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Wavelet Based Identification of a Flexible Structure with Surface Mounted Smart Actuators and Sensors *

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Abstract

In the Ph.D. thesis of Y.C. Pati [2], the theory of rational wavelets was developed for the purpose of system identification. We use this theory in order to study the identification problem of a flexible beam with bonded piezo-electric actuators and sensors. Computational burden is reduced significantly by using the matching pursuit algorithm, due to S. Mallat and Z. Zhang [6].

1 Introduction

In smart structures with a very large number of embedded actuators and sensors, one encounters the problem of inadequate knowledge of physical laws that describes the composite system. This is due in part to the insufficiency of available mathematical models on the behavior of systems with embedded actuators and also due to the extreme complexity of resulting systems. Empirical methods for system identification are expected to offer good solutions for modeling such systems, with the understanding that such schemes ought to take into account the a priori knowledge on the physical behavior, such as appropriate time scales for dynamical effects.

A fundamental limitation of the existing approximation methods for identifying models is that they are not particularly geared to the proper accounting of time-frequency concentration of the transfer functions. The wavelet based approximation scheme proposed in [2], [3] has the potential to overcome this difficiency. We will present experimental results aimed at demonstrating the utility of this method in identification of a cantilevered beam attached to, where the beam is controlled via surface mounted piezo-electric sensors and actuators.

The theory of rational wavelets for system identification applications was developed by Y. C. Pati in his Ph. D. thesis [2]. For the sake of simplicity, let us consider a single input, single output (possible infinite dimensional) linear system. By stable, we mean that the transfer function is an element of the Hardy space of square integrable analytic functions in the open right half of the complex plane. Pati showed that it is possible to construct an affine wavelet frame in this space constructed from an analyzing wavelet, which is a stable rational transfer function which satisfies certain mild conditions. This then allows one to bring in all the machinery that has been developed in the domain of wavelet analysis into the system identification problem. Perhaps the most significant advantage of such an identification scheme would be the ability to use prior knowledge of time-frequency localization properties of the system to be identified.

One of the best-known schemes for system identification that relies on a building-block approach (analogous to wavelets) is the Laguerre basis technique. Here we make comparisons of the wavelet-based identification scheme with Laguerre models. In the interest of speeding up the computation we enhance the methods of [2] by a modification of a fast algorithm due to S. Mallat and Z. Zhang [6], that decomposes any signal into a linear expansion of waveforms that belong to an apriori selected subset. For us this is a subset of the rational wavelet frame, called the dictionary. Algorithm of Mallat and Zhang in [6] is a special case of what is known as a "projection pursuit regression" in the statistics literature [10].

Clearly, the choice of the dictionary has a great deal to do with how compact the decomposition is going to be. For the system identification problem one of the requirements is the real-rationality of the approximation (a real-rational function is a rational one with real coefficients).

We also may use our a priori knowledge of time-frequency localization properties of the system to select the elements of the dictionary in an efficient fashion.

2 Rational Wavelets Based System Identification

Approximation of transfer functions discussed in [2] and [3] is based on a wavelet decomposition of the Hardy space $\mathrm{H}^2(\Pi^+)$, where Π^+ denotes, the half-plane $\Re e\ \ s>0$.

Definition 2.1 *Given a function F which is analytic in Π^+, F is said to belong to the class $\mathrm{H}^2(\Pi^+)$ if*

$$\sup_{x>0}\int_{-\infty}^{\infty}|F(x+iy)|^2\,dy<\infty. \tag{1}$$

$\mathrm{H}^2(\Pi^+)$ is a Banach space with norm (denoted $\|\cdot\|_{H^2}$) defined by (1).

Some of the most basic properties of $\mathrm{H}^2(\Pi^+)$ are captured by the following classical theorem.

Theorem 2.1 *(e.g. [9]) Given $F\in\mathrm{H}^2(\Pi^+)$, the following hold:*

(1) The nontangential limit of F exists at almost every point on the imaginary axis.

(2) The boundary value function of F is in $\mathrm{L}^2(\mathbb{R})$ and,

$$F(x+iy)=\frac{1}{\pi}\int_{\mathbb{R}}F(i\omega)\frac{x}{x^2+(y-\omega)^2}d\omega,\quad x>0$$

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(3) The functions $F_x(y) = F(x+iy)$ converge in L^2 norm to $F(iy)$ as $x \to 0$.

The elements of $H^2(\Pi^+)$ are transfer functions of causal, input-output stable, linear systems. More precisely, we have the classical result,

Theorem 2.2 (Paley-Wiener)
A complex-valued function F is in $H^2(\Pi^+)$ if and only if,

$$F(s) = \int_0^\infty f(t)e^{-st}dt,$$

for some $f \in L^2(0,\infty)$. Furthermore this representation is unique.

By the Paley-Wiener theorem,

$$\frac{1}{2\pi}\int_{\mathbb{R}} F(i\omega)e^{i\omega t}d\omega = 0 \ \text{ for } \ t < 0. \tag{2}$$

Hence boundary values of functions in $H^2(\Pi^+)$ comprise a subspace $\mathcal{D}$ of $L^2(\mathbb{R})$ characterized by (2). We also know by Theorem 2.1 that given the boundary value function $F(i\omega)$, F can be recovered on the right half-plane by the Poisson integral. Also $\|F\|_{H^2} = \|\widetilde{F}\|_{L^2}$ where $\widetilde{F}$ is the nontangential limit of F.

It was shown in [2], [3] that there are stable rational functions that are admissible as analyzing wavelets (in particular, second order rational functions). This then was used to develop a method for approximating transfer functions, which is very similar to the usual wavelet based approximations schemes which have become quite popular in signal analysis.

2.1 Rational Wavelets

We will first describe the notion of wavelets, and then proceed onto defining rational wavelets.

Let Φ be an element of $L^2(\Re)$, and let a_0 and b_0 be positive reals. Define the family, $\{\Phi_{m,n}|(m,n) \in Z \times Z\}$ by,

$$\Phi_{m,n}(s) = a_0^{m/2}\Phi(a_0^m s - nb_0), \tag{3}$$

where a_0 and b_0 are dilation and translation step sizes, respectively and m and n are called dilation and translation levels.

Under some mild conditions on Φ (see e.g. [8]), and for suitable a_0, b_0, this family turns out to be a frame of $L^2(R)$. Now, a given element f of $L^2(R)$ can be expressed (nonuniquely in general) as $f = \sum \alpha_{m,n}\Phi_{m,n}$. Such an expression is called a wavelet decomposition of f, and Φ is the analyzing wavelet or the analyzing wavelet.

In general, one would desire additional properties such as, smoothness of Φ, simplicity of analytical expressions which describe Φ etc. Now let us consider the Hardy space $H^2(\Pi^+)$. One can construct a frame for $H^2(\Pi^+)$ as above. Note that the translations are now done parallel to the imaginary axis (this follows from the way the norm on $H^2(\Pi^+)$ is defined). It is shown in [2] and [3] that it is possible to choose certain stable rational functions (e.g. relative degree two second order systems) as analyzing wavelets in this construction. Such a frame is called a rational wavelet frame. To be explicit, let Φ be a stable rational transfer function, which satisfies certain mild admissibility conditions (see [2] for details). The family

$$\Phi_{m,n}(s) = a_0^{m/2}\Phi(a_0^m s - inb_0), \ m,n \in Z, \tag{4}$$

is called a rational wavelet frame. It is a frame of $H^2(\Pi^+)$.

2.2 Wavelet Based Identification

In the wavelet based system identification problem one first pre-specifies an admissible analyzing wavelet. The natural computational problem is to approximate an experimentally observed transfer function by a finite dimensional approximation obtained as a linear combination of the elements of the frame. It has to be ensured that this approximation has real coefficients, i.e. $\Phi_{m,n}$ and $\Phi_{m,-n}$ must have coefficients that are complex conjugate of each other. It now becomes clear that the translation by $-inb_0$ in the wavelets corresponds to introducing frequency components at nb_0 and dilation by a_0^m corresponds to rescaling the time axis. Therefore, the usual time frequency localization associated with all wavelet schemes correspond to introducing specific frequency components and decay rates in the context of wavelet based identification methodology. This, we believe, is the overwhelming advantage of this method of systems identification scheme for applications such as ours when some information on the time scales and the frequency content of the system to be identified is available.

From a practical viewpoint, one may first restrict attention to a finite subset of the frame called a dictionary. Since $H^2(\Pi^+)$ is a Hilbert space, the identification problem reduces to projecting the experimentally observed transfer function onto the subspace spanned by the elements of the dictionary, and expressing it as a linear combination of the elements. Note however that this representation is nonunique in general due to the inherent redundancy of a frame.

In [2], a least squares algorithm was proposed as a means to carry out this approximation. A time delay model was used as a test case, and it was shown that the wavelet based identification scheme is computationally more attractive than Laguerre based models.

2.3 Laguerre Approximation [2],[5]

Define Φ_m^p as

$$\Phi_m^p = \frac{\sqrt{2p}}{s+p}\left(\frac{s-p}{s+p}\right)^m \ , \ m = 0,1,2,\cdots. \tag{5}$$

It can be shown that the sequence of transfer functions $\{\Phi_m^p\}_{m=0}^\infty$ is an orthonormal basis for $H^2(\Pi^+)$. In the Laguerre method we decompose any $G \in H^2(\Pi^+)$ as a linear combination of elements of the series $\{\Phi_m^p\}_{m=0}^\infty$. i.e.

$$G = \sum_{m=0}^\infty c_m^p(G)\frac{\sqrt{2p}}{s+p}\left(\frac{s-p}{s+p}\right)^m, \quad p > 0. \tag{6}$$

The coefficients $c_m^p(G)$ can be obtained by least squares.

2.4 Matching Pursuit Algorithm

In [6] S. Mallat and Z. Zhang proposed a general algorithm for wavelet decompositions in order to reduce the computational burden. They coined the name *Matching Pursuit Algorithm* to describe it, and it uses the Gram-Schmidt orthogonalization procedure as a model.

Here is a statement of the algorithm.

- *Step 0* Select the analyzing wavelet ϕ with norm equal to one.
- *Step 1* Choose a (finite) set of translation and dilation levels. This will be called a dictionary from which the algorithm will select the translation and dilation levels. The a priori knowledge of time-frequency localization of the signal should help to roughly choose this set. Call this set $\mathbf{D}$.
- *Step 2* Set

$$R^0 f = f; \qquad i = 1.$$

- *Step 3* Find indices $(m_i, n_i) \in \mathbf{D}$ such that

$$|\langle R^{i-1} f,\ \phi_{m_i,n_i}\rangle| \geq |\langle R^{i-1} f,\ \phi_{m,n}\rangle| \quad \forall (m,n) \in \mathbf{D}. \tag{7}$$

- *Step 4* Compute the approximation at level i via

$$f_i = \sum_{k=0}^{i} \langle R^{k-1} f,\ \phi_{m_k,n_k}\rangle \phi_{m_k,n_k}. \tag{8}$$

  If satisfactory, stop. Otherwise continue.

- *Step 5* Compute

$$R^i f = R^{i-1} f - \langle R^{i-1} f,\ \phi_{m_i,n_i}\rangle \phi_{m_i,n_i},$$

  increment i, and go to step 3.

The following observations will help to understand the convergence of the algorithm.

First note that the residuals satisfy the following recursive equation

$$R^n f = \langle R^n f,\ \phi_{\gamma_n}\rangle \phi_{\gamma_n} + R^{n+1} f. \tag{9}$$

Evaluating the inner product on both sides with ϕ_{γ_n} and noting that ϕ_{γ_n} has norm equal to one, shows that $R^{n+1} f$ is orthogonal to ϕ_{γ_n}, which in turn enables us to write,

$$\|R_n f\|^2 = |\langle R^n f,\ \phi_{\gamma_n}\rangle|^2 + \|R^{n+1} f\|^2. \tag{10}$$

This shows that the residual $R^n f$ is monotonically decreasing. Arguing along this line (see e.g. [6] for details) shows that the series,

$$\sum_{n=0}^{\infty} \langle R^n f,\ \phi_{\gamma_n}\rangle \phi_{\gamma_n} \tag{11}$$

converges to the projection of f into the subspace spanned by the dictionary.

There are a few points worth noting that explain the efficiency of the algorithm as far as storage and speed are concerned. First note that in order to choose the indices (m_i, n_i) in the third step, and also to compute the approximation at step 4, we only need the inner product of the residuals with the wavelets. Thus, in step 5 one may directly compute the inner products instead. i.e.

$$\langle R^i f,\ \phi_{m,n}\rangle = \langle R^{i-1} f, \phi_{m,n}\rangle - \langle R^{i-1} f,\ \phi_{m_i,n_i}\rangle \langle \phi_{m_i,n_i},\ \phi_{m,n}\rangle. \tag{12}$$

Note that by doing this we actually do not have to do any integration for computing the inner products (other than just one set of integrations to get $\langle f,\ \phi_{m,n}\rangle$). Also, as far as storage is concerned, note that at each step we only have to keep one set of inner products of the residuals with the wavelets and as we proceed, we can throw away the previous ones.

The third point is that the correlation matrix (with the elements $\langle \phi_{k,l},\ \phi_{m,n}\rangle$) can be computed a priori and in fact this can be done analytically.

$$\begin{aligned} &\langle \phi_{m_1,n_1}(\omega),\ \phi_{m_2,n_2}(\omega)\rangle \\ &= \int \phi_{m_1,n_1}(\omega) a_0^{m_2/2} \overline{\phi(\omega a_0^{m_2} - n_2 b_0)} d\omega \\ &= a_0^{m_2/2} \int \phi_{m_1,n_1}(\omega) \phi(n_2 b_0 - \omega_0^{m_2}) d\omega \\ &= (\phi_{m_1,n_1} * \phi_{m_2,n_2})|_{\omega = \frac{n_2 b_0}{a_0^{m_2}}} \\ &= \mathcal{F}(\check{\phi}_{m_1,n_1}(t) \check{\phi}_{m_2,0}(t))|_{\omega = \frac{n_2 b_0}{a_0^{m_2}}}. \end{aligned} \tag{13}$$

where ˇ indicates Laplace Inverse.

Also, $\phi_{m,n}(t)$ can be written in terms of $\phi(t)$ as

$$\phi_{m,n}(t) = a_0^{-m/2} e^{\frac{i\, n b_0}{a_0^m} t} \phi(\frac{t}{a_0^m}). \tag{14}$$

Substitute for $\phi_{m,n}(t)$ from (14) into (13) to get

$$\begin{aligned} &\langle \phi_{m_1,n_1}(\omega),\ \phi_{m_2,n_2}(\omega)\rangle = a_0^{\frac{-(m_1+m_2)}{2}} \\ &\mathcal{F}(e^{\frac{n_1 b_0}{a_0^{m_1}} it} \check{\phi}(\frac{t}{a_0^{m_1}}) \check{\phi}(\frac{t}{a_0^{m_2}}))|_{\omega = \frac{n_2 b_0}{a_0^{m_2}} - \frac{n_1 b_0}{a_0^{m_1}}}. \end{aligned} \tag{15}$$

3 Experimental Setup

The system to be identified is a cantilevered aluminum beam to which two piezo-electric crystals are bonded. One of them is used as an actuator and the other is used as the sensor, i.e. the input and the output of the system are the supply voltage to one and the output voltage from the other. A power amplifier with two power supplies (operating at $+/- 80v$) delivers power to the piezo material. A signal analyzer (HP 3566 A) applies a swept sine to the input through one of the power amplifiers . The output signal is fed back to the signal analyzer in order to obtain a frequency response plot of the system. The objective of the experiment is to find a finite dimensional model that approximates the transfer function of the system.

A schematic diagram of the experimental setup is shown in fig.(1). The piezo-electric transducer drive circuit is shown in fig.(2).

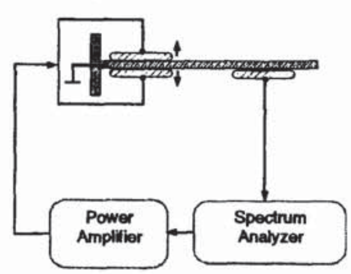

Figure 1: Schematic diagram of the experiment set up

Our frequency range of interest is from $80Hz$ to $1.5KHz$. The gain of the power amplifier is about 43 and

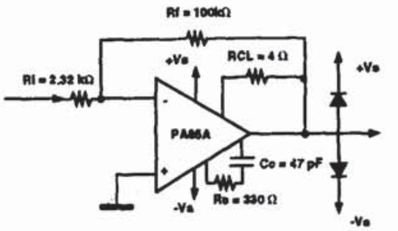

Figure 2: Piezo electric transducer drive circuit

it has a fairly large bandwidth. The frequency response obtained in this way is actually the one from the input of the amplifier to the sensor voltage. The output voltage of the spectrum analyzer is adjustable and has been set to get an output voltage of about 120 *volts peak − to − peak* at the output of the amplifier. At frequencies below $80Hz$ the output is distorted and it is so small that the $60Hz$ line noise is more significant, so we start from $80Hz$. The Bode plot obtained from the experiment is shown in fig.(3). The peaks in the magnitude plot represent the resonant frequencies.

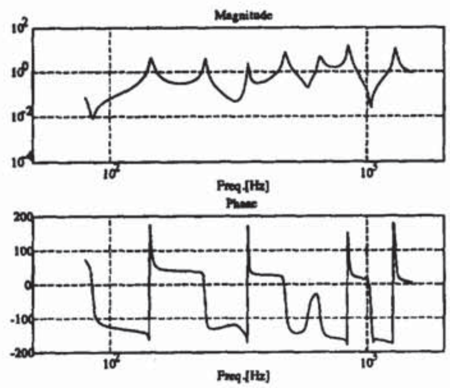

Figure 3: Frequency response obtained from the experiment

4 Computation

The matching pursuit scheme is employed to obtain a rational transfer function that approximates the frequency response of the flexible beam with piezo materials described above. Due to the presence of resonances the frequency response turns out to be fairly well localized in the frequency domain, which is a desirable factor when we want to apply the matching pursuit algorithm with a rational wavelet dictionary. fig. (4) shows the result of an approximation of degree 46. The dashed line represents the empirical frequency response, and the solid line is the the one obtained by the degree 46 approximation. Note that the governing model for the problem is actually a PDE and thus it is not a finite dimensional system. The

ξ	γ	a_0	b_0
1.0	1.60	2.0	3.33

Table 1: Parameters of the analyzing wavelet and translation-dilation step sizes

following function is used as the analyzing wavelet

$$\Psi(s) = \frac{1}{(s+\gamma)^2+\xi}. \tag{16}$$

The dilation and translation step sizes, γ, and ξ are as in Table (1). These values has been chosen to best match the shape of this frequency response.

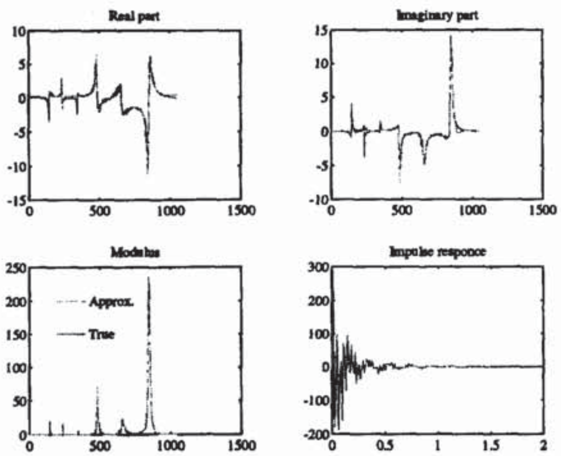

Figure 4: Approximation to the experimental freq. response of order 46

Figure (5) shows the magnitude of the wavelet coefficients obtained from the matching pursuit algorithm. As can be seen in the figure, very few coefficients have a significant magnitude and this is a measure of how well the frequency response is localized in frequency domain. This localization property, in turn, enables us to get a fairly low degree for the approximation.

In Table (2) we have listed the selected elements of the dictionary for an approximation of order 46. Note that each wavelet with nonzero translation adds up four to the order (because we have to include the negative translation too) and those with zero translation just add two. The approximation has the following form:

$$\begin{aligned}\hat{f}(s) \;=\; & \textstyle\sum_{n\neq 0}\sum_m (C_{mn} a_0^{m/2} g(a_0^{m/2}s - inb_0) + \\ & \overline{C_{mn}} a_0^{m/2} g(a_0^{m/2}s + inb0)) + \\ & \textstyle\sum_m C_{m0} a_0^{m/2} g(a_0^{m/2}s).\end{aligned} \tag{17}$$

It should be noted here that the relatively high degree of approximation is a result of trying to get an approximation over a large range of frequencies. In practice for the purpose of control we may not need this level of accuracy. The matching pursuit method has shown to give a better approximation in terms of the model order. Figure (6) shows this fact for the flexible beam example, i.e. for

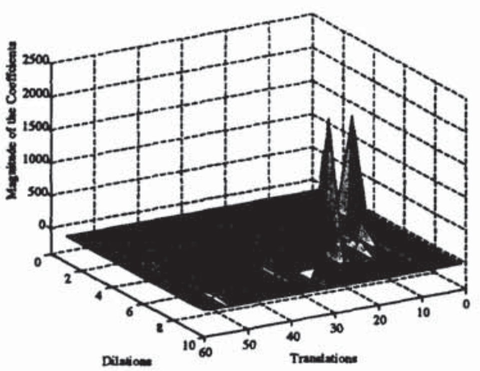

Figure 5: Magnitude of the wavelet coefficients corresponding to the experimental transfer function

Dilation level	Translation level	C_{mn}
1	59	$-0.690 - 3.848i$
0	52	$3.126 + 2.195i$
0	9	$0.409 + 3.033i$
0	29	$2.598 + 0.251i$
-1	15	$-3.007 - 7.586i$
-1	7	$2.814 - 0.275i$
-1	20	$1.737 - 2.216i$
-2	13	$4.712 + 23.738i$
-3	5	$1.660 - 9.231i$
-4	3	$-3.342 - 0.558i$
-4	2	$0.479 - 6.923i$
-5	2	$5.809 + 0.559i$
-6	1	$-2.264 - 2.985i$

Table 2: Selected elements of the dictionary and their coefficients

a certain model order we will get a much lower error with the matching pursuit model. Also for the matching pursuit algorithm the convergence rate is much faster than that of the Laguerre method. In this example we varied the parameter p for Laguerre functions to get the lowest error norm for model order 50 ($p = 210.0$) and then used this value of p to get the graph shown in fig. (6).

5 Conclusion

The matching pursuit method is a simple yet efficient algorithm to compactly represent a waveform/signal. Here, with a little modification, we applied this method to the problem of obtaining an approximation to a given frequency response. In comparison, it has been shown that this method out-performs the classical Laguerre method for system identification and gives a lower normalized error for the same degree of the approximation, at least in the case of the identification problem addresses in this paper. There is a natural question here that one may ask about this method. i.e. since the approximation actually converges to the projection of the transfer function into the space spanned by the transfer functions in the dictionary, why not simply compute the projection using a pseudo inverse and in this way get the answer in one shot? There are two reasons that we do not want to do

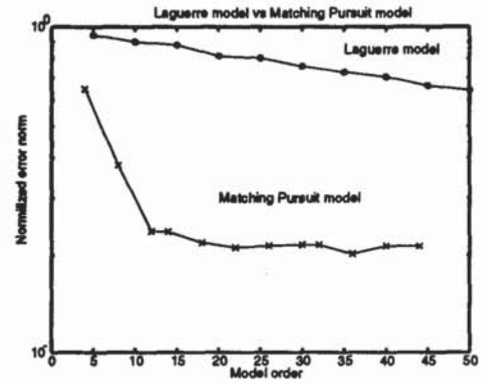

Figure 6: Comparison between Laguerre and matching pursuit models for the experimental set up

this; Firstly, computation of the pseudo inverse is very expensive and requires lots of operations specially when we are dealing with an extremely redundant dictionary. The second reason, which is really most important, is that our objective is to pick those elements in the dictionary that are more important in the sense that they can represent the structure of the signal the best. In this way we have taken advantage of the redundancy of the dictionary and at the same time if we want to only choose, say n elements to represent the system we know when to stop whereas in the case of pseudo inverse there is no clear way to do this.

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