2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Motion Camouflage for Coverage

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Abstract— **Pursuit strategies can lead to cohesive behavior. This idea is explored via consideration of a two-particle mutual pursuit system based on motion camouflage as the underlying strategy. Such a two-particle system can be thought of as a model of a pair of cooperative unmanned aerial vehicles, and in a limiting case as a model of a vehicle in the vicinity of a signal source (beacon). Drawing on reductions by symmetry and associated phase space properties, we show how this particular instance of cyclic pursuit can be exploited to achieve observational requirements such as coverage over physical space.**

I. INTRODUCTION

In observations of nature and in the conduct of human affairs, including competitive sports and warfare, pursuit and evasion phenomena have played a prominent role [1]. Studies of prey capture by echolocating bats [2], and aerial territorial battles among insects [3] [4], indicate that an approximate geometric regularity known as *motion camouflage with respect to infinity* (or constant absolute target direction strategy) is prevalent in the flight behavior of the species involved. It has been suggested that, at least in the case of visual insects such as hoverflies and dragonflies, for reasons of stealth, a pursuer seeks to appear stationary relative to a familiar background feature in the visual field of a target. When the background feature is distant, motion camouflage with respect to infinity would then hold. In these contexts, motion camouflage specifies a submanifold in the joint state space of predator and prey (or pursuer and target of pursuit) that constrains the motion. In [5] a control-theoretic treatment of this phenomenon was initiated, with a focus on planar models. Later papers discuss three-dimensional models and associated feedback laws, as well as questions of competition between various alternative pursuit strategies (see [6] and references therein).

The approach in the current paper as well as in the earlier cited papers is based on interacting particle models of agents in pursuit and evasion. However, here we let a pair of agents cooperate and engage in a form of mutual pursuit, governed by motion camouflage feedback laws, leading to an oscillator system with an interesting Hamiltonian structure (explained in [7]). A motivating problem in this paper is the use of such a cooperative system in a task of comprehensive aerial observation of a region of interest by one or more unmanned aerial vehicles (UAVs). Mutual motion camouflage as a building block for coherent structures (flocks, swarms, schools) is an instance of cyclic pursuit [8] [9] [10], and the exploration of the geometry of this two-particle system is a natural step to further understanding and exploitation in problems such as coverage (see also [11]).

We begin by briefly describing the mutual interaction model in the plane (section II), including reduction by symmetry. Section III is devoted to the limiting case in which one of the particles is stationary, here called the *beacon*. The motion camouflage feedback law, and a modification of the same to include dissipation, are presented. Section IV discusses the exploitation of these ideas in coverage path planning. The case of two mobile agents, which leads to somewhat more complicated coverage issues, is considered in section V.

This research was supported in part by the Air Force Office of Scientific Research under AFOSR grant FA9550710446; by the Army Research Office under ARO grant W911NF0610325; by the ODDR&E MURI2007 Program Grant N000140710734 (through the Office of Naval Research); by the NSF-NIH Collaborative Research in Computational Neuroscience Program (CRCNS2004) NIH-NIBIB grant 1 R01 EB004750-01.

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II. MODELING MUTUAL INTERACTIONS

As in Justh-Krishnaprasad [5] [12], we model each agent as a unit-mass particle moving along twice-differentiable curves in $\mathbb{R}^2$. The motion of the i-th agent is described by its natural Frenet frame [13], defined by position $\mathbf{r_i}$ and orthonormal vectors $\mathbf{x_i}, \mathbf{y_i}$, where $\mathbf{x_i}$ is tangent to the curve traced by the particle, and $\mathbf{y_i} = \mathbf{x_i}^{\perp}$. Each agent is subject to a *gyroscopic* curvature (steering) control, denoted by u_i, which has the property of preserving the speed of motion ν_i. This is appropriate, for example, for the need of UAVs to maintain a certain airspeed in order to remain aloft. The equations of motion are, for $i = 1, 2$:

$$\begin{aligned} \dot{\mathbf{r}}_\mathbf{i} &= \nu_i\, \mathbf{x_i} \\ \dot{\mathbf{x}}_\mathbf{i} &= \nu_i\, u_i\, \mathbf{y_i} \\ \dot{\mathbf{y}}_\mathbf{i} &= -\nu_i\, u_i\, \mathbf{x_i}. \end{aligned} \tag{1}$$

We focus attention on *mutual* interactions between two agents, corresponding to the concept that each agent applies the same control strategy based on the observed state of the other. To account for the possibility that $\nu_1 \neq \nu_2$, we define as mutual gyroscopic interactions those which satisfy:

$$u_1\, \nu_1 = u_2\, \nu_2 = u. \tag{2}$$

When (2) holds, u has a gyroscopic effect on the relative motion vectors ($\mathbf{r} = \mathbf{r_1} - \mathbf{r_2},\ \mathbf{g} = \nu_1\, \mathbf{x_1} - \nu_2\, \mathbf{x_2},\ \mathbf{h} = \mathbf{g}^{\perp}$) as well as on their (scaled) center of mass counterparts ($\mathbf{z} = \mathbf{r_1} + \mathbf{r_2}$, $\mathbf{k} = \nu_1\, \mathbf{x_1} + \nu_2\, \mathbf{x_2}$, $\mathbf{l} = \mathbf{k}^{\perp}$) according to:

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{g} \\ \dot{\mathbf{g}} &= u\, \mathbf{h} \\ \dot{\mathbf{h}} &= -u\, \mathbf{g} \end{aligned} \tag{3}$$

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$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{k}\\ \dot{\mathbf{k}} &= u\,\mathbf{l}\\ \dot{\mathbf{l}} &= -u\,\mathbf{k}\end{aligned}\qquad(4)$$

As a result, mutual interactions have the special property of preserving the magnitude $|\mathbf{g}| = |\mathbf{h}| = \delta$ of the relative velocity vector and the magnitude $|\mathbf{k}| = |\mathbf{l}| = \theta$ of the center of mass velocity vector.

We additionally require the curvature control u to depend only on the relative position and orientation between the particles, i.e. to be SE(2)-invariant (with respect to translations and rotations of the absolute reference frame). Control laws of this kind have the advantage that they can be implemented using only local sensing. We introduce the following *shape variables*, containing information on the shape of the formation composed by the two agents: $\rho = |\mathbf{r}| = (\mathbf{r}\cdot\mathbf{r})^{1/2}$, $\gamma = (1/|\mathbf{r}|)\,(\mathbf{r}\cdot\mathbf{g})$ and $\lambda = (1/|\mathbf{r}|)\,(\mathbf{r}\cdot\mathbf{h})$. The corresponding shape dynamics are:

$$\begin{aligned}\dot{\rho} &= \gamma\\ \dot{\gamma} &= (\delta^2-\gamma^2)/\rho + u\,\lambda\\ \dot{\lambda} &= -\gamma\,\lambda/\rho - u\,\gamma.\end{aligned}\qquad(5)$$

All the control laws discussed in the following sections are of the form $u = u(\rho,\gamma,\lambda)$, hence automatically satisfying the SE(2)-invariance property. The system (5) is a reduction of (3) since $\gamma^2+\lambda^2 = \delta^2 = constant$.

Letting $\zeta = |\mathbf{z}| = (\mathbf{z}\cdot\mathbf{z})^{1/2}$, $\xi = (1/|\mathbf{z}|)\,(\mathbf{z}\cdot\mathbf{k})$ and $\eta = (1/|\mathbf{z}|)\,(\mathbf{z}\cdot\mathbf{l})$, we obtain:

$$\begin{aligned}\dot{\zeta} &= \xi\\ \dot{\xi} &= (\theta^2-\xi^2)/\zeta + u\,\eta\\ \dot{\eta} &= -\xi\,\eta/\zeta - u\,\xi.\end{aligned}\qquad(6)$$

The system (6) is a reduction of (4) since $\xi^2+\eta^2 = \theta^2 = constant$. We showed in [7] that in case of motion camouflage control, the trajectories of the agents can be reconstructed from those of the reduced variables ρ, γ, λ, ζ, ξ, η; the same is also true for the other control laws introduced in this paper.

The model presented is suitable to describe gyroscopic, SE(2)-invariant mutual interactions between two agents, for any values of their (constant) speeds $\nu_i, i = 1,2$. Even as the speed of one of the agents tends to zero (e.g. $\nu_2 \to 0$), and its curvature control grows unbounded ($u_2 \to \infty$), the equations (1) are still well-defined since u_i and ν_i always multiply each other. As a limiting case, the model also captures the dynamics of a single mobile agent moving in the vicinity of a fixed beacon (signal source). Assume that $\nu_2 = 0$; then $\mathbf{g} = \nu_1\,\mathbf{x_1}$ and $\mathbf{h} = \nu_1\,\mathbf{y_1}$ involve only the dynamics of the moving agent, but equations (3), reducible to (5), still apply. In this limiting case there is one-sided interaction since the beacon need not to be an agent itself, but can be for example a natural or artificial landmark of interest which the moving agent uses as a reference for its motion or surveillance.

The "beacon case" is of interest in applications and as a motivating problem setting, and will be discussed first below.

In this setting, the center of mass motion is merely a scaled version of the relative motion.

III. BEACON AND MOTION CAMOUFLAGE

A. Motion Camouflage

The Motion Camouflage (MC) curvature control law takes the form:

$$u = -(\mu/|\mathbf{r}|)\,\left(\mathbf{r}\cdot\dot{\mathbf{r}}^{\perp}\right) = -(\mu/|\mathbf{r}|)\,(\mathbf{r}\cdot\mathbf{h}) = -\mu\,\lambda,\quad(7)$$

where μ is a non-negative constant gain. When $\lambda = 0$ we say that the system is in a state of Motion Camouflage.

In [5], (7) was introduced as a high-gain pursuit law which executes the motion camouflage, or Constant Absolute Target Direction (CATD), strategy displayed by certain animal species in pursuing their targets [2], [3]. Here we study (7) as a law to generate trajectories around a beacon with useful coverage properties. To this end, we drop the requirement for μ to be a high-gain, leaving it instead as a free design parameter. Substituting (7) into (5) yields the closed-loop dynamics:

$$\begin{aligned}\dot{\rho} &= \gamma\\ \dot{\gamma} &= (1/\rho-\mu)\,(\delta^2-\gamma^2)\\ \dot{\lambda} &= (\mu-1/\rho)\,\lambda\,\gamma.\end{aligned}\qquad(8)$$

Equations (8) have been studied in [7], in the context of Mutual Motion Camouflage (for two agents); we recall without proof the relevant results and apply them to the problem at hand.

Theorem 1: Let $(\rho(0),\gamma(0),\lambda(0)) = (\rho_0,\gamma_0,\lambda_0)$ be initial conditions for (8) which satisfy: $\rho_0 > 0$, $-\delta \le \gamma_0 \le \delta$ and $\lambda_0^2+\gamma_0^2 = \delta^2$. Note that $\delta = \nu$, the speed of the moving agent. Then the dynamics (8) fall into one of the following cases:

(a) If $(\rho_0,\gamma_0,\lambda_0) = (1/\mu, 0, \pm\delta)$ then $(\rho(t),\gamma(t),\lambda(t)) = (\rho_0,\gamma_0,\lambda_0)\forall t \ge 0$. These initial conditions correspond to equilibria of (8). The resulting motion of the agent is to follow circular orbits of radius $1/\mu$ around the beacon.

(b) If $\gamma_0 = -\delta$, then $(\rho(t),\gamma(t),\lambda(t)) = (\rho_0-\delta t, -\delta, 0)\forall t \in [0, \rho_0/\delta)$. These initial conditions fall into the *motion camouflage pursuit manifold* (see [6]); the moving agent is in motion camouflage with respect to the beacon, and remains in such state till time ρ_0/δ when it actually collides with the beacon.

(c) If $\gamma_0 = \delta$, then $(\rho(t),\gamma(t),\lambda(t)) = (\rho_0+\delta\,t, \delta, 0)\forall t \ge 0$. These initial conditions fall into the *escape* motion camouflage manifold [6], in which the agent recedes from the beacon while remaining "motion camouflaged" from it.

(d) If $(\rho_0,\gamma_0,\lambda_0) \neq (1/\mu, 0, \pm\delta)$ and $\gamma_0 \neq \pm\delta$, then $(\rho(t),\gamma(t),\lambda(t))$ follows a periodic orbit which:

- corresponds to a level set of the "energy" function:

$$E(\rho,\gamma) = \rho^2\,(\delta^2-\gamma^2)\,e^{-2\mu\rho},\qquad(9)$$

- has period:

$$T = 2\int_{\rho_{min}}^{\rho_{max}} d\rho/\sqrt{\delta^2 - E(\rho_0,\gamma_0)\,e^{2\mu\rho}/\rho^2},\qquad(10)$$

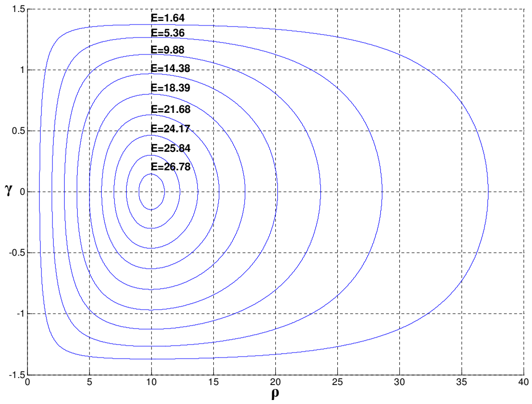

Fig. 1. Phase portrait of (8) when $\mu = 0.1, \delta = \sqrt{2}$

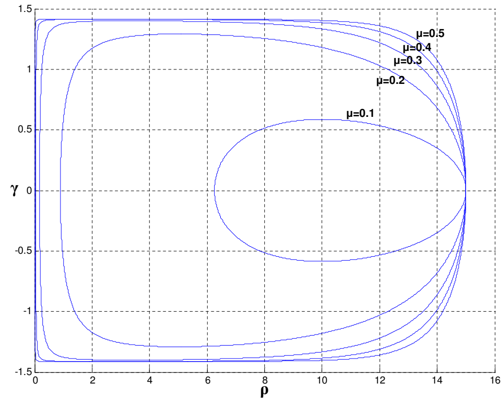

Fig. 2. Orbits of (8) when $(\rho_0, \gamma_0) = (15, 0), \delta = \sqrt{2}$, $\mu \in [0.1, 0.5]$

where ρ_{min} and ρ_{max} are the two positive solutions of:

$$\rho^2\, \delta^2\, e^{-2\mu\rho} = E(\rho_0, \gamma_0), \tag{11}$$

- satisfies: $\lambda(t) = \lambda_0\, \rho_0\, e^{\mu(\rho(t)-\rho_0)}/\rho(t), \quad \forall t \geq 0.$

Except in the special case in which the initial conditions fall in one of cases (a)-(c), we see that the agent orbits around the beacon with distance oscillating between a minimum (strictly positive) value ρ_{min} and a maximum value ρ_{max}.

We also proved in [7] that if we express the relative position $\mathbf{r}$ of the agent with respect to the beacon in polar coordinates $(|\mathbf{r}|, \alpha_{\mathbf{r}})$, then to every period T of the magnitude ρ corresponds a phase shift:

$$\alpha_{\mathbf{r}}(t+T) - \alpha_{\mathbf{r}}(t) = \lambda_0\, \rho_0\, e^{-\mu\rho_0}\, (e^{\mu\rho}/\rho^2)_{avg}, \tag{12}$$

where $(\cdot)_{avg}$ stands for the average over one period. The phase shift is non-zero since $e^{\mu\rho}/\rho^2$ is a strictly positive function and $\lambda_0, \rho_0 \neq 0$ if the initial conditions fall in case (d) of Theorem 1. Therefore the evolution of $\mathbf{r}(t)$ is not T-periodic, and if the right hand side of (12) is not a rational multiple of 2π the agent will fill with time the annular region centered at the beacon and having inner radius ρ_{min} and outer radius ρ_{max}. Motion Camouflage can hence be used to generate region-filling trajectories centered at the beacon.

The values of $\rho_{min} > 0$, important for collision avoidance, and ρ_{max}, which is important if the moving agent must maintain connectivity (visual, radar, etc.) with the beacon, are determined by the values of μ and E. The latter depends on μ but essentially on the initial conditions; hence ρ_{min} and ρ_{max} cannot be chosen at will by changing μ.

Figures 1 and 2 show representative orbits in the (ρ, γ)-space for different initial conditions and different values of μ respectively. Note in Figure 2 that as μ is increased, the trajectories of the moving agent approach closely the motion camouflage manifolds and $\rho_{min} \to 0$; this is consistent with the use of (7) with high-gain as a way to execute the mutual camouflage pursuit strategy.

Remark 1: The Motion Camouflage control law (7) can alternatively be used to achieve abrupt "escape" of the moving agent from the beacon, by using a negative gain μ. Notice in fact from (8) that if $\mu < 0$ then $\ddot{\rho} = \dot{\gamma} > 0\; \forall t \geq 0$, hence $\rho(t) \to \infty, \gamma(t) \to \delta$. In this case the agent recedes from the beacon along the escape motion camouflage manifold.

B. Motion Camouflage with stabilization to a desired orbit

Motion Camouflage alone does not provide enough degrees of freedom to satisfy simultaneously collision avoidance and connectivity requirements; moreover the orbits it produces are not orbitally asymptotically stable and hence not robust to disturbances. Both limitations can be overcome by adding an additional control term to (7), which "adjusts" the energy to the value E_d corresponding to a desired orbit, making it orbitally asymptotically stable.

Theorem 2: Let E_d be a desired value of energy, chosen within the limits of the function (9): $0 < E_d \leq \delta^2\, e^{-2}/\mu^2$. Then the control law:

$$u = u_{MC} + u_{ADJ} = -\mu\,\lambda + k_d\,\lambda\,\gamma\,(E(\rho,\gamma) - E_d), \tag{13}$$

with $k_d > 0$ and $E(\rho, \gamma)$ as in (9), makes the periodic orbit of Motion Camouflage with energy E_d orbitally asymptotically stable with region of attraction $\{(\rho, \gamma, \lambda) : \rho > 0, -\delta < \gamma < \delta, (\rho, \gamma, \lambda) \neq (1/\mu, 0, \pm\delta)\}$.

Proof: If $E(\rho, \gamma) = E_d$, the control law (13) is identical to (7). Hence we still have that the periodic orbit of MC with energy E_d is an invariant manifold. On the other hand all the other periodic orbits of MC (those with $E(\rho, \gamma) \neq E_d$) are not invariant because of the effect of the term u_{ADJ}. The equilibria and motion camouflage invariant manifolds described in Theorem 1 still exist, hence they are excluded from the region of attraction of any other orbit. For all the other initial conditions, we need to prove that the distance between the corresponding integral curves and the periodic orbit with energy E_d converges to 0. Since E is a continuous function of ρ, γ, it suffices to prove that $(E(\rho,\gamma)-E_d)^2 \to 0$:

$$\frac{d}{dt}(E(\rho,\gamma) - E_d)^2 = -K(\rho,\gamma)\,(E(\rho,\gamma) - E_d)^2,$$

where $K(\rho, \gamma) = 4\, k_d(\delta^2 - \gamma^2)\rho^2\gamma^2 e^{-2\mu\rho} \geq 0\, \forall(\rho, \gamma)$.
If the state of the system is outside of equlibria and motion camouflage manifolds, it must fall into either one of the following cases:

1) $\gamma \neq 0 \Rightarrow K(\rho,\gamma) > 0 \Rightarrow d(E(\rho,\gamma) - E_d)^2/dt < 0$ and the distance between the current value of energy and the desired one decreases
2) $\gamma = 0 \Rightarrow K(\rho,\gamma) = 0 \Rightarrow d(E(\rho,\gamma) - E_d)^2/dt = 0$ but $u = u_{MC}$ hence the (ρ,γ)-trajectory follows momentarily the MC periodic orbit associated to the current value of energy, which quickly leads to $\gamma \neq 0$ and hence to case 1.

This proves that for any initial condition in $\{(\rho,\gamma,\lambda) : \rho > 0, -\delta < \gamma < \delta, (\rho,\gamma,\lambda) \neq (1/\mu, 0, \pm\delta)\}$, the distance between the actual energy and the desired one decreases monotonically to 0 i.e. all these initial conditions are attracted by the periodic orbit having energy E_d. ■

The choice of E_d, together with that of μ, allows us to shape the orbits of the agent. In particular the distance between the agent and the beacon at steady state will periodically oscillate between minimum and maximum values given by the two positive solutions to:

$$\rho^2\, \delta^2\, e^{-2\mu\rho} = E_d. \tag{14}$$

Remark 2: The control law (13) can also be used to obtain stable circular motion of the agent around the beacon. If in fact we choose $E_d = E_{max} = \delta^2 e^{-2}/\mu^2$, this desired energy corresponds to one of the equilibria $(\rho,\gamma,\lambda) = (1/\mu, 0, \pm\delta)$ of the reduced dynamics. These equilibria are associated with circular orbits of radius $1/\mu$ centered at the beacon. An alternative, and simpler, control law that achieves stable circular motion is:

$$u = -\mu\,\lambda - k_{eq}\,\gamma\,\lambda^{2k-1}, \tag{15}$$

with k a positive integer and $k_{eq} > 0$.

IV. COVERAGE PATH-PLANNING

The modified MC control law (13) provides a way of designing stable trajectories that can be used to solve coverage path-planning problems as in the following.

Coverage Problem 1: Determine a suitable trajectory for a mobile agent moving at constant speed so that it fully covers an annular region of desired inner radius $\rho_{b,min}$ and desired outer radius $\rho_{b,max}$, centered at a beacon.

A coverage problem of this type could arise for example if the agent must explore (cover) the area surrounding a landmark of interest (beacon) on the basis of local sensing only. The quantity $\rho_{b,max}$ accounts for limited range of sensing, and $\rho_{b,min}$ accounts for safety distance to avoid collisions.

Proposition 1: Coverage Problem 1 can be solved by an agent applying the modified Motion Camouflage control law (13) with parameters:

$$\mu = \frac{\ln(\rho_{b,max}/\rho_{b,min})}{\rho_{b,max} - \rho_{b,min}}, \quad E_d = \rho_{b,min}^2 \delta^2 e^{-2\mu\rho_{b,min}} \tag{16}$$

where $\delta = \nu$, the (constant) speed of the agent.

Proof: We know from section III-B that, after a transient, the trajectories of the agent will converge to the energy level set E_d, and will cover the annular region centered at the beacon and having inner and outer radius given by the solutions to (14). By inspection, one of the solutions when E_d is given by (16) is certainly $\rho_{b,min}$. Express now the other (unknown) solution as $K\ \rho_{b,min}$, i.e. define K as the unknown ratio between the solutions. Since $K\ \rho_{b,min}$ must also satisfy (14), we obtain $K^2 e^{-2\mu K \rho_{b,min}} = e^{-2\mu\rho_{b,min}}$, which when solved for K with μ given by (16), yields exactly $K = \rho_{b,max}/\rho_{b,min}$. ■

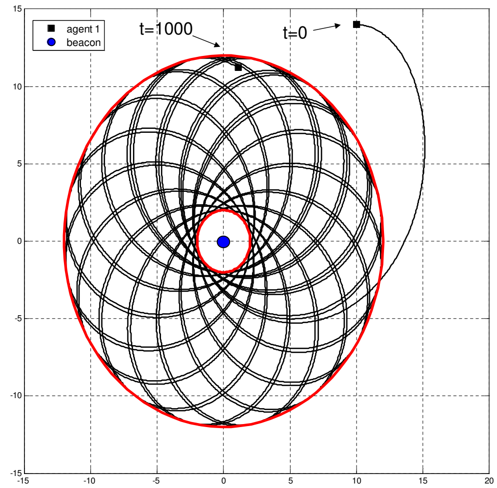

Fig. 3. Solution to Coverage Problem 1 with the modified MC control law (13). The agent starts at position [10,14] and has speed $\nu = 2$. By choosing parameters $\mu = 0.1792$, $E_d = 7.8137$ as prescribed by (16), and $K_d = 0.2$, the agent begins filling the annular region with desired radii $\rho_{b,min} = 2, \rho_{b,max} = 12$.

Remark 3: Proposition 1 holds provided that the initial conditions of the agent (at the moment of activation of (13)) are not in one of cases (a)-(c) of Theorem 1.

Figure 3 shows the results of a simulation in which Coverage Problem 1 is successfully solved by means of the modified Motion Camouflage control law (13).

V. TWO MOBILE AGENTS

After discussing the simpler case of one agent moving about a beacon, we are now ready to consider the more general case of mutual interaction between two agents. The main difference with the beacon case is that now the motion of the agents must be reconstructed not only from their relative motion but also from the non-trivial motion of the center of mass, as determined by the choice of the control law. The reduced center of mass motion equations (6) are coupled to (5) by $u = u(\rho,\gamma,\lambda)$; we consider the same control laws studied for the beacon case, but in the context of mutual interactions.

A. Mutual Motion Camouflage

We define as Mutual Motion Camouflage (MMC) the case in which two agents mutually interact (i.e. their curvature controls satisfy (2)) with the common control law being given by (7). The relative motion between the agents is governed by the same equations (8) studied in section III-A for the beacon case; hence by appeal to those results we can conclude that the relative distance between the agents will periodically oscillate between a minimum value ρ_{min} and a maximum value ρ_{max}, which depend on the choice of μ and the initial conditions. One difference with the beacon case is that the parameter $\delta = |\mathbf{g}|$ here genuinely depends on the initial conditions, in particular initial headings of the agents, whereas in the beacon case it is simply equal to the speed of the mobile agent.

The reconstruction, from the relative motion, of center of mass motion and individual trajectories of the agents is explained in detail for the MMC case in [7]. Here we just recall the main results with a sketch of the proof, with the aim of solving the coverage problem.

Theorem 3: Let $\mathbf{r_1}(0)$, $\mathbf{r_2}(0)$, $\mathbf{x_1}(0)$ and $\mathbf{x_2}(0)$ be the initial positions and headings for the two agents, with respect to an arbitrary absolute reference frame. Assume that $\mathbf{r_1}(0) \neq \mathbf{r_2}(0)$ and $\nu_1\, \mathbf{x_1}(0) \neq \pm\nu_2\, \mathbf{x_2}(0)$, i.e. the agents have some initial spatial separation and their initial velocity vectors are neither identical nor opposite. Denote as $\tilde{\mathbf{r}}_\mathbf{1} = \mathbf{r_1} - \mathbf{z_0}/2$ and $\tilde{\mathbf{r}}_\mathbf{2} = \mathbf{r_2} - \mathbf{z_0}/2$ the positions of the agents with respect to an absolute reference frame which is translated by $\mathbf{z_0}/2$ from the original one, where:

$$\mathbf{z_0} = \mathbf{z}(0) - \sigma^2 \begin{bmatrix} \mathbf{k}^T(0) \\ \mathbf{l}^T(0) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r}(0)\cdot\mathbf{g}(0) \\ \mathbf{r}(0)\cdot\mathbf{h}(0) \end{bmatrix}. \quad (17)$$

Then the evolution in time of $\tilde{\mathbf{r}}_\mathbf{1}$ and $\tilde{\mathbf{r}}_\mathbf{2}$ is given in polar coordinates by:

$$|\tilde{\mathbf{r}}_\mathbf{1}(t)| = K_1\,\rho(t) \quad (18)$$

$$|\tilde{\mathbf{r}}_\mathbf{2}(t)| = K_2\,\rho(t) \quad (19)$$

$$\alpha_{\tilde{\mathbf{r}}_\mathbf{1}}(t) = \tan^{-1}\left(\frac{(a+1)\,\sin\alpha_\mathbf{r}(t) + b\cos\alpha_\mathbf{r}(t)}{(a+1)\,\cos\alpha_\mathbf{r}(t) - b\sin\alpha_\mathbf{r}(t)}\right) \quad (20)$$

$$\alpha_{\tilde{\mathbf{r}}_\mathbf{2}}(t) = \tan^{-1}\left(\frac{(a-1)\,\sin\alpha_\mathbf{r}(t) + b\sin\alpha_\mathbf{r}(t)}{(a-1)\,\cos\alpha_\mathbf{r}(t) - b\sin\alpha_\mathbf{r}(t)}\right) \quad (21)$$

where:
$K_1 = \sqrt{\sigma^2 + 1 + 2\sigma\cos(\alpha_{\tilde{\mathbf{z}}}(0) - \alpha_\mathbf{r}(0))}/2$,
$K_2 = \sqrt{\sigma^2 + 1 - 2\sigma\cos(\alpha_{\tilde{\mathbf{z}}}(0) - \alpha_\mathbf{r}(0))}/2$,
$a = \sigma\cos(\alpha_{\tilde{\mathbf{z}}}(0) - \alpha_\mathbf{r}(0))$,
$b = \sigma\sin(-\alpha_{\tilde{\mathbf{z}}}(0) + \alpha_\mathbf{r}(0))$
are constants which depend on the initial conditions, and $\alpha_{\tilde{\mathbf{z}}}$, $\alpha_\mathbf{r}$ are the polar angle coordinates of $\tilde{\mathbf{z}} = \tilde{\mathbf{r}}_\mathbf{1} + \tilde{\mathbf{r}}_\mathbf{2}$ and $\mathbf{r}$.

Sketch of the proof:

1) Equations (5) and (6), are invariant to arbitrary translations of the absolute reference frame; i.e. they still hold for a set of new variables $\tilde{\rho}$, $\tilde{\gamma}$, $\tilde{\lambda}$, $\tilde{\zeta}$, $\tilde{\xi}$, $\tilde{\eta}$, defined with respect to a translated absolute reference frame. Note that $\tilde{\rho} = \rho$, $\tilde{\gamma} = \gamma$, $\tilde{\lambda} = \lambda$ since these are variables based on relative motion. Hence for any $u = u(\rho, \gamma, \lambda)$ the closed loop equations will also be invariant to translations of the absolute reference frame.
2) If we substitute the MC control law (7) in (5)-(6), then the closed loop system has an invariant manifold $M_\sigma = \{(\rho, \gamma, \lambda, \tilde{\zeta}, \tilde{\xi}, \tilde{\eta}) : \tilde{\zeta} = \sigma\,\rho, \tilde{\xi} = \sigma\,\gamma, \tilde{\eta} = \sigma\,\lambda\}$.
3) There exists a (unique) translation $\mathbf{z_0}/2$ of the absolute reference frame, computed as in (17), which makes the initial conditions fall exactly on the invariant manifold M_σ. Hence with respect to this translated frame we have $\tilde{\zeta}(t) = \sigma\,\rho(t),\ \tilde{\xi}(t) = \sigma\,\gamma(t),\ \tilde{\eta}(t) = \sigma\,\lambda(t)\ \forall t \geq 0$, where $\sigma = \theta/\delta$.
4) From $\rho, \gamma, \lambda, \tilde{\zeta}, \tilde{\xi}, \tilde{\eta}$ it is possible to reconstruct the evolution in polar coordinates of vectors $\mathbf{r}$ and $\tilde{\mathbf{z}}$.
5) Finally (18)-(21) are obtained by expressing $\tilde{\mathbf{r}}_\mathbf{1} = (\tilde{\mathbf{z}} + \mathbf{r})/2$ and $\tilde{\mathbf{r}}_\mathbf{2} = (\tilde{\mathbf{z}} - \mathbf{r})/2$ in polar coordinates, and using the fact that $\dot{\alpha}_{\tilde{\mathbf{z}}}(t) = \tilde{\xi}(t)/\tilde{\zeta}(t) = \lambda(t)/\rho(t) = \dot{\alpha}_\mathbf{r}(t)$, which is a property of (5)-(6) (independently from the choice of the control law).

From Theorem 3, and the fact that $\mathbf{r}(t)$ covers the annular region with inner radius ρ_{min} and outer radius ρ_{max}, we have that the agents cover annular regions, centered at $\mathbf{z_0}/2$ and having radii $(K_1\,\rho_{min}, K_1\,\rho_{max})$ for agent 1 and $(K_2\,\rho_{min}, K_2\,\rho_{max})$ for agent 2.

The dynamics of Mutual Motion Camouflage leads to interesting trajectories with region-filling properties, useful for coverage path-planning applications. As in the beacon case, it is more convenient to make each agent use the modified MC control law with stabilization to a desired periodic orbit (described in section III-B). The advantages are orbital asymptotic stability and an additional degree of freedom (E_d) as explained below.

B. Mutual Motion Camouflage with stabilization to a desired orbit

Consider now the case in which both agents, mutually interacting according to (2), apply the modified MC control law (13). The resulting relative motion is the same as described in section III-B for the beacon case: independent of the initial conditions, the relative motion between the agents will converge to a value of energy equal to E_d. After a transient, the distance between the agents will periodically oscillate between minimum and maximum values given by the two positive solutions to (14). In order to determine the actual trajectories of the agents, we would need in general to derive the center of mass motion and combine it with the relative motion (which is what we did in [7] for the MMC case); nonetheless because of the form of the control law (13), which is equal to MC with an additional term, also SE(2)-invariant, the results of Theorem 3 apply in this case.

Proposition 2: The statement of Theorem 3 is still applicable in the case of mutual interaction between two mobile agents with curvature control (13).

Proof: The first step of the proof of Theorem 3 still applies since (13) is of the form $u = u(\rho, \gamma, \lambda)$. We must further prove that the invariant manifold $M_\sigma = \{(\rho, \gamma, \lambda, \tilde{\zeta}, \tilde{\xi}, \tilde{\eta}) : \tilde{\zeta} = \sigma\,\rho, \tilde{\xi} = \sigma\,\gamma, \tilde{\eta} = \sigma\,\lambda\}$ still exists when we substitute (13) (as opposed to (7)) in (5)-(6).
The resulting closed loop system is:

$$\begin{aligned} \dot{\rho} &= \gamma \\ \dot{\gamma} &= (1/\rho - \mu)\,(\delta^2 - \gamma^2) + \dot{\gamma}_{extra} \\ \dot{\lambda} &= -(1/\rho - \mu)\lambda\,\gamma + \dot{\lambda}_{extra} \end{aligned} \quad (22)$$

$$
\begin{aligned}
\dot{\tilde{\zeta}} &= \tilde{\xi} \\
\dot{\tilde{\xi}} &= (\theta^2 - \tilde{\xi}^2)/\tilde{\zeta} - \mu\,\lambda\,\tilde{\eta} + \dot{\tilde{\xi}}_{extra} \\
\dot{\tilde{\eta}} &= -\tilde{\xi}\,\tilde{\eta}/\tilde{\zeta} + \mu\,\lambda\,\tilde{\xi} + \dot{\tilde{\eta}}_{extra},
\end{aligned}
$$

where $\dot{\gamma}_{extra} = k_d\,\gamma\,(\delta^2 - \gamma^2)(E(\rho,\gamma) - E_d)$, $\dot{\lambda}_{extra} = -k_d\gamma^2\lambda(E(\rho,\gamma)-E_d)$, $\dot{\tilde{\xi}}_{extra} = k_d\gamma\lambda\tilde{\eta}(E(\rho,\gamma)-E_d)$ and $\dot{\tilde{\eta}}_{extra} = -k_d\,\gamma\,\lambda\,\tilde{\xi}\,(E(\rho,\gamma) - E_d)$. Aside from the 'extra' terms, the closed loop system (22) is identical to that of MMC. It is a simple exercise to prove that $\dot{\tilde{\xi}}_{extra} - \sigma\dot{\gamma}_{extra} = \dot{\tilde{\eta}}_{extra} - \sigma\,\dot{\lambda}_{extra} = 0$ when $(\tilde{\zeta},\tilde{\xi},\tilde{\eta}) = \sigma\,(\rho,\gamma,\lambda)$, and therefore the invariant manifold M_σ of MMC is preserved. The remaining steps of the proof of Theorem 3 follow without complications since they do not depend on the choice of the control law. ■

Even in this case the agents will cover (at steady state) annular regions centered at $\mathbf{z_0}/2$ and having radii $(K_1\,\rho_{min}, K_1\,\rho_{max})$ and $(K_2\,\rho_{min}, K_2\,\rho_{max})$ respectively. The values of ρ_{min}, ρ_{max} are fixed by the choice of μ and E_d, independently from the initial conditions, and this can be exploited to achieve certain coverage tasks.

Remark 4: If we choose $E_d = E_{max} = \delta^2 e^{-2}/\mu^2$, the energy value corresponding to equilibria of the relative motion equations, the trajectories of the agents at steady state are circular orbits centered at $\mathbf{z_0}/2$ and having radii K_1/μ for agent 1 and K_2/μ for agent 2. It can be proved that the same is also true if both the agents apply (15).

C. Coverage path-planning with two agents

With the "modified MMC" of section V-B, the (orbitally asymptotically stable) relative motion orbits can be shaped choosing appropriately μ and E_d, and independently from the initial conditions. On the other hand, by setting appropriate initial conditions (positions and directions of motion) before activating the mutual control law, one can also achieve the desired center of motion ($\mathbf{z_0}/2$) and coefficients K_1, K_2 (which together with ρ_{min} and ρ_{max} define the radii of the annular regions covered by the agents). This strategy is well-suited for coverage path-planning problems such as the following.

Coverage Problem 2: Determine suitable trajectories for two mobile agents, which move at constant speeds (not necessarily equal) and must maintain relative distance between $\rho_{d,min}$ and $\rho_{d,max}$, so that they jointly cover an annular region of desired inner radius $\rho_{t,min}$ and desired outer radius $\rho_{t,max}$, centered at a target point $\mathbf{T}$.

Such a problem could arise for example if the agents need to explore (cover) an area centered at a target point (a physical or a virtual beacon), without getting too close or too far from it. In achieving their task, they need to avoid collisions and maintain connectivity at all times, from which arise the requirements on the relative distance.

Proposition 3: Assume that the requirements of Coverage Problem 2 satisfy the following conditions (otherwise one could relax some of the requirements till they do):

$$
\frac{\rho^2_{t,max}}{\rho^2_{d,max}} + \frac{\rho^2_{t,min}}{\rho^2_{d,min}} \geq \frac{1}{2} + \frac{1}{2}\left(\frac{\rho^2_{t,max}}{\rho^2_{d,max}} - \frac{\rho^2_{t,min}}{\rho^2_{d,min}}\right)^2 \quad (23)
$$

$$
\frac{\rho_{t,max}}{\rho_{t,min}} \leq \frac{\rho^2_{d,max}}{\rho^2_{d,min}} \leq \frac{\rho^2_{t,max}}{\rho^2_{t,min}}. \quad (24)
$$

Then Coverage Problem 2 can be solved by the following two-step procedure:

(i) Steer the agents to new positions and orientations which satisfy, at a certain time t',

$$
\begin{aligned}
&\cos(\alpha_{\mathbf{z}-2\mathbf{T}}(t') - \alpha_{\mathbf{r}}(t')) = \pm\frac{1}{\sigma(t')}\left(\frac{\rho^2_{t,max}}{\rho^2_{d,max}} - \frac{\rho^2_{t,min}}{\rho^2_{d,min}}\right) \\
&\sigma(t') = \frac{|\mathbf{k}(t')|}{|\mathbf{g}(t')|} = \sqrt{2\left(\frac{\rho^2_{t,max}}{\rho^2_{d,max}} + \frac{\rho^2_{t,min}}{\rho^2_{d,min}}\right) - 1} \quad (25) \\
&\mathbf{z}(t') = 2\,\mathbf{T} + \sigma(t')^2 \begin{bmatrix} \mathbf{k}^T(t') \\ \mathbf{l}^T(t') \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{r}(t')\cdot\mathbf{g}(t') \\ \mathbf{r}(t')\cdot\mathbf{h}(t') \end{bmatrix}.
\end{aligned}
$$

(ii) Activate at time t' the modified Mutual Motion Camouflage strategy given by (2) and (13), with parameters:

$$
\mu = \frac{\ln(\rho_{d,max}/\rho_{d,min})}{\rho_{d,max} - \rho_{d,min}},\; E_d = \rho^2_{d,min}\delta^2(t')\,e^{-2\mu\rho_{d,min}}. \quad (26)
$$

Proof: We showed in section V-B that when the modified MMC strategy is used, the (steady state) trajectories of the agents cover annular regions centered at a point $\mathbf{z_0}/2$ and having radii $(K_1\,\rho_{d,min}, K_1\,\rho_{d,max})$ and $(K_2\,\rho_{d,min}, K_2\,\rho_{d,max})$ respectively. K_1, K_2 and $\mathbf{z_0}$ depend only on the initial conditions at the moment of activation of the control law, as expressed in (17) and Theorem 3. Imposing conditions (25), we make $\mathbf{z_0}/2 = \mathbf{T}$ so that the annular regions are actually centered at the target point $\mathbf{T}$. Moreover we make the inner and outer radii of the union of the annular regions covered by the agents equal to $\rho_{t,min}$ and $\rho_{t,max}$ respectively. The left inequality in (24) guarantees that the annular regions covered by the agents intersect each other, so that the coverage task is jointly accomplished by the two agents. The other inequalities in (23)-(24) are needed to guarantee that the equations in (25) are well-defined. The choice of sign in the first equation of (25) defines which of the agents will cover the outermost region and which the innermost. Finally the modified MMC strategy is activated at time t' with the choice of parameters which satisfies the relative motion requirements $(\rho_{d,min}, \rho_{d,max})$. ■

Figure 4 shows the results of a simulation in which Coverage Problem 2 is successfully solved by the procedure described in Proposition 3.

We conclude this section providing some insight on how the step (i) prescribed by Proposition 3 can be accomplished. There are many different choices of positions and velocities which satisfy equations (25). The second equation prescribes a certain value for σ, i.e. the ratio between the magnitudes of center of mass velocity $\mathbf{k}$ and relative velocity $\mathbf{g}$. This value

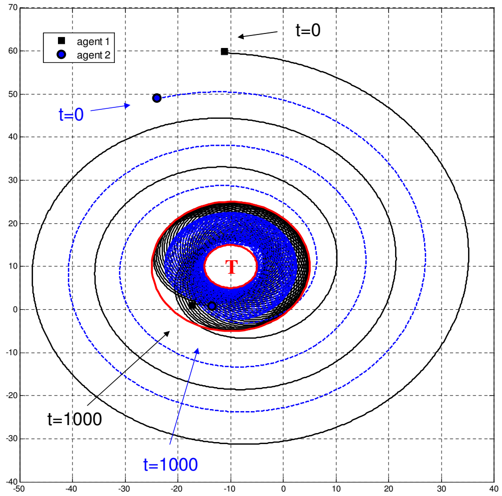

Fig. 4. Solution to Coverage Problem 2 using Proposition 3. The target is at [-10,10] and the desired annular region to cover has radii $\rho_{t,min} = 5$, $\rho_{t,max} = 15$; the relative distance must satisfy $\rho_{d,min} = 2$, $\rho_{d,max} = 5$. The agents, having speeds $\nu_1 = 0.8$ and $\nu_2 = 1.2$, start from initial conditions that satisfy (25) and mutually interact through the control law (13). By choosing parameters $\mu = 0.3054$, $E_d = 1.3204$ as prescribed by (26), and $K_d = 3$, the agents begin filling the annular regions with radii $(6, 15)$ and $(5, 12.5)$ respectively, jointly completing the task.

is also equal to the ratios $\tilde{\zeta}/\rho$, $\tilde{\xi}/\gamma$ and $\tilde{\eta}/\lambda$, where $\tilde{\zeta}, \tilde{\xi}, \tilde{\eta}$ are defined with respect to the absolute reference frame centered at $\mathbf{T}$; initial conditions satisfying (25) fall in fact within the invariant manifold M_σ of section V-A. The first equation provides instead a requirement on the difference between the polar angles of $\mathbf{r}$ and $\tilde{\mathbf{z}}$, and the third equation is equivalent to the requirement that: $\alpha_{\mathbf{k}} - \alpha_{\mathbf{g}} = \alpha_{\tilde{\mathbf{z}}} - \alpha_{\mathbf{r}}$. Hence the only conditions imposed by (25) are on the relations between $\mathbf{r}$ and $\tilde{\mathbf{z}}$ and between $\mathbf{g}$ and $\mathbf{k}$ (and also obviously between $\mathbf{h}$ and $\mathbf{l}$). Nevertheless there is complete freedom in the choice of the relative vectors $\mathbf{r}$ and $\mathbf{g}$; one could choose for instance to have $\mathbf{r}(t') = \mathbf{r}(0)$ and $\mathbf{g}(t') = \mathbf{g}(0)$, i.e. to preserve the initial relative configuration, or choose a $\mathbf{g}(t')$ with the desired magnitude $\delta(t')$ so to affect the relative motion orbits. Given any choice of the pair $\mathbf{r}(t'), \mathbf{g}(t')$, one can derive the corresponding vectors $\tilde{\mathbf{z}}(t')$, $\mathbf{k}(t')$ and $\mathbf{l}(t')$, and then the individual positions and velocities for the agents, which satisfy (25).

The problem then reduces to finding for each agent a steering law which drives it to this new position and velocity, from the initial ones. Since the agent moves at constant speed, this problem is equivalent to finding the curve which connects two points on the plane each with specified velocity direction. This is the *Dubins problem*, whose solution is well-known when curvature bounds are imposed (see section 13.5 of [14]). The trajectories found in this way for each agent must be checked for possible collisions; if they are not collision-free, the process must be repeated with a different choice of $\mathbf{r}(t'), \mathbf{g}(t')$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper we have discussed a program of using pursuit laws of a specific type, inspired by phenomena in nature, to accomplish tasks of coverage by cooperative unmanned systems (modeled here as planar interacting particles). We have made use of reduction and phase portrait properties in carrying out this program in a basic example of a two-particle system. Further work is under way to investigate many-particle analogs of cohesion by pursuit.

VII. ACKNOWLEDGMENTS

The authors wish to thank Eric Justh and Kevin Galloway for valuable discussions.

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