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Approximation of Stable Linear Systems via Rational Wavelets[1]

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ABSTRACT

In this paper we present a new approach to the approximation (and identification) of transfer functions of certain classes of stable linear systems using the theory of frames. Specifically we work with frames generated in $H^2(\Pi^+)$ by the action of the affine group on rational wavelets. It is well-known that the affine group Aff(1) and the group Sl(2) act on transfer functions via certain natural scaling processes. The present paper exploits one such scaling action and some recent results of Daubechies to construct decompositions and approximations to transfer functions of infinite dimensional systems. A systematic computational method based on time-frequency concentrations has been developed to implement our ideas.

1 INTRODUCTION

The theory of frames in Hilbert spaces has proved to be a useful means to understand wavelet transforms based on nonorthogonal "basis functions" (c.f. Daubechies [2]). One of the main goals of this paper is to show that using this theory, and wavelet transforms based on the affine group Aff(1), one can construct decompositions of certain classes of transfer functions. Our approach is to generate frames from a single rational function by taking its translates and dilates. This is followed by a projection operation to obtain transfer functions that are Laplace transforms of real weighting patterns.

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It is well-known that the affine group Aff(1) acts on transfer functions in several ways. These were first investigated in the paper [1]. Two such actions are given by

(a) $g(s) \longrightarrow g(\alpha s + \beta)$

(b) $g(s) \longrightarrow \frac{\alpha g(s)}{1+\alpha\beta g(s)}$

The first action can be imbedded in an action of the group Sl(2) on transfer functions. The latter is due to Brockett and is perhaps not as well-known. This has probably further implications along the lines of Bargmann's theory of representations of Sl(2) on various Banach spaces of analytic functions. In the setting of this paper, a third action involving translates along the imaginary axis is used to create frames from a single (rational) function. This is the main result of this paper.

A primary application of this work is in the area of input-output approximation and identification of infinite-dimensional (e.g. distributed parameter) linear systems. To this end, we have developed a software environment that incorporates visualization tools to ascertain which frame elements to keep and which are to be discarded to produce a prescribed level of approximation. We present an example involving approximation of a delay system.

2 THE HARDY SPACE $H^2(\Pi^+)$

The class of transfer functions of interest to us is the Hardy space $H^2(\Pi^+)$, where Π^+denotes, the half-plane $\Re e \;\; s > 0$.

Definition 2.1 *Given a function F which is analytic in Π^+, F is said to belong to the class $H^2(\Pi^+)$if*

$$\sup_{x>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \qquad (2.1)$$

$H^2(\Pi^+)$ is a Banach space with norm (denoted $\|\cdot\|_{H^2}$) defined by (2.1).

Some of the most basic properties of $H^2(\Pi^+)$are captured by the following theorem (c.f. [5]).

Theorem 2.1 *Given $F \in H^2(\Pi^+)$, the following hold:*

(1) The nontangential limit of F exists at almost every point on the imaginary axis.

(2) The boundary value function of F is in $L^2(\mathbb{R})$ and,

$$F(x+iy) = \frac{1}{\pi} \int_{\mathbf{R}} F(i\omega) \frac{x}{x^2 + (y-\omega)^2} d\omega,$$

for $x > 0$.

(3) The functions $F_x(y) = F(x+iy)$ converge in L^2 norm to $F(iy)$ as $x \to 0$.

The elements of $H^2(\Pi^+)$ are transfer functions of causal input-output stable linear systems. More precisely, we have the classical result,

Theorem 2.2 (Paley-Wiener) *A complex-valued function F is in $H^2(\Pi^+)$ if and only if,*

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt,$$

for some $f \in L^2(0, \infty)$. Furthermore this representation is unique.

By the Paley-Wiener theorem,

$$\frac{1}{2\pi} \int_{\mathbf{R}} F(i\omega) e^{i\omega t} d\omega = 0 \;\; \text{for} \;\; t < 0. \qquad (2.2)$$

Hence boundary values of functions in $H^2(\Pi^+)$ comprise a subspace $\mathcal{D}$ of $L^2(\mathbb{R})$ characterized by (2.2). We also know by Theorem 2.1 that given the boundary value function $F(i\omega)$, F can be recovered on the right half-plane by the Poisson integral. Also $\|F\|_{H^2} = \|\widetilde{F}\|_{L^2}$ where $\widetilde{F}$ is the nontangential limit of F.

3 FRAMES AND WAVELETS

Frames, which were first introduced by Duffin and Schaeffer in [3], are natural generalizations of orthonormal bases for Hilbert spaces.

Definition 3.1 *Given a Hilbert space $\mathcal{H}$ and a sequence of vectors $\{h_n\}_{n=-\infty}^{\infty} \subset \mathcal{H}$, $\{h_n\}_{n=-\infty}^{\infty}$ is called a frame if there exists constants $A > 0$ and $B < \infty$ such that*

$$A\|f\|^2 \leq \sum_n |< f, h_n >|^2 \leq B\|f\|^2, \qquad (3.1)$$

for every $f \in \mathcal{H}$. A and B are called the frame bounds.

Remarks:

- A frame $\{h_n\}$ with frame bounds $A = B$ is called a *tight frame*.
- Every orthonormal basis is a tight frame with $A = B = 1$.
- A tight frame of unit-norm vectors for which $A = B = 1$ is an orthonormal basis.

Definition 3.2 *Given a frame $\{h_n\}$ in the Hilbert space $\mathcal{H}$, with frame bounds A and B, we can define the* frame operator *$S : \mathcal{H} \to \mathcal{H}$ as follows. For any $f \in \mathcal{H}$,*

$$Sf = \sum_n < f, h_n > h_n. \qquad (3.2)$$

The following theorem lists some properties of the frame operator which we shall find useful. Proofs of these and other related properties of frames can be found in [4] or [2].

Theorem 3.1

(1) S is a bounded linear operator with $AI \leq S \leq BI$, where I is the identity operator in $\mathcal{H}$.

(2) S is an invertible operator with $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

(3) Since $AI \leq S \leq BI$ implies that $\|I - \frac{2}{A+B}S\| \leq 1$, S^{-1} can be computed via the Neumann series,

$$S^{-1} = \frac{2}{A+B}\sum_{k=0}^{\infty}\left(I - \frac{2}{A+B}S\right)^k. \quad (3.3)$$

(4) The sequence $\{S^{-1}h_n\}$ is also a frame, called the dual frame, *with frame bounds B^{-1} and A^{-1}.*

(5) Given any $f \in \mathcal{H}$, f can be expressed in terms of frame elements as

$$\begin{aligned} f &= \sum < f, S^{-1}h_n > h_n \\ &= \sum < f, h_n > S^{-1}h_n. \quad (3.4) \end{aligned}$$

For the Hilbert space $L^2(\mathbb{R})$, frames can be constructed from affine wavelets. Recall that the affine group (or $ax + b$ group), denoted as Aff(1) has the left regular representation on $L^2(\mathbb{R})$ given by,

$$\begin{aligned} D_a f(x) &= a^{1/2} f(ax) \quad a > 0 \\ T_b f(x) &= f(x - b). \end{aligned}$$

A function ψ is said to be an *analyzing wavelet* or *mother wavelet* for $L^2(\mathbb{R})$ if

$$\left\{\psi_{m,n}(x) = D_{a_0^m} T_{nb_0}\psi(x) = a_0^{m/2}\psi(a_0^m x - nb_0)\right\}$$

is a frame for $L^2(\mathbb{R})$ for fixed constants $a_0 > 0$ and b_0. In that case any $f \in L^2(\mathbb{R})$ can be represented as

$$f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_{m,n}\psi_{m,n}. \quad (3.5)$$

Here $\{c_{m,n}\} \in \ell^2(\mathbb{Z}\times\mathbb{Z})$ and the representation (3.5) is a frame decomposition or wavelet decomposition.

Given a function ψ in $L^2(\mathbb{R})$ satisfying the admissibility condition

$$\int_{\mathbf{R}} \frac{\left|\widehat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty, \quad (3.6)$$

where $\widehat{\psi}$ is the Fourier transform of ψ, under mild hypotheses it is possible to determine values of a_0 and b_0 such that the sequence $\{\psi_{m,n}\}$ forms a frame for $L^2(\mathbb{R})$. In this case we say that (ψ, a_0, b_0) generates an affine frame for $L^2(\mathbb{R})$. Numerically, a_0 and b_0 can be determined by application of a theorem of Daubechies [2].

Satisfying (3.6) imposes the requirement that $\widehat{\psi}(0) = 0$, and since $\psi \in L^2(\mathbb{R})$, under additional smoothness assumptions, this makes the analyzing wavelet approximately a bandpass function. The concepts of time and frequency concentrations of a function can be precisely defined (c.f. [8]), but roughly speaking these are intervals in the time or frequency domains which contain 'most' of the energy in the signal. Let $\Omega(\psi) = [\omega_0(\psi), \omega_1(\psi)]$ denote the frequency concentration of ψ and $R(\psi) = [t_0(\psi), t_1(\psi)]$ ($t_0 \geq 0$) denote the time concentration of Ψ. Then ψ is a function which is concentrated in the time frequency plane on the rectangle $\mathcal{Q} = \Omega(\psi) \times R(\psi)$. Recalling the dilation property of the Fourier transform,

$$f(ax) \xrightarrow{\mathcal{F}} a^{-1}\widehat{f}(a^{-1}\omega),$$

we see that each of the frame elements $\psi_{m,n}$ are concentrated on rectangles

$$\begin{aligned} \mathcal{Q}_{m,n} = \ & [a_0^{-m}(t_0(\psi) + nb_0), a_0^{-m}(t_1(\psi) + nb_0)] \\ & \times[a_0^m\omega_0(\psi), a^m\omega_1(\psi)]. \end{aligned}$$

(Due to the symmetry about $\omega = 0$ for real-valued functions, only positive frequencies are considered.) Thus each coefficient $c_{m,n}(f)$ in (3.5) provides information about f in localized regions of the time-frequency plane.

It is now possible to speak of frames in $H^2(\Pi^+)$generated by affine wavelets. Let $RH^2(\Pi^+)$ denote the class of real-rational functions in $H^2(\Pi^+)$. Thus by the Paley-Wiener theorem $RH^2(\Pi^+)$ represents transfer functions of causal finite-dimensional linear systems with square integrable weighting patterns. Of particular interest to us is the case where the analyzing wavelet Ψ belongs to the class $RH^2(\Pi^+)$.

For any $F \in H^2(\Pi^+)$ define the restriction of F to vertical lines in the right half-plane by,

$$F_x(y) = F(x + iy), \qquad x > 0.$$

Also define the Fourier transform along vertical lines in $\Pi^+$by,

$$\widehat{F}_x(u) = \frac{1}{2\pi}\int_{\mathbf{R}} F(x+iy)e^{iuy}dy$$

for $F \in \mathrm{H}^2(\Pi^+)$.

Now let $\Psi \in \mathrm{RH}^2(\Pi^+)$ be our candidate for an analyzing wavelet, and suppose that for any $x > 0$ Ψ satisfies the admissibility condition,

$$C_{\Psi_x} = \int_{\mathbf{R}} \frac{\left|\widehat{\Psi}_x(u)\right|^2}{|u|}du < \infty, \qquad x > 0. \quad (3.7)$$

Under these assumptions, a family of continuous wavelet transforms can be defined on $\mathrm{H}^2(\Pi^+)$. Let,

$$\Psi^{(a,b)}(s) = a^{1/2}\Psi(as-ib), \qquad a,b \in \mathbb{R}, \quad a > 0.$$

Then for any $F \in \mathrm{H}^2(\Pi^+)$ define the continuous wavelet transform on the line $\Re e\ s = x$ by,

$$W_x f(a,b) = \int_{\mathbf{R}} F(x+iy)\overline{\Psi^{(a,b)}(x+iy)}dy. \quad (3.8)$$

Inversion of this transform is accomplished by,

$$\begin{aligned} &F(x+iy) \\ &= \frac{1}{C_{\Psi_x}}\int_{\mathbf{R}}\int_{\mathbf{R}} W_x F(a,b)\Psi^{(a,b)}(x+iy)\frac{da\ db}{a^2}. \end{aligned}$$

To define a discrete wavelet transform with respect to Ψ, let

$$\Psi_{m,n}(s) = a_0^{m/2}\Psi(a_0^m s - inb_0), \ a_0 > 0, \ \Re e\ s = x.$$

Since Ψ satisfies the admissibility condition, it is possible to determine parameters $a_0 > 0$ and b_0 such that the family $\{\Psi_{m,n}\}_{m,n\in\mathbf{Z}}$, is a *frame* for $\mathcal{D}$ which is a closed subspace of $\mathrm{L}^2(\mathbb{R})$. Thus for any $F \in \mathrm{H}^2(\Pi^+)$ we have that

$$F(x+iy) = \sum_m \sum_n \left\langle F, S^{-1}\Psi_{m,n}\right\rangle \Psi_{m,n}, \quad (3.9)$$

where S is the *frame operator* associated with the frame $\{\Psi_{m,n}\}$.

Example 1: Consider the function

$$\Psi(s) = \frac{1}{(s+\gamma)^2+\xi^2}.$$

It is admissible for $\Re e\ s \geq 0$. It is shown in [7] that for $\gamma = 5$ and $\xi = 1$, (Ψ, a_0, b_0) generates and affine frame for $\mathrm{H}^2(\Pi^+)$, for $a_0 = 2$, and $0 < b_0 \leq 16.5$.

4 WS DECOMPOSITIONS

Note that although the analyzing wavelet $\Psi \in \mathrm{RH}^2(\Pi^+)$, we have that $\Psi_{m,n} \notin \mathrm{RH}^2(\Pi^+)$, for $n \neq 0$. Hence arbitrary truncations of the series (3.9) are not necessarily real-rational functions. However it is easily seen that functions of the form $G^{m,n}(s) = \alpha\Psi_{m,n}(s) + \overline{\alpha}\Psi_{m,-n}(s), \ n \neq 0$, where $\overline{\alpha}$ denotes the complex conjugate of α, are real-rational. Furthermore in [7, 9] it is shown that for $F, \Psi \in \mathrm{H}^2_{\mathbf{R}}(\Pi^+)$ = Laplace transforms of real-valued functions in $\mathrm{L}^2(0,\infty)$,

$$c_{m,n} = \left\langle F, S^{-1}\Psi_{m,n}\right\rangle = \overline{\langle F, S^{-1}\Psi_{m,-n}\rangle} = \overline{c}_{m,-n}. \quad (4.1)$$

The above two observations lead to following decomposition theorem for $\mathrm{H}^2_{\mathbf{R}}(\Pi^+)$.

Theorem 4.1 *Let $\Psi \in \mathrm{RH}^2(\Pi^+)$, be an admissible analyzing wavelet, such that (Ψ, a_0, b_0), generates an affine frame $\{\Psi_{m,n}\}$, for $\mathrm{H}^2(\Pi^+)$. Then any F in $\mathrm{H}^2_{\mathbf{R}}(\Pi^+)$ may be represented as,*

$$F = \sum_m \sum_{n=0}^{\infty} F^{m,n}, \quad (4.2)$$

where each $F^{m,n}$ is a real-rational function defined by,

$$\begin{aligned} F^{m,0} &= \left\langle F, S^{-1}\Psi_{m,0}\right\rangle \Psi_{m,0} \\ F^{m,n} &= \left\langle F, S^{-1}\Psi_{m,n}\right\rangle \Psi_{m,n} + \\ &\quad \overline{\langle F, S^{-1}\Psi_{m,n}\rangle}\Psi_{m,-n}, \ n = 1,2,\ldots \end{aligned}$$

for $m \in \mathbb{Z}$.

We shall refer to (4.2) as a *wavelet system* (WS) decomposition of $F \in \mathrm{H}^2_{\mathbf{R}}(\Pi^+)$.

4.1 Rational WS Approximation

Given a transfer function $F \in \mathrm{H}^2_{\mathbf{R}}(\Pi^+)$, and its WS decomposition, a rational approximation $\widetilde{F}$, to F, may be constructed as a truncated WS series, i.e

$$\widetilde{F}(s) = \sum_{(m,n)\in\mathcal{J}} F^{m,n}(s),$$

where $\mathcal{J}$ is a suitably chosen *finite* index set, and $F^{m,n}$, are real-rational functions as in Theorem

4.1. By selecting the index set $\mathcal{J}$, to include the terms with the largest wavelet coefficients $\alpha_{m,n} = \langle F, S^{-1}\Psi_{m,n}\rangle$, a coarse bound on the approximation error may be computed in terms of the norm of the coefficients in the selected terms and the ratio B/A of the frame bounds (c.f. [7]). A key feature of WS rational approximations is that time-frequency localization properties of the underlying wavelets can result in fairly compact representations which in turn lead to low-order approximants.

5 NUMERICAL RESULTS

For comparison purposes, we can also consider rational approximation methods based on truncations of the Fourier-Laguerre expansions of transfer functions (c.f. [6]). The class of Laguerre orthonormal bases $\{\Phi_m^p\}_{m=0}^{\infty}$ for $H^2(\Pi^+)$, are defined by,

$$\Phi_m^p = \frac{\sqrt{2p}}{s+p}\left(\frac{s-p}{s+p}\right)^m, \; p > 0, \; m = 0, 1, \ldots \tag{5.1}$$

Hence any $F \in H^2(\Pi^+)$, may be decomposed with respect to the orthonormal basis $\{\Phi_m^p\}_{m=0}^{\infty}$, and truncations of such series decompositions may be used as rational approximants. In this section we apply both WS methods and Laguerre function methods to a delay system.

Consider the infinite-dimensional (delay) system with transfer function,

$$G(s) = \frac{e^{-s\tau}}{s^2 + q_1 s + q_2}.$$

We applied both WS and Laguerre techniques to approximating this system with $\tau = 2.0$, $q_1 = 1.25$, $q_2 = 0.40625$. In both Laguerre and WS methods, there exits tradeoffs between representing long delays and short time constants (c.f. [7]). The above values of τ, q_1 and q_2, were chosen such that the resulting approximation problem is fairly challenging for both WS and Laguerre methods. Figure 1 is a plot of the magnitudes of the coefficients in the WS decomposition of the above transfer function. Note that the coefficients are well concentrated, and that the delay is visually evident from the distribution

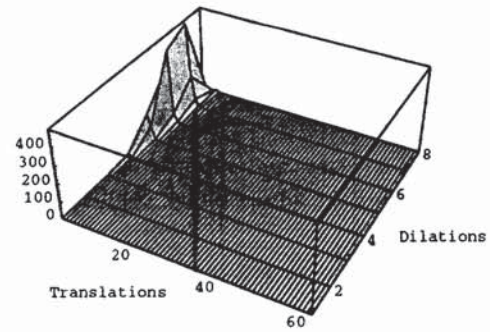

Figure 1: Magnitude of WS coefficients for delay system example.

of coefficients over dilation levels. The normalized time-domain approximation error for both WS approximation and Laguerre approximation is plotted in Figure 2 versus model order.

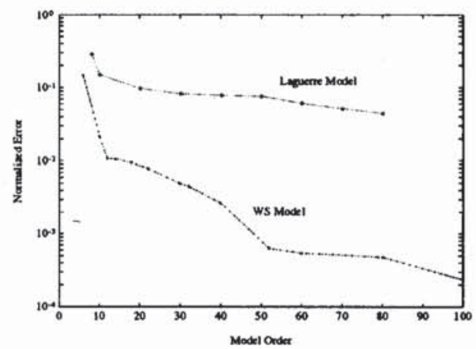

Figure 2: Approximation error for delay system example.

6 CONCLUSIONS AND DISCUSSION

In this paper we discussed the use of rational wavelet decompositions of the Hardy space $H^2(\Pi^+)$, in problems of rational approximation of linear systems. The requirement of real-rationality of the approximants led to the wavelet system (WS) decomposition via a regrouping of the wavelet expansion. Numerical comparisons of the methods of the current paper with methods based on the classical Laguerre functions are promising. Truncated WS representations may also find application as 'linear-in-parameters' black-box model sets for system identification (c.f. [7]).

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