

# Extremal Collective Behavior

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**Abstract**—Curves and natural frames can be used for describing and controlling motion in both biological and engineering contexts (e.g., pursuit and formation control). The geometry of curves and frames leads naturally to a Lie group formulation where coordinated motion is represented by interacting particles on Lie groups - specifically,  $SE(2)$  or  $SE(3)$ . Here we consider a particular type of optimal control problem in which the interactions between particles arise from a cost function dependent on each particle's steering, and which penalizes steering differences between the particles (expressed via the graph Laplacian). With this choice of cost function, we are able to perform Lie-Poisson reduction. Furthermore, we are able to derive a closed-form expression (using Jacobi elliptic functions) for certain special solutions of the coupled multi-particle problem on  $SE(2)$ .

## I. INTRODUCTION

It is convenient to decompose the problem of multi-vehicle coordinated control into two tasks: trajectory generation (based on pursuit, formation flight, or some other mission-guided strategy), and trajectory tracking (the role assigned to the autopilot). Similarly, in analyzing data and testing hypotheses about biological motion it is convenient to separate the task of trajectory reconstruction (based on measured data) from the problem of trajectory generation by the animal (as it implements a particular strategy) [5], [16]. Inspired by considerations of high-speed vehicles and UAVs, the task of trajectory generation can be formulated in terms of constant-speed particle motion, or equivalently, in terms of curves and moving frames. By packaging the curve and frame equations as left-invariant systems on  $SE(2)$  (for planar curves) or  $SE(3)$  (for curves in three-dimensional space), we are led to interaction laws for particles in Lie groups as a basis for designing coordinated vehicle trajectories for tasks such as pursuit, formation flight, and boundary tracking [8], [9], [17], [11], [18]. These tasks have been formulated using *relative shape*, i.e., the system has a group symmetry, which the Lie group formulation allows us to exploit.

In this paper, we retain the Lie group formulation of curves and moving frames for describing trajectories of multiple coordinated vehicles, but we consider a different type of strategy: collective optimization of trajectories subject to fixed endpoint conditions. From a practical standpoint, this

type of strategy could be appropriate if it is desired to coordinate the arrival of particular vehicles at particular locations at scheduled points in time. Here we consider minimizing the cumulative “steering energy” for the individual particles, along with “steering difference energy” for multiple particles. For a single particle, minimizing the steering energy is a reasonable objective for certain types of vehicles (particularly high-speed vehicles), because turning requires extra fuel consumption, reduces dynamic stability (e.g., leading to roll-over in a high-speed ground vehicle), and places more demands on an autopilot than traveling in a straight line. For multiple vehicles in proximity to each other, and moving at roughly the same speed (and in roughly the same direction), minimizing the differences among their steering controls is a technique for reducing the likelihood of collision. The advantage of penalizing steering control differences, rather than inter-vehicle distances, is that Lie-Poisson reduction can be applied to simplify the description of the interacting particle dynamics, which is the main theme of this paper.

To understand the advantage of Lie-Poisson reduction, it is helpful to consider the various types of reduction used in mechanics [7], [15]. Poisson reduction can be applied to various mechanics problems involving coupled rigid bodies to reduce the dimension of the space in which solutions need to be computed. For example, the overall rigid motion symmetry in a coupled two-body problem considered in [6], [12] allows Poisson reduction from the state space

$$T^*SO(3) \times T^*SO(3) \cong SO(3) \times SO(3) \times so^*(3) \times so^*(3)$$

to the reduced space  $SO(3) \times so^*(3) \times so^*(3)$ . However, in mechanical problems it is unusual to have further reduction associated with Lie-Poisson (in this example, to  $so^*(3) \times so^*(3)$ ). There are examples of such Lie Poisson reduction in physics, e.g., nonlinear optical polarization dynamics [2], [13], but here we focus on optimal control problems which admit Lie Poisson reduction.

This paper is organized as follows. We first formulate a general optimal control problem for multiple particles in an arbitrary matrix Lie group. The particles are cost-coupled through a connected, undirected graph. The Maximum Principle is applied, and the first-order necessary conditions for regular extremals are derived. Furthermore, the Lie-Poisson reduced equations are computed. Next, we specialize to  $SE(2)$ , and express the reduced equations in a more concrete form. For a single particle in  $SE(2)$ , the reduced equations can be solved in closed form using Jacobi elliptic functions - indeed, this is a simple example from elastica theory [7]. What we show is that there are special

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solutions for the multi-particle reduced dynamics which can also be written in closed form using Jacobi elliptic functions. The conclusion is that for certain special fixed-endpoint conditions, the optimal multi-particle steering law (where the cost function includes both “steering energy” and “steering difference energy” terms) consists of steering controls for the individual particles which are all proportional to one another.

**Dedication:** It is a pleasure to dedicate this work to Tudor Stefan Ratiu on the occasion of his sixtieth birthday.

## II. OPTIMAL CONTROL PROBLEM

### A. Problem formulation

Consider a connected, undirected graph with vertices  $v_1, v_2, \dots, v_N \in \mathcal{V}$ , without self loops, and denote the degree of vertex  $v_i$  by  $d(v_i)$ ,  $i = 1, \dots, N$ . The adjacency matrix  $A$  is then defined by  $a_{ij} = 1$  if vertices  $v_i$  and  $v_j$  are connected, and  $a_{ij} = 0$  otherwise,  $i, j = 1, \dots, N$ . We define the degree matrix  $D = \text{diag}(d(v_1), \dots, d(v_N))$ , and the graph Laplacian by  $B = D - A$ , where we note that  $A$ ,  $D$ , and  $B$  are symmetric matrices [4].

We seek to minimize

$$\mathcal{L} = \int_0^T L(\xi_1(t), \dots, \xi_N(t)) dt, \quad (1)$$

subject to the controlled dynamics  $\dot{g}_k = g_k \xi_k$ ,  $g_k \in G$ , a matrix Lie group (having real-valued entries), with fixed endpoints  $g_k(0) = g_{k0}$ ,  $g_k(T) = g_{kT}$ ,  $k = 1, \dots, N$ , where the Lagrangian

$$L(\xi_1, \dots, \xi_N) = \frac{1}{2} \left( \sum_{k=1}^N |\xi_k|^2 + \chi \sum_{k=1}^N \sum_{j=1}^N a_{kj} |\xi_k - \xi_j|^2 \right), \quad (2)$$

with  $\chi > 0$  a constant. We use the trace norm  $|\xi|^2 = \text{tr}(\xi^T \xi)$ , and inner product  $\langle \xi, \eta \rangle = \text{tr}(\xi^T \eta)$ ,  $\xi, \eta \in \mathfrak{g}$ , where  $\mathfrak{g}$  is the Lie algebra associated with  $G$ . We have

$$\sum_{k=1}^N \sum_{j=1}^N a_{kj} |\xi_k - \xi_j|^2 = 2 \sum_{k=1}^N \sum_{j=1}^N b_{kj} \langle \xi_j, \xi_k \rangle, \quad (3)$$

where we have used the symmetry of  $A$ , and thus we can rewrite (2) as

$$L(\xi_1, \dots, \xi_N) = \frac{1}{2} \sum_{k=1}^N |\xi_k|^2 + \chi \sum_{k=1}^N \left\langle \xi_k, \sum_{j=1}^N b_{kj} \xi_j \right\rangle. \quad (4)$$

**Remark:** Note that the expression  $(\xi_k - \xi_j)$  is used in the Lagrangian (2), rather than  $(\xi_k - \text{Ad}_{g^{-1}} \xi_j)$ ,  $g = g_j^{-1} g_k$ , which would be more natural from a mechanics viewpoint. The attitude we take here is to posit a Lagrangian, solve the corresponding optimal control problem, and then assess whether the resulting analysis (specifically, the form of the Lie-Poisson reduced dynamics) provides any useful insight regarding control of particle collectives.

We take  $\xi_k$  to be affine in the control vector  $u_k$  for each  $k$ , i.e., we take

$$\xi_k = X_q + \sum_{i=1}^m u_{ki} X_i, \quad k = 1, \dots, N, \quad (5)$$

where  $u_k = (u_{k1}, \dots, u_{km}) \in \mathbb{R}^m$ ,  $\{X_1, X_2, \dots, X_n\}$  is a basis (assumed to be orthonormal with respect to the trace inner product) for the Lie algebra  $\mathfrak{g}$ ,  $m < n$ , and  $q \in \{(m+1), \dots, n\}$ . Thus, the system is underactuated and has drift. (Replacing  $X_q$  in (5) with zero yields a driftless system.) With the substitution (5) we can write  $L = L(u_1, \dots, u_N)$ .

### B. Maximum Principle

Restricting attention to regular extremals of the fixed endpoint problem posed in the previous subsection, we define the pre-hamiltonian

$$H(p, g, u) = \langle p, g \xi_u \rangle - L(u) \quad (6)$$

where  $p \in T_g^* G^N$ , the cotangent space at  $g$ , and  $G^N$  denotes the direct product of  $N$  copies of the Lie group  $G$ . Here

$$g = \text{diag}(g_1, g_2, \dots, g_N), \quad \xi_u = \text{diag}(\xi_1, \xi_2, \dots, \xi_N), \quad (7)$$

i.e.,  $g$  and  $\xi_u$  are block-diagonal matrices, and  $u = (u_1, u_2, \dots, u_N)$  is the control vector (of length  $mN$ ). Then (6) can be written as

$$\begin{aligned} H(p, g, u) &= H(p_1, \dots, p_N, g_1, \dots, g_N, u_1, \dots, u_N) \\ &= \left( \sum_{k=1}^N \langle p_k, g_k \xi_k \rangle \right) - L(u_1, \dots, u_N), \end{aligned} \quad (8)$$

where  $p_k \in T_{g_k}^* G$  for  $k = 1, \dots, N$ . The Maximum Principle then states that if  $u_1, \dots, u_N$  minimizes  $\mathcal{L}$ ,  $g_1, \dots, g_N$  denotes the corresponding trajectory in  $G^N$ , and the only extremals of  $\mathcal{L}$  are regular extremals, then

$$\begin{aligned} H(p_1, \dots, p_N, g_1, \dots, g_N) \\ = \sup_{v_k \in \mathbb{R}^m, k=1, \dots, N} H(p_1, \dots, p_N, g_1, \dots, g_N, v_1, \dots, v_N), \end{aligned} \quad (9)$$

for a.e.  $t \in [0, T]$ , where  $H$  is the hamiltonian [7], [14]. The first-order necessary condition for (9) is  $\partial H / \partial u_{ki} = 0$ , or

$$\begin{aligned} \frac{\partial}{\partial u_{ki}} \left[ \left( \sum_{j=1}^N \langle p_j, g_j \xi_j \rangle \right) - L(u_1, \dots, u_N) \right] \\ = \left\langle p_k, g_k \frac{\partial \xi_k}{\partial u_{ki}} \right\rangle - \frac{\partial L}{\partial u_{ki}} = 0, \end{aligned} \quad (10)$$

for  $k = 1, \dots, N$  and  $i = 1, \dots, m$ , where  $(u_1, \dots, u_N)$  denote the optimal controls.

Using (5), we have  $\partial \xi_k / \partial u_{ki} = X_i$ , and from (4), we compute

$$\frac{\partial L}{\partial u_{ki}} = u_{ki} + 2\chi \sum_{j=1}^N b_{jk} u_{ji}. \quad (11)$$

Also,

$$\left\langle p_k, g_k \frac{\partial \xi_k}{\partial u_{ki}} \right\rangle = \langle p_k, g_k X_i \rangle = \langle \mu_k, X_i \rangle = \mu_{ki}, \quad (12)$$

where the translation to the identity of  $p_k$  is given by  $\mu_k \in \mathfrak{g}^*$ , the dual space of the Lie algebra of  $G$ , and

$$\mu_k = \sum_{i=1}^n \mu_{ki} X_i^b, \quad (13)$$

where  $\{X_1^b, X_2^b, \dots, X_n^b\}$  is the dual basis to  $\{X_1, X_2, \dots, X_n\}$ . Thus, (10) becomes

$$\mu_{ki} = u_{ki} + 2\chi \sum_{j=1}^N b_{jk} u_{ji}, \quad k = 1, \dots, N, \quad i = 1, \dots, m. \quad (14)$$

Defining  $\tilde{\mu}_k = [\mu_{k1} \quad \mu_{k2} \quad \dots \quad \mu_{km}]^T$ , we have

$$\tilde{\mu}_k = u_k + 2\chi \sum_{j=1}^N b_{jk} u_j, \quad (15)$$

or

$$\begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix} = ((\mathbb{I}_N + 2\chi B) \otimes \mathbb{I}_m) \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad (16)$$

where  $\otimes$  denotes the Kronecker product. For convenience, we define

$$\Psi = ((\mathbb{I}_N + 2\chi B) \otimes \mathbb{I}_m)^{-1} = (\mathbb{I}_N + 2\chi B)^{-1} \otimes \mathbb{I}_m, \quad (17)$$

assuming that the inverse exists. All of the eigenvalues of  $B$  are real and nonnegative, including (at least) one zero eigenvalue [4]. Therefore,  $\Psi$  is guaranteed to exist for  $\chi > 0$ . We then have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \Psi \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix}, \quad (18)$$

and substituting the optimal controls back into the hamiltonian gives

$$\begin{aligned} H &= \sum_{k=1}^N \langle \mu_k, \xi_k \rangle - L(u_1, \dots, u_N) \\ &= \sum_{k=1}^N \langle \mu_k, \xi_k \rangle - \left( \frac{1}{2} \sum_{k=1}^N |\xi_k|^2 + \chi \sum_{k=1}^N \left\langle \xi_k, \sum_{j=1}^N b_{kj} \xi_j \right\rangle \right). \end{aligned} \quad (19)$$

After some calculation, we obtain

$$H = \sum_{k=1}^N \mu_{kq} + \frac{1}{2} [\tilde{\mu}_1 \quad \dots \quad \tilde{\mu}_N] \Psi \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix} - \frac{1}{2} N, \quad (20)$$

where our assumption that  $B$  is symmetric implies that  $\Psi$  is also symmetric.  $H$ , being independent of  $g$ , permits reduction.

### C. Lie-Poisson reduction

The process of Lie-Poisson reduction takes the original system on  $(T^*G)^N$  and reduces it to a system on  $(\mathfrak{g}^*)^N$ , with the reduced variables defined as  $\mu_k$ ,  $k = 1, \dots, N$ . The reduced hamiltonian has already been computed as (20): we can ignore the constant term and write

$$h = \sum_{k=1}^N \mu_{kq} + \frac{1}{2} [\tilde{\mu}_1 \quad \dots \quad \tilde{\mu}_N] \Psi \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_N \end{bmatrix}, \quad (21)$$

where we recall that  $\tilde{\mu}_k$  encodes the first  $m$  components of  $\mu_k$  for each  $k = 1, \dots, N$ .

Defining  $\check{\mu} = [\mu_1 \quad \mu_2 \quad \dots \quad \mu_N]^T$ , so that  $\check{\mu}$  is a vector of length  $\check{n} = Nn$ , the dynamics for  $\check{\mu}$  are

$$\dot{\check{\mu}} = \Lambda(\check{\mu}) \nabla h = \Lambda(\check{\mu}) \begin{bmatrix} \partial h / \partial \check{\mu}_1 \\ \partial h / \partial \check{\mu}_2 \\ \vdots \\ \partial h / \partial \check{\mu}_{\check{n}} \end{bmatrix}, \quad (22)$$

with

$$\Lambda(\check{\mu}) = -\sum_{k=1}^n \begin{bmatrix} \mu_{1k} & 0 & \dots & 0 \\ 0 & \mu_{2k} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \mu_{Nk} \end{bmatrix} \otimes \begin{bmatrix} \Gamma_{11}^k & \Gamma_{12}^k & \dots & \Gamma_{1n}^k \\ \Gamma_{21}^k & \Gamma_{22}^k & \dots & \Gamma_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n1}^k & \Gamma_{n2}^k & \dots & \Gamma_{nn}^k \end{bmatrix}, \quad (23)$$

where  $\Gamma_{ij}^k$  are the structure constants for  $\mathfrak{g}$ , and all of the coupling between the  $\mu_k$ ,  $k = 1, \dots, N$ , in the reduced equations are due to  $\nabla h$ . Due to the block diagonal structure of (23), each Casimir, invariant on  $\mathfrak{g}^*$ , associated with  $G$  contributes  $N$  Casimirs to the reduced system (22).

### III. SPECIALIZATION TO $G = SE(2)$

By specializing to  $G = SE(2)$ , we can carry the calculation further. Consider  $N$  coupled particles in  $SE(2)$ : specifically, equation (5) becomes

$$\xi_k = X_2 + u_k X_1, \quad k = 1, \dots, N, \quad (24)$$

where  $u_k$  is a scalar steering control for each  $k$ , and the basis we use for the Lie algebra  $se(2)$  is

$$X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (25)$$

which is normalized so that  $|X_1| = |X_2| = |X_3| = 1$  with respect to the trace norm.

From (21) we have

$$h = \sum_{k=1}^N \mu_{k2} + \frac{1}{2} [\mu_{11} \quad \dots \quad \mu_{N1}] \Psi \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{N1} \end{bmatrix}. \quad (26)$$

The Lie-Poisson reduced dynamics are given by (22) and (23), where now

$$\Lambda(\check{\mu}) = -\frac{1}{\sqrt{2}} \begin{bmatrix} \Omega_1 & 0 & \dots & 0 \\ 0 & \Omega_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \Omega_N \end{bmatrix}, \quad (27)$$

with

$$\Omega_k = \begin{bmatrix} 0 & \mu_{k3} & -\mu_{k2} \\ -\mu_{k3} & 0 & 0 \\ \mu_{k2} & 0 & 0 \end{bmatrix}, \quad k = 1, \dots, N. \quad (28)$$

Also,

$$\frac{\partial h}{\partial \vec{\mu}} = \begin{bmatrix} \partial h / \partial \mu_1 \\ \partial h / \partial \mu_2 \\ \vdots \\ \partial h / \partial \mu_N \end{bmatrix}, \quad \partial h / \partial \mu_k = \begin{bmatrix} \partial h / \partial \mu_{k1} \\ 1 \\ 0 \end{bmatrix}, \quad (29)$$

$k = 1, \dots, N$ , with

$$\begin{bmatrix} \partial h / \partial \mu_{11} \\ \partial h / \partial \mu_{21} \\ \vdots \\ \partial h / \partial \mu_{N1} \end{bmatrix} = \Psi \begin{bmatrix} \mu_{11} \\ \mu_{21} \\ \vdots \\ \mu_{N1} \end{bmatrix}, \quad (30)$$

and  $\Psi = (\mathbb{I}_N + 2\chi B)^{-1}$ .

### A. Motivation

The optimization problem we pose for coupled particles in  $SE(2)$  can be viewed as an optimal control problem for framed curves in  $\mathbb{R}^2$ , where the objective is to minimize the curvature (and curvature differences) subject to fixed endpoint conditions. Framed curves in  $\mathbb{R}^2$  can be defined by the system

$$\dot{\mathbf{r}} = \mathbf{x}, \quad \dot{\mathbf{x}} = \mathbf{y}u, \quad \dot{\mathbf{y}} = -\mathbf{x}u, \quad (31)$$

where  $\mathbf{r} \in \mathbb{R}^2$  is the position vector,  $\mathbf{x} \in \mathbb{R}^2$  is the unit tangent vector to the trajectory,  $\mathbf{y} \in \mathbb{R}^2$  is the unit normal vector, and  $u$  is the plane curvature (i.e., the steering control). Here, we assume unit-speed motion, so that time  $t$  is also the arc length parameter. (Note that  $u$  in (31) differs from  $u_k$  in (24) by a factor of  $1/\sqrt{2}$ , due to our normalization of the basis for  $se(2)$ .) An alternative way to express (31) is

$$\dot{\theta} = u, \quad \dot{\mathbf{r}} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad (32)$$

where  $\theta$  is the angle associated with the unit tangent vector to the trajectory.

### B. Single-particle optimization problem

For a single particle in  $SE(2)$ , consider minimization of

$$\mathcal{L} = \frac{1}{2} \int_0^T |\xi_u(t)|^2 dt = \frac{1}{2} \int_0^T (1 + u(t)^2) dt, \quad (33)$$

subject to the dynamics  $\dot{g} = g\xi_u$ ,  $\xi_u = X_2 + X_1u$ , and the endpoint conditions  $g(0) = g_0$ ,  $g(T) = g_T$ . For this single particle fixed endpoint problem we have the Lie-Poisson reduced dynamics

$$\frac{d}{dt} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & 0 \\ \mu_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ 1 \\ 0 \end{bmatrix}, \quad (34)$$

along with the conserved quantities

$$h = \mu_2 + \frac{1}{2}\mu_1^2, \quad c = \frac{1}{2}(\mu_2^2 + \mu_3^2), \quad (35)$$

the (reduced) hamiltonian and Casimir function, respectively [14]. (Here  $\mu_1$ ,  $\mu_2$ , and  $\mu_3$  are the scalar components of the single vector  $\mu \in se(2)^*$ .) We thus have

$$\ddot{\mu}_1 = -\frac{1}{\sqrt{2}}\dot{\mu}_3 = \frac{1}{2}\mu_2\mu_1 = \frac{1}{2} \left( h - \frac{1}{2}\mu_1^2 \right) \mu_1. \quad (36)$$

For certain values of  $h$ , this second-order cubic equation has elliptic function solutions [3]. To obtain these solutions explicitly, we introduce the variables

$$\mu_1(t) = \sigma y(\nu(t - \eta)), \quad (37)$$

where  $\sigma > 0$ ,  $\eta$ , and  $\nu > 0$  are constant. Then

$$\sigma\nu^2 y''(\nu(t - \eta)) - \frac{1}{2}h\sigma y(\nu(t - \eta)) + \frac{1}{4}\sigma^3 y(\nu(t - \eta))^3 = 0, \quad (38)$$

where  $y''(\cdot)$  denotes the second derivative of  $y(\cdot)$  with respect to its argument. Defining  $m$  such that

$$m = \frac{\sigma^2}{8\nu^2} \text{ and } 1 - 2m = -\frac{h}{2\nu^2}, \quad (39)$$

we obtain

$$y'' + (1 - 2m)y + 2my^3 = 0, \quad (40)$$

which has as its solution the Jacobi elliptic function

$$y = \text{cn}(\nu(t - \eta), m), \quad (41)$$

provided  $0 \leq m \leq 1$ . (Note that this  $m$  is unrelated to the earlier definition of  $m$  as the number of control inputs per particle in  $G$ . Henceforth, we use  $m$  to refer only to the parameter of the Jacobi elliptic function.) Thus, we may conclude that solutions to (36) are of the form

$$\mu_1(t) = \left( 2\sqrt{2} \nu \sqrt{m} \right) \text{cn}(\nu(t - \eta), m), \quad m = \frac{1}{2} \left( 1 + \frac{h}{2\nu^2} \right), \quad (42)$$

with  $\nu^2 \geq |h|/2$ . We then have the optimal control  $u$  given by  $u = \mu_1$ .

**Remark:** For  $G = SE(2)$ , we can express the dynamics as (32) with fixed endpoints

$$\theta(0) = \theta_0, \quad \mathbf{r}(0) = \mathbf{r}_0, \quad \theta(T) = \theta_T, \quad \mathbf{r}(T) = \mathbf{r}_T. \quad (43)$$

Thus, the endpoint conditions involve not only the position endpoints in the plane,  $\mathbf{r}_0$  and  $\mathbf{r}_T$ , but also the tangent vectors to the trajectory at the endpoints,  $\theta_0$  and  $\theta_T$ . Note that (42) has three parameters,  $\nu$ ,  $\eta$ , and  $m$ , as degrees of freedom for meeting the endpoint conditions (which also represent three degrees of freedom, because without loss of generality we can take  $\mathbf{r}_0 = (0, 0)$  and  $\theta_0 = 0$ ). It is clear that a solution satisfying a given set of endpoint conditions (43) exists provided  $|\mathbf{r}_T - \mathbf{r}_0| < T$ . It is thus reasonable to speak of optimal solutions, provided  $|\mathbf{r}_T - \mathbf{r}_0| < T$ .

### C. Special class of coupled solutions

For the coupled system of  $N > 1$  particles in  $SE(2)$ , we have not found a closed-form solution like (42) that applies in general. However, there are special solutions to (22) with  $G = SE(2)$  which do take the form of Jacobi elliptic functions.

Suppose  $\chi$  and  $B$  are fixed, and consider the following class of candidate solutions, analogous to (42):

$$\mu_{k1} = \sigma_k \text{cn}(\nu(t - \eta), m), \quad k = 1, \dots, N, \quad (44)$$

where  $\sigma_k > 0$ ,  $k = 1, \dots, N$ ,  $\nu, \eta$ , and  $m$  are given constants. Then

$$\begin{bmatrix} \partial h / \partial \mu_{11} \\ \partial h / \partial \mu_{21} \\ \vdots \\ \partial h / \partial \mu_{N1} \end{bmatrix} = \Psi \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{bmatrix} \text{cn}(\nu(t - \eta), m), \quad (45)$$

and from (26) we have

$$h = \sum_{k=1}^N \mu_{k2} + \frac{1}{2} [\sigma_1 \cdots \sigma_N] \Psi \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix} \text{cn}(\nu(t - \eta), m)^2. \quad (46)$$

We also have  $N$  Casimirs of the form  $c_k = \frac{1}{2}(\mu_{k2}^2 + \mu_{k3}^2)$ ,  $k = 1, \dots, N$ .

We would like to find all possible choices of  $\sigma_1, \dots, \sigma_N, \nu, \eta$ , and  $m$  for which (44) solves system (22) with  $G = SE(2)$ . However, we leave this problem for future work, and instead here ask the simpler question: given fixed positive values for  $\sigma_1, \dots, \sigma_N$ , how do they constrain the remaining constants, and what form must the controls  $u_1, \dots, u_N$  take?

Suppose that  $\beta_1, \beta_2, \dots, \beta_N$ , positive constants, satisfy

$$[\sigma_1 \cdots \sigma_N] \text{diag}(\beta_1, \dots, \beta_N) \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix} = [\sigma_1 \cdots \sigma_N] \Psi \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix}. \quad (47)$$

(Note that for  $\sigma_1, \dots, \sigma_N$  given, this amounts to one scalar equation relating the  $N$  variables  $\beta_1, \dots, \beta_N$ .) Then we can define

$$h_k = \mu_{k2} + \frac{1}{2} \beta_k \mu_{k1}^2, \quad k = 1, \dots, N, \quad (48)$$

and it follows that  $h = \sum_{k=1}^N h_k$ . Furthermore,

$$\begin{bmatrix} \partial h / \partial \mu_{11} \\ \vdots \\ \partial h / \partial \mu_{N1} \end{bmatrix} = \text{diag}(\beta_1, \dots, \beta_N) \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix} \text{cn}(\nu(t - \eta), m), \quad (49)$$

and we obtain the reduced dynamics

$$\begin{bmatrix} \dot{\mu}_{k1} \\ \dot{\mu}_{k2} \\ \dot{\mu}_{k3} \end{bmatrix} = -\frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \mu_{k3} & -\mu_{k2} \\ -\mu_{k3} & 0 & 0 \\ \mu_{k2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_k \mu_{k1} \\ 1 \\ 0 \end{bmatrix}, \quad (50)$$

for  $k = 1, \dots, N$ , which leads to

$$\ddot{\mu}_{k1} = \frac{\beta_k}{2} \left( h_k - \frac{1}{2} \beta_k \mu_{k1}^2 \right) \mu_{k1}, \quad k = 1, \dots, N. \quad (51)$$

But (51) is satisfied by (44) when

$$h_k = \text{constant}, \quad m = \frac{\beta_k \sigma_k^2}{4\nu^2}, \quad 1 - 2m = -\frac{\beta_k h_k}{2\nu^2}, \quad (52)$$

for all  $k = 1, \dots, N$ . Note that (52) implies

$$\beta_j \sigma_j = \beta_k \sigma_k, \quad j, k = 1, \dots, N, \quad (53)$$

which imposes  $N - 1$  independent constraints on  $\beta_1, \dots, \beta_N$ . Along with (47), there are a total of  $N$  independent scalar equations for the  $N$  quantities  $\beta_1, \dots, \beta_N$  we seek to solve

for (given  $\sigma_1, \dots, \sigma_N$ ). If a solution  $\beta_1, \dots, \beta_N > 0$  to these equations exists, we say that the collection  $\sigma_1, \dots, \sigma_N$  is admissible.

To summarize, the reduced dynamics for the coupled system of  $N$  particles (with  $\chi$  and  $B$  fixed a priori) is given by (22). However, we focus on special solutions to (22), having the form (44), where the collection of positive constants  $\sigma_1, \dots, \sigma_N$  is admissible. Then  $\nu, \eta$ , and  $m$  exist (indeed,  $\eta$  is arbitrary, and we can choose  $\nu$  so that  $0 \leq m \leq 1$ ), such that (44) is in fact a solution to (22). Finally, because the optimal controls can be related to the reduced variables through

$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \Psi \begin{bmatrix} \mu_{11} \\ \vdots \\ \mu_{N1} \end{bmatrix}, \quad (54)$$

we have

$$\begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = \Psi \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{bmatrix} \text{cn}(\nu(t - \eta), m). \quad (55)$$

#### D. Interpretation of special solutions

The special solutions (55) have the property that each particle's steering control is proportional to every other particle's steering control. We only expect such special solutions to apply to a thin set of endpoint conditions for a system of  $N$  particles. However, the definition of this space of special solutions is not vacuous: there are examples of sets of endpoint conditions consistent with coupled optimal solutions having the form (55).

In fact, despite applying to only a thin set of endpoint conditions, there is a reasonable amount of freedom in (55) through the choice of  $\sigma_1, \dots, \sigma_N$ . We can generate collections of trajectories using (55), and then state that among all possible collections of curves satisfying the same endpoint conditions, the trajectories we have generated satisfy the (necessary) condition for optimality of the coupled system.

#### E. Numerical example

Suppose we take  $N = 2$ ,  $\chi = 1$ ,

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad (56)$$

$\sigma_1 = 1$ , and  $\sigma_2 = 1/2$ . Then we have

$$\Psi = (\mathbb{I}_N + 2\chi B)^{-1} = \begin{bmatrix} 1 + 2\chi & -2\chi \\ -2\chi & 1 + 2\chi \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}. \quad (57)$$

From

$$\begin{aligned} \beta_1 \sigma_1^2 + \beta_2 \sigma_2^2 &= [\sigma_1 \quad \sigma_2] \Psi \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \\ \implies \beta_1 + \frac{1}{4} \beta_2 &= \frac{1}{10} \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \frac{23}{40}, \end{aligned} \quad (58)$$

and  $\beta_1 \sigma_1 = \beta_2 \sigma_2 \implies \beta_1 = \frac{1}{2} \beta_2$ , we obtain  $\beta_1 = 23/60$ ,  $\beta_2 = 23/30$ . Choosing  $T = 6$ ,  $m = \frac{1}{2}$ , and  $\eta = 0$ , we have

$$\nu = \frac{\beta_1 \sigma_1}{2m} = \frac{23}{60}, \quad (59)$$

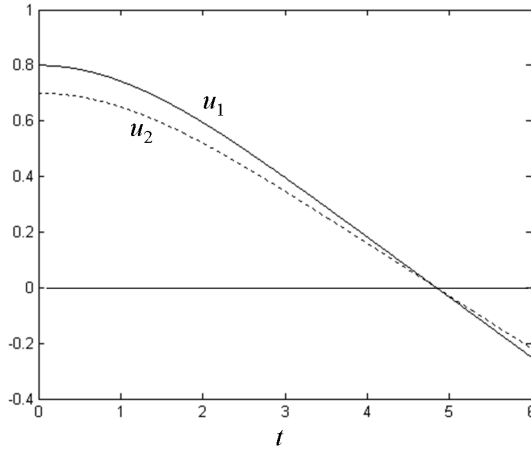


Fig. 1. Controls  $u_1(t)$  and  $u_2(t)$  given by (60) plotted vs. time  $t$ .

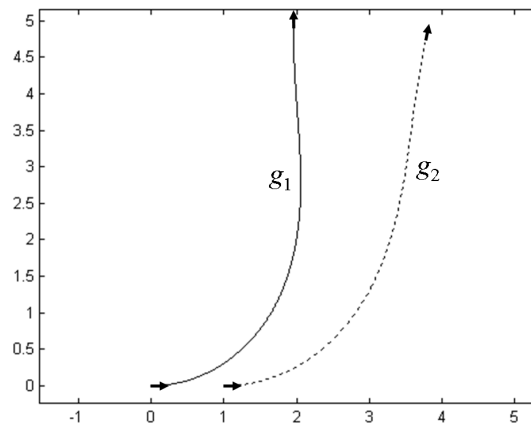


Fig. 2. Planar trajectories corresponding to the steering controls in figure 1. The short bold arrows indicate the corresponding fixed endpoint conditions.

so that, finally,

$$\begin{aligned} u_1(t) &= \frac{1}{5} (3\mu_{11} + 2\mu_{21}) = \frac{4}{5} \text{cn}(\nu(t - \eta), m), \\ u_2(t) &= \frac{1}{5} (2\mu_{11} + 3\mu_{21}) = \frac{7}{10} \text{cn}(\nu(t - \eta), m), \end{aligned} \quad (60)$$

for  $t \in [0, T]$ . Figure 1 shows  $u_1(t)$  and  $u_2(t)$  given by (60). Figure 2 shows the corresponding planar trajectories.

Two caveats about this example merit emphasis. First, we have chosen initial headings which are aligned; other than through boundary conditions, the steering controls do not feel the effects of relative positions and headings between the particles, and so there is no guarantee of noncollision in this setting. Second, the endpoint conditions in this example are such that the special solutions (based on proportional elliptic function steering controls) apply; for arbitrary endpoint conditions, these special solutions need not apply, and we must resort to solving (22), e.g., numerically.

#### IV. DIRECTIONS FOR FUTURE WORK

For planar trajectories, the correspondence between a particle in  $\mathbb{R}^2$  moving along a (smooth) trajectory, and a particle (i.e., a group element) evolving in  $SE(2)$  is unambiguous.

Associated with the particle trajectory in  $\mathbb{R}^2$ , there is an orthonormal frame consisting of a unit tangent vector and unit normal vector, and these vectors along with the position vector map in a one-to-one fashion to elements of  $SE(2)$ . A particle in  $\mathbb{R}^3$  moving along a (smooth) trajectory has an intrinsically defined unit tangent vector and corresponding normal plane, but this is not enough to have a one-to-one mapping to elements of  $SE(3)$ . This introduces subtleties in interpreting results for moving particles in  $SE(3)$  in terms of curves and frames in  $\mathbb{R}^3$ . Single-particle fixed endpoint optimal control problems on  $SE(3)$  (with generalizations to  $SE(n)$ ) are explored in [10].

In his recent work [1] Brockett investigates the collective and individual behaviors of a set of identical agents with nonlinear dynamics, all subject to identical controls, and asks questions about collective response to a coordination signal broadcast by a leader. In such a setting he shows that nonlinearity plays a critical role. While his focus is on closed loop effects (whereas our subject is open loop optimal control), it may be useful to examine the results of section III.C on special proportional controls from the perspective of his paper.

#### REFERENCES

- [1] R. W. Brockett, "On the control of a flock by a leader," *Proc. Steklov Institute of Mathematics*, **268**: 49-57, 2010.
- [2] D. David, D.D. Holm, and M.V. Tratnik, "Hamiltonian chaos in nonlinear optical polarization dynamics," *Physics Reports* **187**(6): 281-367, 1990.
- [3] H.T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover (Reprint), 1962.
- [4] J.A. Fax and R.M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Automatic Cont.*, **49**(9): 1465-1476, 2004.
- [5] K. Ghose, T.K. Horiuchi, P.S. Krishnaprasad, and C.F. Moss, "Echolocating Bats Use a Nearly Time-Optimal Strategy to Intercept Prey," *PLoS Biology*, **4**(5):0865-0873, 2006.
- [6] R. Grossman, P.S. Krishnaprasad, and J.E. Marsden, "The dynamics of two coupled rigid bodies," in *Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits*, F.M.A. Salam and M.L. Levi, eds., pp. 373-378, SIAM, 1998.
- [7] V. Jurdjevic, *Geometric control theory*, Cambridge Univ. Press, 1997.
- [8] E.W. Justh and P.S. Krishnaprasad, "Equilibria and steering laws for planar formations," *Sys. Ctrl. Lett.*, **52**(1): 25-38, 2004.
- [9] E.W. Justh and P.S. Krishnaprasad, "Natural frames and interacting particles in three dimensions," *Proc. 44th IEEE Conf. Decision and Control*, pp. 2841-2846, 2005 (see also arXiv:math.OA/0503390v1).
- [10] E.W. Justh and P.S. Krishnaprasad, "Optimal natural frames," *Communications in Information and systems*, in press, 2010.
- [11] E.W. Justh and P.S. Krishnaprasad, "Steering laws for motion camouflage," *Proc. R. Soc. A*, **462**(2076): 3629-3643, 2006.
- [12] P.S. Krishnaprasad, "Eulerian many-body problems," In *Contemporary Mathematics*, **97**: 187-208, AMS, 1989.
- [13] P.S. Krishnaprasad, "Lie Poisson structure, dual spin spacecraft and asymptotic stability," *Nonlinear Analysis: Theory, Methods and Applications* **9**(10): 1011-1035, 1985.
- [14] P.S. Krishnaprasad, "Optimal control and Poisson reduction," Institute for Systems Research Technical Report T.R. 93-87, 1993.
- [15] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd ed., Springer, 1999.
- [16] P.V. Reddy, *Steering laws for Pursuit*, M.S. Thesis, Dept. of Electrical and Computer Eng., Univ. of Maryland, 2007.
- [17] P.V. Reddy, E.W. Justh and P.S. Krishnaprasad, "Motion camouflage in three dimensions," *Proc. 45th IEEE Conf. Decision and Control*, pp. 3327-3332, 2006 (see also arXiv:math.OA/0603176v1).
- [18] F. Zhang, E.W. Justh, and P.S. Krishnaprasad, "Boundary following using gyroscopic control," *Proc. 43rd IEEE Conf. Decision and Control*, pp. 5204-5209, 2004.