt}$. Replacing α and t by the expressions in (3.7) we obtain from (3.5),

$$(\frac{d\beta(\tau)}{d\tau})^2 - \lambda^2\mu^2 v_0^2 = -\frac{4}{5}\mu^2 k_1(\frac{1}{K})^{3/2}\frac{1}{\sqrt{\lambda}}\beta(\tau)^{5/2}, \qquad (3.8)$$

Let us now choose λ and μ such that,

$$\lambda^2\mu^2 v_0^2 = 1, \qquad \frac{4}{5}\mu^2 k_1(\frac{1}{K})^{3/2}/\sqrt{\lambda} = 1. \qquad (3.9)$$

The equation(3.8) now takes the universal form,

$$\frac{d\beta(\tau)}{d\tau} = \sqrt{1-\beta(\tau)^{5/2}}. \qquad (3.10)$$

subject to the initial condition $\beta(0) = 0$. This completes the construction. □

The ordinary differential equation (3.10) has a unique solution with the given initial condition, and can be solved numerically. The result is plotted in Figure 3. Once the solution of this universal equation is obtained, any impact problem can be immediately solved by using the following relation. From the equation (3.9), we can solve for λ and μ respectively,

$$\lambda = [(\frac{4}{5})^2 \frac{k_1^2}{v_0^4 K^3}]^{1/5}, \qquad \mu = [(\frac{5}{4})^2 \frac{K^3}{v_0 k_1^2}]^{1/5}. \quad (3.11)$$

Let τ_{max} be the time before the solution of (3.10) cross the time axis once again. Then the impact duration T for the original problem is,

$$T = \mu \tau_{max} = [(\frac{5}{4})^2 \frac{K^3}{v_0 k_1^2}]^{1/5} \tau_{max}.$$

The impact force can be obtained from,

$$f(\tau) = (\frac{1}{K\lambda})^{3/2} \beta(\tau)^{3/2} = (\frac{5}{4} \frac{v_0^2}{k_1 K})^{3/5} \beta(\tau)^{3/2}. \quad (3.12)$$

By scaling back the time from τ to t, we can re-

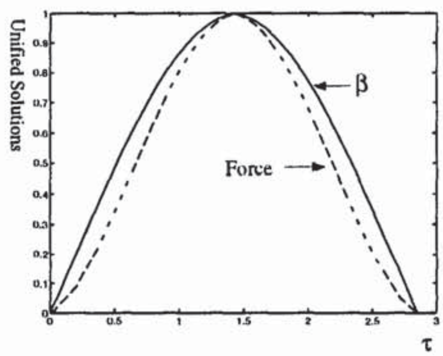

Figure 3: Unified Solutions of Equation 3.10

cover the real force $f(t)$. Let us apply this method to the examples discussed in section 2.

Case A: The model parameters are given at Table 1, and the impacting bodies are assumed to be made of steel. Figure 4 shows that the approximate solution is practically indistinguishable from the exact solution (numerically computed with the small increment method).

Remark 3.1 *The first-order approximation has an interesting physical interpretation. Note that $k_1 = (1/m_1 + 1/m_2)$. This means that we treat the two flexible bodies as if they were two single rigid bodies colliding in free space. When the impact duration is relatively large (because of low-speed impact, softer materials etc.), effects of other factors such as damping cannot be ignored. The first-order approximation will introduce some errors.*

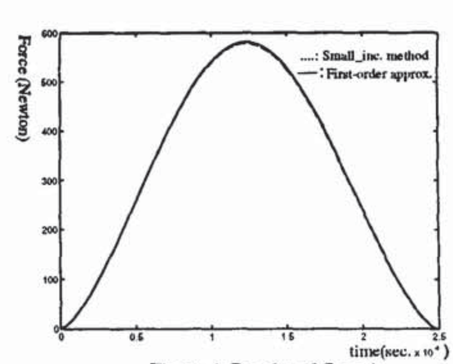

Figure 4: Results of Case A

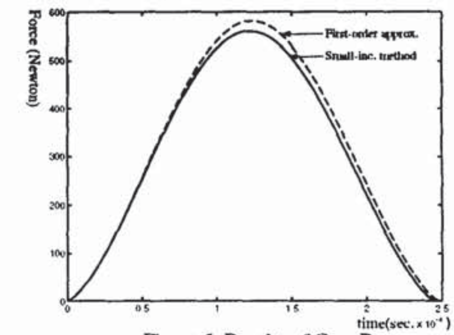

Figure 5: Results of Case B

Case B: For the purpose of comparison, we increased the damping coefficients c_i 10 times larger than before. Results are shown in Figure 5. Solution obtained from the first-order approximation now begins to deviate from the exact solution. This motivates us to consider the development of a second order approximation.

3.2. Second-Order Approximation

The second-order approximation of the function $\bar{h}(t)$ is $k_1 t + \frac{1}{2}k_2 t^2$, where $k_1 \neq 0$, $k_2 \neq 0$, and the approximation of $h_0(t)$ is $v_0 t + \frac{1}{2}k_0 t^2$. We plug these approximations into the nonlinear equation (2.6) to obtain,

$$K[f(t)]^{2/3} = v_0 t + \frac{1}{2}k_0 t^2 - \int_0^t [k_1(t-\tau) + \frac{1}{2}k_2(t-\tau)^2] f(\tau) d\tau. \quad (3.13)$$

Using the relation (2.2), replace the $f(\cdot)$ by $\alpha(\cdot)$,

$$\alpha(t) = v_0 t + \frac{1}{2}k_0 t^2 - \int_0^t (\frac{1}{K})^{3/2} [k_1(t-\tau) + \frac{1}{2}k_2(t-\tau)^2] \alpha^{3/2}(\tau) d\tau.$$

Hence we obtain,

$$\frac{d^3\alpha(t)}{dt^3} = -\frac{3}{2}k_1' \sqrt{\alpha(t)} \frac{d\alpha(t)}{dt} - k_2' \alpha(t)^{3/2}, \quad (3.14)$$

where, $k_1' = k_1(\frac{1}{K})^{3/2}$, *and* $k_2' = k_2(\frac{1}{K})^{3/2}$.
Let us introduce two new variables,

$$\beta(\cdot) = \lambda\alpha(\cdot), \qquad t = \mu\tau. \tag{3.15}$$

From equation (3.14) we obtain,

$$\frac{d^3\beta(\tau)}{d\tau^3} = -\frac{3k_1'}{2}\frac{\mu^2}{\sqrt{\lambda}}\sqrt{\beta(\tau)}\frac{d\beta(\tau)}{d\tau} - k_2'\frac{\mu^3}{\sqrt{\lambda}}\beta(\tau)^{3/2} \tag{3.16}$$

Let us first assume that $k_2 > 0$, choose the parameters λ and μ such that,

$$\frac{3}{2}k_1'\frac{\mu^2}{\sqrt{\lambda}} = 1, \qquad k_2'\frac{\mu^3}{\sqrt{\lambda}} = 1. \tag{3.17}$$

Equation (3.16) now becomes a universal differential equation,

$$\frac{d^3\beta(\tau)}{d\tau^3} = -\sqrt{\beta(\tau)}\frac{d\beta(\tau)}{d\tau} - \beta(\tau)^{3/2}. \tag{3.18}$$

with the initial conditions given by,

$$\beta(0) = 0, \frac{d\beta(\tau)}{d\tau}|_{\tau=0} = \lambda\mu v_0 = \gamma_1,$$

$$\frac{d^2\beta(\tau)}{d\tau^2}|_{\tau=0} = \lambda\mu^2 k_0 = \gamma_2. \tag{3.19}$$

Hence, we have a universal ordinary differential equation (3.18), and a two parameters family of initial datas, parametrized by γ_1 and γ_2. □
The equation(3.17) can be solved for μ and λ respectively,

$$\mu = \frac{3}{2}\frac{k_1'}{k_2'} = \frac{3}{2}\frac{k_1}{k_2}, \quad \lambda = (\frac{3}{2})^3\frac{k_1'^4}{k_2'^2}.$$

$$\gamma_1 = (\frac{3}{2})^4\frac{k_1'^5}{k_2'^3}v_0, \quad \gamma_2 = (\frac{3}{2})^5\frac{k_1'^6}{k_2'^4}k_0. \tag{3.20}$$

Once again, the ODE (3.18) with two parameters γ_1 and γ_2 can be numerically solved. A table can be created to store the results with different γ_1 and γ_2. Any impact problem can be immediately solved by using data from the table. Obviously, the higher-order approximation may increase the accuracy but at the expense of computational load. Let us apply the second-order approximation to the examples of section 2.
Case C: For the purpose of comparsion, we increased the damping coefficients c_i 50 times. It is easy to show that $k_2 = -(c_1/m_1^2 + c_2/m_2^2) < 0$; For given model parameters, a table is generated with two variables γ_1 and γ_2. It is seen in Figure 6 that the result obtained by using the second order approximation agrees with true results with a high degree of accuracy.

4. Conclusion

Approximations have been developed for the analysis of dynamics of flexible bodies undergoing impact. The first-order approximation yields a special function which can be used for analytical and computational purposes. This approximation seems to give results which are very close to the exact solutions. A second order approximation was developed as well, and it leads to a two parameter family of ordinary differential equations of which the solutions contain universal features of impact problems. In a future paper, we will discuss the case of two ODE's in closed loop force control.

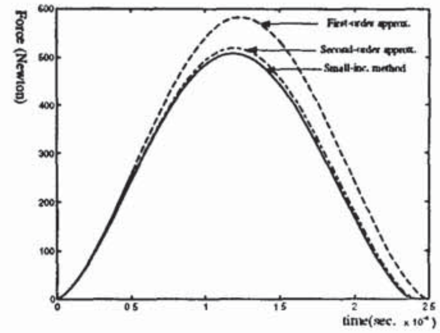

Figure 6. Solutions by First, Second-order Approx. and Small-inc. Method

References

[1] R.Volpe and P.Khosla. A theoretical and experimental investigation of impact control for manipulators. *Int. J. Robot. Research*, 12:351–365, 1993.

[2] B.V.Chapnik, G.R. Heppler, and J.D. Aplevich. Controlling the impact response of a one-link flexible robotic arm. In *Proc. of the 1990 IEEE int. conf. of Rob. and Auto.*, pages 1444–1449, 1990.

[3] K. Youcef-Toumi and D. A .Gutz. Impact and force control. In *Proc. of the 1989 IEEE int. conf. of Rob. and Auto.*, pages 410–416, 1989.

[4] I.Salmatjidis. *Impact and Force Control of Flexible Manipulators.* Master's Thesis, Univ. of Maryland, College Park, 1991.

[5] Y. Wang and M.T. Mason. Modeling impact dynamics for robotic operations. In *Proc. of the 1992 IEEE int. conf. of Rob. and Auto.*, pages 678–685, 1987.

[6] A.Yigit, A.Ulsoy, and R.Scott. Dynamics of a radially rotating beam with impact. *ASME J. of Vibration and Acoustics*, 112:1391–1413, 1990.

[7] B.Chapnik, G.Heppler, and J.Aplevich. Modeling impact on a one-link flexible robotic arm. *IEEE Trans on Robotics and Automation*, 7:666–672, 1991.

[8] J.K. Parker and F.W. Paul. Controlling impact forces in pneumatic robot hand design. *ASME J. of Dynamic Systems, Measurement and Control*, 9:328–334, 1987.

[9] W.Goldsmith. *Impact.* Edward Arnold Ltd, London, 1960.

[10] E.H.Lee. The impact of a mass striking a beam. *J. Appl. Mechanics*, 7:A129–A138, 1940.

[11] K.L. Johnson. *Contact Mechanics.* Cambridge University Press, 1985.

[12] Q.F. Wei, P.S. Krishnaprasad, and W.P. Dayawansa. Modeling of impact on a flexible beam. In *Proc. of the 32th Conference on Decision and Control(CDC)*, pages 1377–1382, 1993.

[13] Y.Xu and R.Paul. On position compensation and force control stability of a robot with a compliant wrist. In *Proc. of the 1988 IEEE int. conf. of Rob. and Auto.*, pages 1173–1178, 1988.