Approximate Inversion of the Preisach Hysteresis Operator With Application to Control of Smart Actuators

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***Abstract*—Hysteresis poses a challenge for control of smart actuators. A fundamental approach to hysteresis control is inverse compensation. For practical implementation, it is desirable for the input function generated via inversion to have regularity properties stronger than continuity. In this paper, we consider the problem of constructing right inverses for the Preisach model for hysteresis. Under mild conditions on the density function, we show the existence and weak-star continuity of the right-inverse, when the Preisach operator is considered to act on Hölder continuous functions. Next, we introduce the concept of regularization to study the properties of approximate inverse schemes for the Preisach operator. Then, we present the fixed point and closest-match algorithms for approximately inverting the Preisach operator. The convergence and continuity properties of these two numerical schemes are studied. Finally, we present the results of an open-loop trajectory tracking experiment for a magnetostrictive actuator.**

***Index Terms*—Approximate inversion, closest-match algorithm, electro-active polymers, fixed point iteration algorithm, hysteresis, magnetostriction, piezoelectricity, Preisach operator, regularization, shape memory alloys, smart actuators.**

I. INTRODUCTION

SMART materials, e.g., magnetostrictives, piezoceramics, and shape memory alloys (SMAs), exhibit strong coupling between applied electromagnetic/thermal fields and strains that can be exploited for actuation and sensing. Hysteresis in smart materials, however, poses a significant challenge in smart material actuators (also called *smart actuators*). Models for hysteresis in smart materials can be classified into those that are physics-based and those that are phenomenology-based. Physics-based models use principles of thermodynamics to obtain constitutive relationships between conjugate variables. Such examples include the Jiles–Atherton model [1] and the ferromagnetic hysteresis model [2], [3], where hysteresis is considered to arise from pinning of domain walls on defect sites. The most popular hysteresis model used for magnetic materials has been the Preisach operator [4], and it has been used lately to model the hysteresis phenomenon in piezoelectrics [5], magnetostrictive materials [6], [7], shape-memory alloys [8], [9], and electro-active polymers [7]. The Preisach operator is a model of the phenomenological type. Although in general, the Preisach operator does not provide physical insight into the problem, it is capable of producing behaviors similar to those of physical systems [4]. It is of great interest to the smart structures and controls community because of its utility in developing low-order models that can be used for designing real-time controllers.

A fundamental idea in coping with hysteresis is inverse compensation (see, e.g., [5] and [10]–[12]), as illustrated in Fig. 1. If one can construct an approximate right inverse $\hat{W}^{-1}$ of the hysteresis operator W, then the output y of W will approximately equal the reference trajectory y_{ref}.

This paper deals with approximate inversion of the Preisach operator Γ, where u is required to be a Hölder continuous function. It contains five contributions: a) the proof of weak-star continuity of the inverse acting on the space of Hölder continuous functions, under a mild and easily verifiable condition on the Preisach density function; b) the formulation of regularization for the inversion problem; c) the development of a *fixed point* iteration algorithm and its convergence analysis; d) the development of the *closest-match* algorithm and its convergence analysis; and e) experimental validation of the *closest-match* algorithm. These contributions are briefly discussed next.

Brokate and Sprekels [13] prove the existence and continuity of the inverse of the Preisach operator when the domain is the space of continuous functions, under very mild conditions on the density function. Visintin [14] proves a theorem on the weak-star continuity of the inverse, when the domain is the space of Hölder continuous functions, under very strong sufficient conditions on the density function that are not easily verifiable. Fig. 7 shows an identified (in a nonparametric manner) density function for a magnetostrictive actuator [7]. The density function has a value zero on a large area of the Preisach domain and this implies that Visintin's condition will not be satisfied for this actuator. We need a theorem for the weak-star continuity of the inverse operator acting on spaces of Hölder continuous functions that only depends on the density conditions close to the diagonal on the Preisach plane. Such a theorem would be in the same spirit as [13, Cor. 2.11.21] for the continuity of the inverse operator acting on the space of continuous functions. We present a theorem in Section II that concludes the results of Visintin's theorem under mild conditions on the density function (these conditions are still stronger than Brokate and Sprekels'

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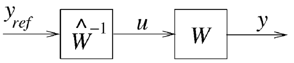

Fig. 1. Illustration of inverse compensation.

conditions, as expected). The utility of this theorem to control engineers is that the conditions can be easily verified.

In [15], we showed that the approximate inverse to an *incrementally strictly increasing* (ISI) Preisach operator can be computed numerically. For $u_1, u_2 \in C[0,T]$, consider the ordering $u_1 \geq u_2$ if and only if $u_1(t) \geq u_2(t)$ for all $t \in [0,T]$. Then the Preisach operator is said to be incrementally strictly increasing [16] (ISI) if there exist constants $k_1, k_2 > 0$ such that $k_1(u_1 - u_2) \leq \Gamma[u_1] - \Gamma[u_2] \leq k_2(u_1 - u_2)$. This definition is different from the *piecewise strictly increasing* operator (PSI) defined by Brokate and Sprekels. A Preisach operator is said to be piecewise strictly increasing if $(\Gamma[u](T) - \Gamma[u](0))(u(T) - u(0)) \geq 0$ for a monotone input $u \in C[0,T]$. Under the mild condition that the density function is integrable and nonzero almost everywhere on a strip of positive width along the diagonal on the Preisach plane, it is easy to show that the corresponding Preisach operator is PSI. The ISI condition requires very stringent conditions on the density function. For example, if the density function took a constant positive value on the set $\alpha_{\min} \leq \beta \leq \alpha \leq \alpha_{\max}$ in the Preisach plane, then it is ISI. We have shown in [15] and [17] that the Fixed Point iteration: $u_{n+1} = u_n + (1/k_2)(y - \Gamma(u_n))$ converges to a function u^* that satisfies $\Gamma[u^*] = y$ via a contraction. Leang and Devasia [18] apply this result to the positioning of piezoelectric actuators. In this paper, under the (significantly) milder condition of PSI Preisach operators, we show the convergence of the same scheme without using a contraction argument.

We are led to the space of Lipschitz continuous functions as we would like the solution of the inverse problem, to have regularity properties stronger than just continuity. For example, in the case of inductors or transformers with a ferromagnetic core, the Preisach operator is usually considered to map the axial magnetic field $H(t)$ function to the axial magnetization $M(t)$ (see [4])

$$\Gamma[H](t) = M(t). \tag{1}$$

The electro-motive force across the terminals of the inductor is then proportional to the time-derivative of $B(t) = \mu_0(H(t) + M(t))$, where μ_0 is the permittivity of free-space. Therefore, it is desirable for $H(t)$ which is the solution to (1) to be a differentiable function of time. Similar considerations apply to other situations where one uses the Preisach operator, for example, piezoelectricity. Now, Rademacher's Theorem states that a function that is Lipschitz on an open subset of $\mathbf{R}^n$ is almost everywhere differentiable on that subset in the sense of the Lebesgue measure [19], and so it is reasonable to seek Lipschitz continuous functions as solutions to the inverse problem. Consideration of Lipschitz functions is also motivated by constraints on implementation of control signals often encountered in practice.

Our theorem (see Theorem 2.2) shows that the inverse maps generic functions in the space of Hölder continuous functions on $[0,T]$ denoted by $C^{0,\nu_2}[0,T]$ to the space $C^{0,\nu_1}[0,T]$ where $(\nu_1/\nu_2) \leq (1/2)$. This result implies that in general, even if the desired output function is differentiable, the input function does not need to be a Lipschitz continuous function. For engineering reasons, if one wishes to obtain a Lipschitz continuous function as the (approximate) inverse of a Hölder continuous function, an operation called *mollification* [20] has to be carried out. The natural question that arises then is the following: If the desired output function is changed by a small amount either due to noise or by design, then how "close" is the resulting mollified solution to the original mollified solution (and in what sense)? This is a question of enormous engineering importance, and to discuss it, we develop the notion of *regularization* for solving the inversion problem in Section III. Two approximate inversion algorithms for the Preisach operator are then developed. Both algorithms use the PSI property of a Preisach operator. In Section IV, we present the fixed point algorithm to approximately invert the Preisach operator, and study its convergence and continuity properties under the PSI condition. Next, the closest-match algorithm is developed and analyzed in Section V. The latter algorithm is applied to tracking control of a magnetostrictive actuator, and experimental results are reported to demonstrate its efficacy.

II. PREISACH OPERATOR AND ITS INVERSE

To fix the notation and the problem setup, the Preisach operator Γ and some known results are reviewed first in Section II-A. Section II-B then studies the weak-star continuity of Γ^{-1} in the space of Hölder continuous functions under the weak condition. Let I be a closed interval, $\nu \in (0,1]$, and $T > 0$. The following notation will be used to denote different function spaces:

- $C[0,T]$: space of continuous functions on $[0,T]$;
- $C_m[0,T]$: space of monotone, continuous functions on $[0,T]$;
- $C_{pm}[0,T]$: space of piecewise monotone, continuous functions on $[0,T]$;
- $C_I[0,T]$: space of continuous functions taking values in I, i.e., $u(t) \in I, \forall\, u \in C_I[0,T], \forall\, t \in [0,T]$;
- $C_{pm,I}[0,T] : C_{pm}[0,T] \cap C_I[0,T]$;
- $C^{0,\nu}[0,T]$: space of Hölder continuous functions on $[0,T]$, i.e., $\forall\, u \in C^{0,\nu}[0,T], \forall\, t_1, t_2 \in [0,T]$
$$\sup_{0 \leq t_1, t_2 \leq T} \frac{|u(t_2) - u(t_1)|}{|t_2 - t_1|^\nu} \leq C_0$$
for some constant C_0.

Other spaces such as $C_I^{0,\nu}[0,T]$ are defined analogously to the definition of $C_I[0,T]$ from $C[0,T]$. In this paper, the following two norms are heavily used:

$$\|u\|_\infty \triangleq \sup_{0 \leq t \leq T} |u(t)| \qquad \forall\, u \in C[0,T] \text{ and}$$
$$\|u\|_{0,\nu} \triangleq \|u\|_\infty + \sup_{0 \leq t_1, t_2 \leq T} \frac{|u(t_2) - u(t_1)|}{|t_2 - t_1|^\nu}$$

for $u \in C^{0,\nu}[0,T]$.

A. Preisach Operator

A detailed treatment on the Preisach operator can be found in [4], [13], and [14]. For a pair of thresholds (β, α) with $\beta \leq \alpha$, consider a delayed relay $\hat{\gamma}_{\beta,\alpha}[\cdot,\cdot]$ (called a Preisach *hysteron*), as illustrated in Fig. 2. For $u \in C[0,T)$ and an initial configuration $\zeta \in \{-1, 1\}$, $v = \hat{\gamma}_{\beta,\alpha}[u, \zeta]$ is defined as, for $t \in [0,T]$

$$v(t) \triangleq \begin{cases} -1, & \text{if } u(t) < \beta \\ 1, & \text{if } u(t) > \alpha \\ v(t^-), & \text{if } \beta \leq u(t) \leq \alpha \end{cases}$$

where $v(0^-) = \zeta$ and $t^- \triangleq \lim_{\epsilon > 0, \epsilon \to 0} t - \epsilon$.

Define *the Preisach plane* $\mathcal{P}_0 \triangleq \{(\beta, \alpha) \in \mathbb{R}^2 : \beta \leq \alpha\}$, where $(\beta,\alpha) \in \mathcal{P}_0$ is identified with $\hat{\gamma}_{\beta,\alpha}$. For $u \in C[0,T]$ and a Borel measurable configuration ζ_0 of all hysterons, $\zeta_0 : \mathcal{P}_0 \to \{-1, 1\}$, the output of the Preisach operator Γ is defined as

$$\Gamma[u, \zeta_0](t) = \int_{\mathcal{P}_0} \mu(\beta,\alpha) \hat{\gamma}_{\beta,\alpha}[u, \zeta_0(\beta,\alpha)](t) d\beta d\alpha \tag{2}$$

for some Borel measurable function μ, called the *Preisach density function*. It is assumed in this paper that $\mu \geq 0$; μ has a compact support $\mathcal{P}$; and is an integrable function, that is $\mu \in L^1(\mathcal{P})$.

For each $t \in [0,T]$, $\mathcal{P}$ can be divided into two regions

$$\mathcal{P}_-(t) \triangleq \{(\beta,\alpha) \in \mathcal{P} | \text{ output of } \hat{\gamma}_{\beta,\alpha} \text{ at } t \text{ is } -1\}$$
$$\mathcal{P}_+(t) \triangleq \{(\beta,\alpha) \in \mathcal{P} | \text{ output of } \hat{\gamma}_{\beta,\alpha} \text{ at } t \text{ is } +1\}$$

so that $P = \mathcal{P}_-(t) \cup \mathcal{P}_+(t)$. Equation (2) can be rewritten as

$$\Gamma[u, \zeta_0](t) = \int_{\mathcal{P}_+(t)} \mu(\beta,\alpha) d\beta d\alpha - \int_{\mathcal{P}_-(t)} \mu(\beta,\alpha) d\beta d\alpha. \tag{3}$$

It can be easily shown [4], [13] that each of $\mathcal{P}_-$ and $\mathcal{P}_+$ is a connected set, and that the output of the Preisach operator is determined by the boundary between $\mathcal{P}_-$ and $\mathcal{P}_+$. The boundary is also called the *memory curve*, since it provides information about the state of Γ. Thus the initial state function ζ_0 can instead be replaced by a memory curve in the Preisach plane. Using the transform: $r = (\alpha - \beta)/2$ and $s = (\alpha + \beta)/2$ one can describe the memory curve as a function $(r, \psi(r))$ defined on a compact region $[0, r_{\max}]$. The set of *admissible memory curves* can then be defined as [13]

$$\Psi_0 \triangleq \{\phi | \phi : \mathbf{R}_+ \to \mathbf{R}, |\phi(r) - \phi(\bar{r})| \leq |r - \bar{r}|$$
$$\forall\, r, \bar{r} \geq 0, R_{\text{supp}}(\phi) < +\infty\}$$

where

$$R_{\text{supp}}(\phi) \triangleq \sup\{r | r \geq 0, \phi(r) \neq 0\}.$$

The memory curve ψ_{-1} at $t = 0$ is called the *initial memory curve* and hereafter it will be put as the second argument of the Preisach operator. Note that $\psi_{-1}(0)$ equals the last input value of Γ. The Preisach density will be denoted as $\omega(\cdot,\cdot)$ in the (r,s) coordinates. In this paper, both coordinate systems, (β,α) and (r,s), are used depending on whichever is more convenient; similarly, both $\mu(\beta,\alpha)$ and $\omega(r,s)$ will be used for the Preisach density.

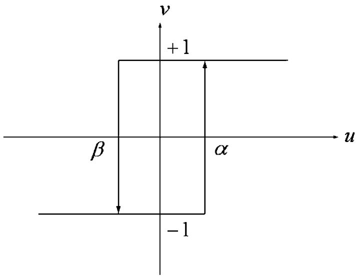

Fig. 2. Illustration of an elementary Preisach hysteron.

Let the input signal take values in $I = [u_{\min}, u_{\max}]$, that is, $u(t) \in I, \forall\, t \in [0,T]$. Define the function $\chi(\cdot)$ on $[0, u_{\max} - u_{\min}]$

$$\chi_I(x) \triangleq \inf\{|\Gamma[u, \psi_{-1}](T) - \Gamma[u, \psi_{-1}][0]| :$$
$$\psi_{-1} \in \Psi_0, u \in C_m[0,T], |u(T) - u(0)| = x\}.$$

The function $\chi(\cdot)$ is continuous and monotonically increasing under our basic hypothesis on μ. It is easy to check that if $\chi(x) > 0$ for all $x > 0$, then Γ is PSI. Let J be the smallest interval that contains the output values of Γ when the input u takes values in I. It can be shown that if $\chi_I(x) > 0, \forall\, x > 0$, then $\Gamma[\cdot, \psi_{-1}] : C_I[0,T] \to C_J[0,T]$ is invertible and the inverse operator is also continuous [13], [14]. Furthermore, Visintin [14] shows that, if $\forall\, x \in [0, u_{\max} - u_{\min}]$

$$\chi_I(x) \geq C x^{(\nu_2/\nu_1)} \tag{4}$$

for $0 < \nu_1, \nu_2 \leq 1$, then the inverse of $\Gamma[\cdot, \psi_{-1}]$ maps C_J^{0,ν_2} into C_I^{0,ν_1} and it is weak-star continuous.

B. Milder Condition for the Weak-Star Continuity of the Inverse

Condition (4) is strong since it needs to hold for all $x \in [0, u_{\max} - u_{\min}]$. It is hard to verify directly also, as it is posed in terms of $\chi_I(x)$. In this section, a weaker condition in terms of the Preisach density function $\omega(\cdot,\cdot)$ is shown to lead to the weak-star continuity of the inverse.

Before proceeding, we sketch the construction of the weak-star topology on $C^{0,\nu}[0,T]$, $0 < \nu < 1$. A function f in the space $C^{0,\nu}(\mathbf{R})$, $0 < \nu < 1$, can be expanded using a Faber–Schauder basis as described in [21, p. 40]. Thus, we have a map $\Phi : C^{0,\nu}[0,T] \to l^\infty$ given by $\Phi(f) = \{a_{m,n}\}$. Its adjoint Φ^* maps elements in l^1 that describe the weak-star topology on l^∞ to the dual of $C^{0,\nu}[0,T]$. These functionals $\Phi^*(z)$, $z \in l^1$, define the weak-star topology of $C^{0,\nu}[0,T]$. In Section III, we will define a distance metric for the weak-star topology of $C^{0,\nu}[0,T]$ based on this construction. It is well known that this topology is coarser than the norm topology on $C^{0,\nu}[0,T]$ defined using $\|f\|_{0,\nu}$.

The following three lemmas will be used in proving the main result of this section.

Lemma 2.1: Let $\xi \geq 0$, $\epsilon > 0$. If the Preisach density $\omega(r,s) \geq C r^\xi$, for some $C > 0$, for almost every $(r,s) \in R_\epsilon = [0,\epsilon] \times [u_{\min} - \epsilon, u_{\max} + \epsilon]$, then $\chi_I(x) \geq K x^{\xi+2}$ for $0 \leq x \leq 2\epsilon$, for some $K > 0$.

Proof: Let $\bar{r} = (x/2)$. For $x \in [0, 2\epsilon]$

$$\begin{aligned}\chi_I(x) &= 2 \inf_{s_0 \in [u_{\min}, u_{\max}]} \int_0^{\bar{r}} \int_{s_0-((x/2)-r)}^{s_0+((x/2)-r)} \omega(r,s)\,ds\,dr \\ &\geq 2 \int_0^{\bar{r}} (x-2r) C r^{\xi} dr \\ &= \frac{C}{(1+\xi)(2+\xi)2^{\xi}} x^{\xi+2}.\end{aligned}$$

□

Lemma 2.2: [14] Let X, Y, S_1, S_2 be metric spaces such that $S_1 \subset X$ and $S_2 \subset Y$ with continuous injections. Let $f : X \to Y$ be continuous and such that it maps relatively compact subsets of S_1 into relatively compact subsets of S_2 (with respect to the topologies of S_1 and S_2). Then, $f : S_1 \to S_2$ is continuous with respect to the topologies of S_1 and S_2.

Lemma 2.3: Let $0 = T_0 < T_1 < \cdots < T_N = T$ be a uniform partition of $[0, T]$ such that $\Delta_i = [T_{i-1}, T_i]$; $i = 1, \ldots, N$, has length δ. Let $f_i \in C^{0,\nu}(\Delta_i)$, $0 < \nu \leq 1$, $i = 1, \ldots, N$, with $\|f_i\|_{0,\nu} \leq K$ and $f_i(T_i) = f_{i+1}(T_i)$. Then, the function obtained by concatenation $f = \sum_i f_i I_{\Delta_i}$, where I_{Δ_i} is the indicator function of Δ_i, belongs in $C^{0,\nu}[0, T]$ and $\|f\|_{0,\nu} \leq (1 + N^{1-\nu})K$.

Proof: As $\|f_i\|_{0,\nu} \leq K$, we have $|f_i(t)| \leq K$ and $|f_i(t) - f_i(t')| \leq K|t - t'|^{\nu}$ for $t, t' \in \Delta_i$. This implies $|f(t)| \leq K$, $\forall\, t \in [0, T]$. Next, for $t \in \Delta_1$ and $t' \in \Delta_N$

$$\begin{aligned}|f(t) - f(t')| &\leq |f(t) - f(T_1)| + |f(T_1) - f(T_2)| \\ &\quad + \cdots + |f(T_{N-1} - f(t')| \\ &\leq K|t - T_1|^{\nu} + \cdots + K|T_{N-1} - t'|^{\nu}.\end{aligned}$$

We wish to find a constant $\bar{L}$ such that the sum $a_1^{\nu} + a_2^{\nu} + \cdots + a_N^{\nu} \leq \bar{L}(a_1 + \cdots + a_N)^{\nu}$ where $a_1, \ldots, a_N \geq 0$ and $0 < \nu \leq 1$. Dividing by $(a_1 + \cdots + a_N)^{\nu}$ one obtains the following function on the left-hand side: $g(p_1, \ldots, p_N) = p_1^{\nu} + \cdots + p_N^{\nu}$ where $p_1, \ldots, p_N \geq 0$ and $\sum_{i=1}^{N} p_i = 1$. This function is maximized by $p_i = (1/N)$ for all i and the maximum value is $N(1/N^{\nu}) = N^{1-\nu}$. Thus, $\bar{L} = N^{1-\nu}$ and

$$|f(t) - f(t')| \leq N^{1-\nu} K |t - t'|^{\nu}.$$

For t and t' in other intervals Δ_i, one can proceed similarly and arrive at the same inequality. Therefore

$$\begin{aligned}\|f\|_{0,\nu} &= \|f\|_{\infty} + \sup_{t \neq t'; t, t' \in [0,T]} \frac{|f(t) - f(t')|}{|t - t'|^{\nu}} \\ &\leq K(1 + N^{1-\nu}).\end{aligned}$$

□

Before presenting our main theorem on Γ^{-1}, we summarize the continuity properties of the operator Γ under certain conditions on ω. The utility of this theorem is that it combines results in [13] and [14] under a common condition on the density function. These are the same conditions needed on ω for our main result. It must be noted that these conditions are slightly stronger than those of [13, Prop. 2.4.11 and Cor. 2.11.21], and weaker than [14, Th. 3.9].

Theorem 2.1: Let $\Gamma[\cdot, \psi_{-1}]$ be a Preisach operator with domain $I = [u_{\min}, u_{\max}]$, where $\psi_{-1} \in \Psi_0$. Assume that the density function $\omega(r, s)$ has compact support; is integrable; is nonnegative; and $\omega(r, s) \geq C r^{\xi}$ for almost every $(r, s) \in R_{\epsilon} = [0, \epsilon] \times [u_{\min} - \epsilon, u_{\max} + \epsilon]$, where $C > 0$, $\xi \geq 0$, and $\epsilon > 0$. Then

1) $\Gamma[\cdot, \psi_{-1}] : C_I[0, T] \to C_J[0, T]$ is Lipschitz continuous;
2) $\Gamma[\cdot, \psi_{-1}] : C_I^{0,\nu}[0, T] \to C_J^{0,\nu}[0, T]$ is weak-star continuous, where $0 < \nu \leq 1$;
3) $\Gamma[\cdot, \psi_{-1}] : C_I[0, T] \to C_J[0, T]$ is invertible, and its inverse can be extended to a continuous operator $\Gamma^{-1}[\cdot, \psi_{-1}] : C_J[0, T] \to C_I[0, T]$.

Proof: By the conditions on the density, the Preisach operator $\Gamma[\cdot, \psi_{-1}] : C_I[0, T] \to C_J[0, T]$ is PSI and is Lipschitz continuous (by [13, Th. 2.4.11]). They also show that Γ maps norm-bounded sets in $C_I^{0,\nu}[0, T]$ to norm-bounded sets in $C_J^{0,\nu}[0, T]$. As these sets are compact in the weak-star topology, Lemma 2.2 yields the weak-star continuity of Γ. To show the last statement, note that $\chi_I(x) > 0$ by Lemma 2.1 for $x \in (0, 2\epsilon]$. As $\chi_I(x)$ is a continuous, increasing function of x and so $\chi_I(x) > 0$ for all $x \in (0, b - a)$, the proof of [13, Th. 2.11.20] applies here. □

Under the same conditions on ω as in the previous theorem, we would like to show the existence and continuity of the inverse for the Preisach operator acting between spaces of Hölder continuous functions. The following theorem is our main result.

Theorem 2.2: Assume that the Preisach density function $\omega(r, s)$ has compact support, $\omega \geq 0$, and $\omega(r, s) \geq C r^{\xi}$ for almost every $(r, s) \in R_{\epsilon} = [0, \epsilon] \times [u_{\min} - \epsilon, u_{\max} + \epsilon]$, where $C > 0$, $\xi \geq 0$, $\epsilon > 0$. Then, for any $\psi_{-1} \in \Psi_0$, $\Gamma^{-1}[\cdot, \psi_{-1}]$ is weak-star continuous from $C_J^{0,\nu_2}[0, T]$ to $C_I^{0,\nu_1}[0, T]$, where $\nu_2 \in (0, 1]$ and $\nu_1 = (\nu_2/\xi + 2)$.

Proof: Let $y \in C_J^{0,\nu_2}[0, T]$ with $\|y\|_{0,\nu_2} \leq K$. By Theorem 2.1, $\Gamma[\cdot, \psi_{-1}]$ is invertible and there exists $u \in C_I[0, T]$ such that $\Gamma[u, \psi_{-1}] = y$. We will show that u belongs in $C_I^{0,\nu_1}[0, T]$.

Partition $[0, T]$ uniformly such that $0 = T_0 < T_1 \cdots < T_N = T$ and $T_i - T_{i-1} \leq \delta$ where $i = 1, \ldots, N$. The choice of δ will be described shortly. Restrict y to the intervals $\Delta_i = [T_{i-1}, T_i]$; $i = 1, \ldots, N$, and obtain the functions y_i. Similarly restricting u to Δ_i one obtains u_i. Define the function

$$\operatorname{osc}(v; [a, b]) \triangleq \max_{t \in [a,b]} v(t) - \min_{t \in [a,b]} v(t)$$

for $v \in C[0, T]$ and $[a, b] \subset [0, T]$. Note that

$$\chi_I(\operatorname{osc}(u_i; [t, t'])) \leq \operatorname{osc}(y; [t, t']) \qquad \forall\, [t, t'] \subset \Delta_i \tag{5}$$

by [13, Lemma 2.11.18]. As $\|y\|_{0,\nu_2} \leq K$, for $t, t' \in \Delta_i$

$$|y(t) - y(t')| \leq K|t - t'|^{\nu_2} \leq K\delta^{\nu_2} \tag{6}$$

and, hence

$$\operatorname{osc}(y; \Delta_i) \leq K\delta^{\nu_2} \tag{7}$$

which by (5) implies

$$\chi_I(\operatorname{osc}(u_i; \Delta_i)) \leq K\delta^{\nu_2}. \tag{8}$$

From Lemma 2.1

$$\chi_I(x) \geq Cx^{\xi+2}, \qquad x \in [0, 2\epsilon]. \tag{9}$$

Now, choose $\delta > 0$ small enough so that

$$K\delta^{\nu_2} \leq C(2\epsilon)^{\xi+2}.$$

This together with (8), (9), and the monotone increasing property of $\chi_I(\cdot)$, implies

$$\operatorname{osc}(u_i; \Delta_i)) \leq 2\epsilon.$$

Note that the choice of δ fixes the number of partitions N.

Next, for $t, t' \in \Delta_i$; $i = 1 \ldots, N$, (5) and (9) yield

$$\begin{aligned} C|u_i(t) - u_i(t')|^{\xi+2} &\leq \chi_I(\operatorname{osc}(u_i, [t, t'])) \\ &\leq \operatorname{osc}(y; [t, t']) \text{ (by (5))} \qquad (10) \\ &\leq K|t - t'|^{\nu_2} \text{ (as } \|y\|_{0,\nu_2} \leq K) \qquad (11) \end{aligned}$$

which leads to

$$\begin{aligned} |u_i(t) - u_i(t')| &\leq \left(\frac{K}{C}\right)^{1/(\xi+2)} |t - t'|^{\nu_2/(\xi+2)} \\ &= \left(\frac{K}{C}\right)^{1/(\xi+2)} |t - t'|^{\nu_1}. \qquad (12) \end{aligned}$$

Finally, using Lemma 2.3, one gets $\|u\|_{0,\nu_1} \leq K_1$ for some $K_1 > 0$. This implies that $\Gamma^{-1}[\cdot, \psi_{-1}]$ maps norm-bounded sets in $C_J^{0,\nu_2}[0,T]$ to norm-bounded sets in $C_I^{0,\nu_1}[0,T]$. As these sets are compact in the respective weak-star topologies of $C^{0,\nu_i}[0,T]$, $i = 1, 2$, we apply Lemma 2.2 to Γ^{-1} with $X = C_J[0,T]$; $Y = C_I[0,T]$; $S_1 = C_J^{0,\nu_2}[0,T]$; and $S_2 = C_I^{0,\nu_1}[0,T]$, to obtain the weak-star continuity of Γ^{-1}. □

Let $0 < \nu_1 < \nu_2 \leq 1$. As $C^{0,\nu_2}[0,T] \subset C^{0,\nu_1}[0,T]$, the linear functionals on $C^{0,\nu_1}[0,T]$ are also linear functionals on $C^{0,\nu_2}[0,T]$. As a result, the weak-star topology on $C^{0,\nu_2}[0,T]$ (denoted by τ_2) is finer than the topology (denoted by τ_1) inherited from the weak-star topology of $C^{0,\nu_1}[0,T]$. This implies that weak-star compact sets of $C^{0,\nu_2}[0,T]$ remain compact in the topology τ_1 [22]. Denote the weak-star topology of $C^{0,\nu_1}[0,T]$ by τ.

Corollary 2.1: Suppose that Γ is a Preisach operator with a density function that satisfies the conditions of Theorem 2.2. Let $U = \Gamma^{-1}\left[C_J^{0,\nu_2}, \psi_{-1}\right]$ and $(\nu_2/\nu_1) = \xi + 2$. Then, the maps $\Gamma^{-1} : \left(C_J^{0,\nu_2}[0,T], \tau_2\right) \to (U, \tau)$, and $\Gamma : (U, \tau) \to \left(C_J^{0,\nu_2}[0,T], \tau_1\right)$ are continuous maps.

Proof: By Theorem 2.2, $\Gamma^{-1} : \left(C_J^{0,\nu_2}[0,T], \tau_2\right) \to (U, \tau)$ is continuous, as $U \subset C_I^{0,\nu_1}[0,T]$. To show the second statement, observe that the map $\Gamma : (U, \tau) \to \left(C_J^{0,\nu_1}[0,T], \tau\right)$ is continuous by Theorem 2.1. But we must have $\Gamma : U \to C_J^{0,\nu_2}[0,T]$ by the definition of U. So $\Gamma : (U, \tau) \to \left(C_J^{0,\nu_2}[0,T], \tau_1\right)$ is continuous, by the definition of τ_1. □

Thus, the composition

$$\Gamma \circ \Gamma^{-1} : \left(C_J^{0,\nu_2}[0,T], \tau_2\right) \to \left(C_J^{0,\nu_2}[0,T], \tau_1\right)$$

is continuous, as τ_2 is finer than τ_1. Note that we cannot infer a similar statement had we considered the composition $\Gamma^{-1} \circ \Gamma$. Thus, we are naturally led to the concept of *right inverses* of Preisach operators and fortunately, that is what is needed in applications.

III. REGULARIZATION

The objective of this section is to study approximate solution methods for the operator equation

$$\Gamma[u, \psi_{-1}] = y \tag{13}$$

where $y \in C[0,T]$. Since the condition $\chi_I(x) > 0$ for $x > 0$ guarantees the existence of a continuous inverse for $\Gamma[\cdot, \psi_{-1}] : C_I[0,T] \to C_J[0,T]$, theoretically there is no need for any regularization if one is looking for just a continuous input function. However, for implementation of the inverse in numerical and physical experiments, it is desirable that the input generated via inversion has certain regularity properties, for example, Lipschitz continuity. The two algorithms to be discussed later in this paper result in Lipschitz continuous functions u as approximate solutions to (13) for $y \in C[0,T]$. On the other hand, the proof of Theorem 2.2 shows that a piecewise strictly increasing Preisach operator has an inverse that maps generic functions in $C_J^{0,\nu_2}[0,T]$ to functions in $C_I^{0,\nu_1}[0,T]$ with $(\nu_1/\nu_2) \leq (1/2)$, which rules out the possibility of getting a Lipschitz continuous u in general. This raises the issue of how to evaluate an approximate inversion scheme in terms of the convergence to the exact inverse. For this purpose, it is useful to define a norm on approximate inverses by the following procedure.

As $C^{0,\nu}[0,T]$, $0 < \nu < 1$, is isomorphic to l^∞, the weak-star topology on $C^{0,\nu}[0,T]$ is defined by a countable family of seminorms. On the other hand, $C^{0,1}[0,T]$ is isomorphic to L^∞ and so its weak-star topology is also defined by a countable family of seminorms [21]. Using these seminorms, one can define equivalent metrics on $C^{0,\nu}[0,T]$, $0 < \nu \leq 1$ such that convergence in any of the metrics is equivalent to convergence in the weak-star topology [23, page 14]. Denote any one of the metrics so obtained on $C^{0,\nu_i}[0,T]$, where $i = 1, 2$ and $0 < \nu_1 < \nu_2 \leq 1$, by $d_i(\cdot, \cdot)$. A key observation is that these metrics are *translation invariant*, that is, $d_i(x + c, y + c) = d_i(x, y)$ since they are defined using seminorms.

One would like to define an (induced) norm for Γ^{-1} in studying the convergence of approximation schemes. Putting the inverse operator and various approximate inverses in a vector space would facilitate the use of tools available to vector spaces. This can be achieved by appropriately shifting the input and the output of Γ. To be specific, considering that the inputs must have the initial condition $u(0) = \psi_{-1}(0)$ and the outputs must have the same initial value $z_0 = \Gamma[u; \psi_{-1}](0)$, we define the sets $\bar{I} = \{v - \psi_{-1}(0) \mid v \in I\}$, and $\bar{J} = \{w - z_0 \mid w \in J\}$, and the maps

$$\begin{aligned} \bar{\Gamma}[\cdot, \psi_{-1}] &: C_{\bar{I}}^{0,\nu_1}[0,T] \to C_{\bar{J}}^{0,\nu_2}[0,T] \\ &\bar{u} \mapsto \bar{y} = \Gamma[\bar{u} + \psi_{-1}(0), \psi_{-1}](t) - z_0 \qquad (14) \\ \bar{\Gamma}^{-1}[\cdot, \psi_{-1}] &: C_{\bar{J}}^{0,\nu_2}[0,T] \to C_{\bar{I}}^{0,\nu_1}[0,T] \\ &\bar{y} \mapsto \bar{u} = \Gamma^{-1}[\bar{y} + z_0, \psi_{-1}] - \psi_{-1}(0). \qquad (15) \end{aligned}$$

By translation invariance of d_i, one has $d_1(u_1 - \psi_{-1}(0), u_2 - \psi_{-1}(0)) = d_1(u_1, u_2)$ and $d_2(y_1 - z_0, y_2 - z_0) = d_2(y_1, y_2)$.

It can be verified that $\bar{\Gamma}^{-1}[\cdot, \psi_{-1}]$ belongs in the vector space $\mathcal{S}$ (with field $\mathbf{R}$) of maps $S : C^{0,\nu_2}[0,T] \to C^{0,\nu_1}[0,T]$ that satisfy $S[\theta_2](t) = \theta_1 \ \forall\, t \in [0,T]$, where θ_i are the zero-functions in $C^{0,\nu_i}[0,T]; i = 1,2$. The zero element Θ on $\mathcal{S}$ is simply the element that maps all $\bar{y} \in C^{0,\nu_2}[0,T]$ to θ_1. On $\mathcal{S}$, we can define the norm

$$\|S\|_{\mathcal{S}} = \sup_{\substack{\bar{y}_1, \bar{y}_2 \in C^{0,\nu_2}[0,T] \\ \bar{y}_1 \neq \bar{y}_2}} \frac{d_1(S[\bar{y}_1], S[\bar{y}_2])}{d_2(\bar{y}_1, \bar{y}_2)}. \tag{16}$$

Convergence of approximate inverse schemes can be discussed using this norm.

Definition 3.1: Let $\bar{\Gamma}$ be defined by (14). A *regularization strategy* for $\bar{\Gamma}$ is a family of operators

$$R_\epsilon[\cdot, \psi_{-1}] : C_{\bar{J}}[0,T] \to C_{\bar{I}}^{0,\nu_1}[0,T], \qquad \epsilon > 0$$

such that

1) $\forall\, \bar{y} \in C_{\bar{J}}[0,T]$

$$\lim_{\epsilon \to 0} \bar{\Gamma} \circ R_\epsilon[\bar{y}, \psi_{-1}] = \bar{y}; \tag{17}$$

2)

$$\lim_{\epsilon \to 0} d_1(R_\epsilon[\bar{y}, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]) = 0 \tag{18}$$

uniformly on bounded sets of $C_{\bar{J}}^{0,\nu_2}[0,T]$.

In other words, one requires point-wise convergence for $\bar{y} \in C_{\bar{J}}[0,T]$ and weak-star convergence for $\bar{y} \in C_{\bar{J}}^{0,\nu_2}[0,T]$. Obviously, R_ϵ with domain restricted to functions in $C^{0,\nu_2}[0,T]$ is in $\mathcal{S}$. The following elementary lemmas hold for the family $\{R_\epsilon\}$.

Lemma 3.1: If $\|R_\epsilon - \bar{\Gamma}^{-1}\|_{\mathcal{S}} \to 0$, as $\epsilon \to 0$, then $\lim_{\epsilon \to 0} d_1(R_\epsilon[\bar{y}, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]) = 0$ uniformly on bounded sets of $C_{\bar{J}}^{0,\nu_2}[0,T]$.

Proof: Consider the bounded set $\mathcal{M} = \{\bar{y} \mid d_2(\bar{y}, 0) \le M\}$. Now

$$\begin{aligned} &d_1(R_\epsilon[\bar{y}, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]) \\ &\quad \le d_1\left((R_\epsilon - \bar{\Gamma}^{-1})[\bar{y}, \psi_{-1}], 0\right) \\ &\quad\quad \text{(by the translation invariance of } d_1\text{)} \\ &\quad \le \|R_\epsilon - \bar{\Gamma}^{-1}\|_{\mathcal{S}} d_2(\bar{y}, 0) \quad \text{(by the definition of } \|\cdot\|_{\mathcal{S}}\text{)} \\ &\quad \le M \|R_\epsilon - \bar{\Gamma}^{-1}\|_{\mathcal{S}}. \end{aligned}$$

So given an $\epsilon_0 > 0$, there exists an $\bar{\epsilon} > 0$ such that: If $0 < \epsilon \le \bar{\epsilon}$, then $d_1(R_\epsilon[\bar{y}, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]) < \epsilon_0$ for all $y \in \mathcal{M}$. ☐

This lemma shows that (18) is weaker than norm-convergence. Since $\Gamma^{-1} : C^{0,\nu_2}[0,T] \to C^{0,\nu_1}[0,T]$ is weak-star continuous, one would like the approximating family to have a similar property. The next lemma studies the weak-star continuity properties of the family $\{R_\epsilon\}$.

Lemma 3.2: Let $\|\bar{\Gamma}^{-1}\|_{\mathcal{S}}$ be bounded and $\{R_\epsilon\}$ be a regularization strategy for $\bar{\Gamma}$. Then given $\epsilon_0 > 0$ and a bounded set $\mathcal{M}$, there exists an $\bar{\epsilon} > 0$ and $\delta > 0$ such that: If $0 < \epsilon \le \bar{\epsilon}$; $\bar{y}_1, \bar{y}_2 \in \mathcal{M}$; and $d_2(\bar{y}_1, \bar{y}_2) < \delta$, then $d_1(R_\epsilon[\bar{y}_1, \psi_{-1}], R_\epsilon[\bar{y}_2, \psi_{-1}]) < \epsilon_0$.

Proof: Let $\mathcal{M} = \{\bar{y} \mid d_2(\bar{y}, 0) \le M\}$. Then, given $\epsilon_0 > 0$, there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon \le \bar{\epsilon}$, we have $d_1(R_\epsilon[\bar{y}, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]) < (\epsilon_0/3)$, for all $\bar{y} \in \mathcal{M}$. Therefore, for $\bar{y}_1, \bar{y}_2 \in \mathcal{M}$

$$\begin{aligned} &d_1(R_\epsilon[\bar{y}_1, \psi_{-1}], R_\epsilon[\bar{y}_2, \psi_{-1}]) \\ &\quad \le d_1(R_\epsilon[\bar{y}_1, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}_1, \psi_{-1}]) \\ &\quad\quad + d_1(\bar{\Gamma}^{-1}[\bar{y}_1, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}_2, \psi_{-1}]) \\ &\quad\quad + d_1(R_\epsilon[\bar{y}_2, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}_2, \psi_{-1}]) \\ &\quad < 2\frac{\epsilon_0}{3} + d_1(\bar{\Gamma}^{-1}[\bar{y}_1, \psi_{-1}], \bar{\Gamma}^{-1}[\bar{y}_2, \psi_{-1}]) \\ &\quad \le 2\frac{\epsilon_0}{3} + \|\bar{\Gamma}^{-1}\|_{\mathcal{S}} d_2(\bar{y}_1, \bar{y}_2) \\ &\quad < \epsilon_0, \ \text{for } d_2(\bar{y}_1, \bar{y}_2) < \delta \end{aligned}$$

where $\delta > 0$ is chosen as $\delta = \epsilon_0 / (3\|\bar{\Gamma}^{-1}\|_{\mathcal{S}})$. ☐

This lemma shows that verifying the boundedness $\bar{\Gamma}^{-1}$ is sufficient to ensure weak-star continuity-like properties of the regularization strategy. It also shows that one should not try to prove the weak-star continuity of R_ϵ for any fixed $\epsilon > 0$, but rather consider the family $\{R_\epsilon\}$ as a whole.

IV. FIXED-POINT ITERATION-BASED APPROXIMATE INVERSION

In this section, an approximate inversion algorithm is proposed based on successive iteration. The point-wise convergence condition for a regularization strategy (17) is proved under the same conditions on the density function as in Theorems 2.1 and 2.2. The second condition (18) is much more difficult to prove, and we will consider it in future research.

First, consider the case that the desired output function is monotone. Let $C_{m^+,J}[0,T]$ denote the space of nondecreasing, continuous functions on $[0,T]$ taking values in J, and $C^{0,1}_{m^+,J}[0,T]$ denote those functions in $C_{m^+,J}[0,T]$ that are Lipschitz continuous. We consider the equation $\Gamma[u, \psi_{-1}] = y$ where $\psi_{-1} \in \Psi_0$ and $y \in C_{m^+,J}[0,T]$ (and $y \in C^{0,1}_{m^+,J}[0,T]$) in Proposition 4.1. Analogous results are true if $C_{m^-,J}([0,T])$ and $C^{0,1}_{m^-,J}([0,T])$ (the space of nonincreasing functions) are considered.

Proposition 4.1: Assume that the Preisach density function $\omega(r,s)$ has compact support; is integrable; is nonnegative; and $\omega(r,s) \ge C r^\xi$ for almost every $(r,s) \in R_\epsilon = [0,\epsilon] \times [u_{\min} - \epsilon, u_{\max} + \epsilon]$, where $C > 0, \xi \ge 0$, and $\epsilon > 0$. Let k_2 denote the Lipschitz constant for Γ. Let $\psi_{-1} \in \Psi_0$ with the corresponding output y_0. For $y \in C_{m^+,J}[0,T]$ with $y(0) = y_0$, consider the following algorithm:

$$\begin{cases} u^{(n+1)} = u^{(n)} + \frac{y - \Gamma[u^{(n)}, \psi_{-1}]}{k_2}, & n \ge 0 \\ u^{(0)} \equiv \psi_{-1}(0). \end{cases} \tag{19}$$

Then, the following hold.

1) For any $n \ge 0$, $u^{(n)} \in C_{m^+,I}[0,T]$; and if $y \in C^{0,1}_{m^+,J}[0,T]$, $u^{(n)} \in C^{0,1}_{m^+,I}[0,T]$.

2) As $n \to \infty$, $u^{(n)}$ converges pointwise to $u^* \in C_{m^+,I}[0,T]$ with $\Gamma[u^*, \psi_{-1}] = y$.

3) For $\epsilon > 0$, let N_ϵ be the smallest integer satisfying $N_\epsilon \ge (k_2(u_{\max} - u_{\min})/\epsilon)$. Then

$$\|\Gamma\left[u^{(N_\epsilon)}, \psi_{-1}\right] - y\|_\infty \le \epsilon.$$

4) As $n \to \infty$, we have $u^{(n)} \to u^*$ uniformly on $[0,T]$.

Proof:

1) Under the hypothesis on the density function, it is clear from Theorem 2.1 that $\Gamma : C_I[0,T] \to C_J[0,T]$ is Lipschitz continuous. We will first show $u^{(n)} \in C_{m^+,I}[0,T], \forall\, n$. Then, we will show that $u^{(n)}$ is Lipschitz continuous provided $y \in C^{0,1}_{m^+,J}[0,T]$.

Clearly, $u^{(n)} \in C_I[0,T], \forall\, n$. We use induction to show $u^{(n)} \in C_{m^+,I}[0,T]$. Since $u^{(0)}$ is a constant function, it is nondecreasing. Now, suppose that for some $n \geq 0, u^{(n)}$ is nondecreasing. This, together with the Lipschitz continuity of Γ, implies, for $0 \leq t_1 \leq t_2 \leq T$

$$\Gamma\left[u^{(n)}, \psi_{-1}\right](t_2) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t_1) \leq k_2\left(u^{(n)}(t_2) - u^{(n)}(t_1)\right). \tag{20}$$

Using (19), we have

$$\begin{aligned} u^{(n+1)}(t_2) &- u^{(n+1)}(t_1) \\ &= \frac{y(t_2) - y(t_1)}{k_2} + u^{(n)}(t_2) - u^{(n)}(t_1) \\ &\quad - \frac{\Gamma\left[u^{(n)}, \psi_{-1}\right](t_2) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t_1)}{k_2} \\ &\geq \frac{y(t_2) - y(t_1)}{k_2} \ (\text{by } (20)) \\ &\geq 0, \ (\text{since } y \in C_{m^+,J}[0,T]) \end{aligned}$$

and, therefore, $u^{(n+1)}$ is nondecreasing.

Next, we show that $u^{(n)}$ is Lipschitz continuous for every n, if $y \in C^{0,1}_{m^+,J}[0,T]$, again by induction. Note that $u^{(0)}$ is Lipschitz continuous and $\Gamma : C^{0,1}_I[0,T] \to C^{0,1}_J[0,T]$ by Theorem 2.1. Hence $\Gamma\left[u^{(0)}, \psi_{-1}\right]$ is Lipschitz continuous, and by (19) $u^{(1)}$ is Lipschitz continuous. Furthermore, if we assume $u^{(n)}$ to be Lipschitz continuous, the same arguments imply that $u^{(n+1)}$ is Lipschitz continuous. Thus, $u^{(n)}$ is Lipschitz continuous for every n, by induction.

2) Consider the sequence $\{u^{(n)}\}$. As $y \geq y_0 = \Gamma\left[u^{(0)}, \psi_{-1}\right](t)$, we have $u^{(1)} \geq u^{(0)}$. In the preceding, the inequality $f \geq g$ is said to be true, if and only if $f(t) \geq g(t)$ for all $t \in [0,T]$. Suppose $u^{(n)} \geq u^{(n-1)}$ for some $n \geq 1$. From (19)

$$u^{(n+1)} = u^{(n)} + \frac{y - \Gamma\left[u^{(n)}, \psi_{-1}\right]}{k_2} \tag{21}$$

$$u^{(n)} = u^{(n-1)} + \frac{y - \Gamma\left[u^{(n-1)}, \psi_{-1}\right]}{k_2}. \tag{22}$$

The Lipschitz continuity of the operator $\Gamma[\cdot, \psi_{-1}]$ implies

$$\Gamma\left[u^{(n)}, \psi_{-1}\right] - \Gamma\left[u^{(n-1)}, \psi_{-1}\right] \leq k_2(u^{(n)} - u^{(n-1)}). \tag{23}$$

Subtracting (22) from (21), and using (23), we get $u^{(n+1)} \geq u^{(n)}$. Note that $u^{(n)}(t) > u^{(n-1)}(t)$ if and only if $y(t) > \Gamma\left[u^{(n-1)}, \psi_{-1}\right](t)$ by (19). For each $t \in [0,T]$, as $\{u^{(n)}(t)\}$ is a monotone increasing sequence bounded by $u_{\max}$, the sequence $u^{(n)}(t) \to u^*(t)$ as $n \to \infty$. Hence, $\{u^{(n)}\}$ converges pointwise to some u^*. By the continuity of $\Gamma[\cdot, \psi_{-1}]$, the sequence $\{\Gamma[u^{(n)}, \psi_{-1}]\} \to \Gamma[u^*, \psi_{-1}]$. By (19), $u^* = u^* + (y - \Gamma[u^*, \psi_{-1}])/k_2$ which implies $\Gamma[u^*, \psi_{-1}] = y$. Now, we have $u^* \in C_I[0,T]$ due to the condition on the density function and item 3) of Theorem 2.1, and $u^* \in C_{m^+,I}[0,T]$ because each $u^{(n)}$ is monotone and the set $C_{m^+,I}[0,T]$ is a closed subspace of $C_I[0,T]$.

3) If for some $t \in [0,T]$, $\left|y(t) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t)\right| > \epsilon$, then $u^{(n+1)}(t) - u^{(n)}(t) > \epsilon/k_2$. Since $\left|y(t) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t)\right|$ is nonincreasing with n, and $u^{(n)}(t) - \psi_{-1}(0)$ is bounded by $u_{\max} - u_{\min}$, one concludes that after N_ϵ iterations, $\left|y(t) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t)\right| \leq \epsilon$ for every t.

4) By Lemma 2.1 and the assumption on $\omega(r,s)$, we have $\chi_I(x) \geq K x^{\xi+2}$ for $0 \leq x \leq 2\epsilon$, for some $K > 0$. Hence

$$\left|y(t) - \Gamma\left[u^{(n)}, \psi_{-1}\right](t)\right| \geq K\left|u^{(n)}(t) - u^*(t)\right|^{\xi+2}. \tag{24}$$

From item 3), $\left\|y - \Gamma\left[u^{(n)}, \psi_{-1}\right]\right\|_\infty \to 0$ as $n \to \infty$. Equation (24) then implies the uniform convergence of $\{u^{(n)}\}$ to u^*. □

Based on Proposition 4.1, the following algorithm (see illustration in Fig. 3) can be used to generate an approximate inverse $u_\epsilon \in C^{0,1}_I[0,T]$ for $y \in C_J([0,T])$ such that $\|\Gamma[u_\epsilon, \psi_{-1}] - y\|_\infty \leq \epsilon$.

```
Fixed Point Algorithm:
• Step 1. Pick $y' \in C^{0,1}_{pm,J}[0,T]$ such that $\|y' - y\|_\infty \leq \epsilon' \triangleq \epsilon/2$, and the variation in each monotone section of $y'$ is at least $\epsilon'$.
  Let $0 = T_0 < T_1 < T_3 < \cdots < T_{2N-1} < T_{2N+1} = T$ be the standard partition for $y'$. We will shortly define the times $T_2, T_4, \ldots, T_{2N}$.
• Step 2. On $[T_0, T_1]$, run the algorithm (19) (at most $N_{\epsilon'}$ times) until $\left\|y' - \Gamma\left[u^{(n)}, \psi_{-1}\right]\right\|_\infty \leq \epsilon'$. Set
      $u_\epsilon(t) = u^{(n)}(t)$,    for $t \in [T_0, T_1]$.
• Step 3. Let $T_2 \geq T_1$ be the smallest time instant such that $y'(T_2) = \Gamma[u_\epsilon, \psi_{-1}](T_1)$. $T_2$ is well defined considering Step 1. Set $u_\epsilon(t) \equiv u_\epsilon(T_1)$ on $(T_1, T_2)$;
• Step 4. Run (19) $N_{\epsilon'}$ times on $[T_2, T_3]$ with $u^{(0)} \equiv u_\epsilon(T_1)$, which defines $u_\epsilon$ on $[T_2, T_3]$;
• Step 5. Continue Steps 3 and 4 until $u_\epsilon$ is defined up to the final time $T$.
```

As in Section III, for $t \in [0,T]$, define

$$\bar{y}(t) \triangleq y(t) - y(0) \text{ and } \bar{u}_\epsilon(t) \triangleq u_\epsilon(t) - u(0). \tag{25}$$

Define

$$\begin{aligned} R_\epsilon[\cdot, \psi_{-1}] : C_{\bar{J}}[0,T] &\to C^{0,1}_{\bar{I}}[0,T] \\ \bar{y} &\mapsto \bar{u}_\epsilon \end{aligned} \tag{26}$$

where u_ϵ is the result of the fixed point algorithm. Let $y_\epsilon = \Gamma[u_\epsilon, \psi_{-1}]$ and $\bar{\Gamma}[\cdot, \psi_{-1}]$, $\bar{\Gamma}^{-1}[\cdot, \psi_{-1}]$ be defined as in (14) and (15).

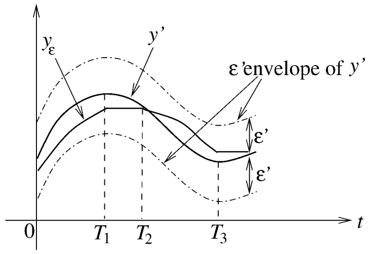

Fig. 3. Illustration of the fixed-point iteration-based inverse algorithm.

One can establish the following regularization-type properties for the scheme R_ϵ.

Theorem 4.1: Assume that the density function of the Preisach operator Γ satisfies the conditions of Proposition 4.1. Let $\epsilon > 0$. Then

1) $\forall\, \bar{y} \in C_{\bar{J}}[0,T]$,

$$\lim_{\epsilon \longrightarrow 0} \bar{\Gamma} \circ R_\epsilon[\bar{y}, \psi_{-1}] = \bar{y}; \tag{27}$$

2) $\forall\, \phi \in L^1[0,T]$

$$\lim_{\epsilon \longrightarrow 0} < R_\epsilon[\bar{y}, \psi_{-1}] - \Gamma^{-1}[\bar{y}, \psi_{-1}], \quad \phi \geq 0 \tag{28}$$

uniformly for $\bar{y}$ on bounded sets of $C_{\bar{J}}^{0,1}[0,T]$.

Proof: Given $\bar{y} \in C_{\bar{J}}[0,T]$, choose $\bar{y}' \in C_{pm,\bar{J}}[0,T]$ according to step 1). By Proposition 4.1, on the time-intervals $[T_{2n}, T_{2n+1}]$ where $n = 0, \ldots N$, we have: $|\bar{y}_\epsilon(t) - \bar{y}'(t)| = |y_\epsilon(t) - y'(t)| \leq \epsilon' = \epsilon/2$. On the time intervals $[T_{2n+1}, T_{2n+2}]$ where $n = 0, \ldots N-1$, u_ϵ is simply a constant, and by step 3) of the fixed point algorithm: $|\bar{y}_\epsilon(t) - \bar{y}'(t)| = |y_\epsilon(t) - y'(t)| \leq \epsilon'$. Thus, $\|\bar{y}_\epsilon - \bar{y}'\|_\infty = \|y_\epsilon - y'\|_\infty \leq \epsilon'$, and then

$$\|\bar{y}_\epsilon - \bar{y}\|_\infty = \|y_\epsilon - y\|_\infty \leq \|y_\epsilon - y'\|_\infty + \|y - y'\|_\infty \leq \epsilon.$$

Hence, the scheme $\{R_\epsilon\}$ satisfies the first condition (17) for a regularization strategy.

Next, let $y \in C_J^{0,1}[0,T]$, and $\bar{y}$ be given by (25). Let $\|y\|_{0,1} \leq M$. Pick $y' \in C_{pm,J}^{0,1}[0,T]$ such that $\|y - y'\|_{0,1} \leq \delta$. Then, $\|y'\|_{0,1} \leq \|y' - y\|_{0,1} + \|y\|_{0,1} \leq M + \delta$. The finest partition needed for all such functions y' is one with intervals of length $\epsilon'/(M+\delta)$. Therefore, the upper bound on the number of iterations needed for convergence (to within ϵ in the sup-norm) is $((M+\delta)TN_{\epsilon'})/\epsilon'$. Thus, we have uniform convergence on bounded sets in $C_{\bar{J}}^{0,1}[0,T]$.

By Theorem 2.2, $\bar{\Gamma}^{-1} : C_{\bar{J}}^{0,1}[0,T] \rightarrow C_{\bar{I}}^{0,\nu}[0,T]$ where $\nu = 1/(\xi + 2)$. This implies that $\bar{u}^* = \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]$ belongs in $C_{\bar{I}}^{0,\nu}[0,T]$, even though $\bar{u}_\epsilon \in C_{\bar{I}}^{0,1}[0,T]$. As $C_{\bar{I}}^{0,\nu}[0,T] \subset L_{\bar{I}}^{\infty}[0,T]$, we have $L_{\bar{I}}^{1}[0,T] \subset C_{\bar{I},w*}^{0,\nu}[0,T]$, where $C_{\bar{I},w*}^{0,\nu}[0,T]$ denotes the weak-star dual of $C_{\bar{I}}^{0,\nu}[0,T]$. Let ϕ be an element of $L_{\bar{I}}^{1}[0,T]$. Since $\|\bar{u}_\epsilon - \bar{u}^*\|_\infty \rightarrow 0$ as $\epsilon \rightarrow 0$, we have

$$\begin{aligned} \langle \bar{u}_\epsilon - \bar{u}^*, \phi \rangle &= \int_0^T (\bar{u}_\epsilon(t) - \bar{u}^*(t))\phi(t)dt \\ &\leq \|\bar{u}_\epsilon - \bar{u}^*\|_\infty \|\phi\|_1 \\ &\rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned} \tag{29}$$

□

The previous result falls slightly short of showing that R_ϵ is a regularization scheme. In order to show R_ϵ is a regularization scheme, (29) must hold for all $\phi \in C_{\bar{I},w*}^{0,\nu}[0,T]$. This is a question that needs to be further investigated in the future.

V. DISCRETIZATION-BASED APPROXIMATE INVERSION

In this section, a discretization-based approximate inversion scheme is discussed. The discretization results in a discretized Preisach operator, an approximate inverse of which can be efficiently constructed by the so called *closest-match algorithm.* Experimental results on trajectory tracking of a magnetostrictive actuator based on this algorithm will also be presented.

A. The Closest-Match Algorithm

There are two discretization steps involved, discretization of the input range $I = [u_{\min}, u_{\max}]$ and discretization of the time interval $[0,T]$. Discretize $[u_{\min}, u_{\max}]$ uniformly into $L+1$ levels and denote the resulting set of discrete input values as $U_L = \{\hat{u}_i, i = 1, \ldots, L+1\}$, where

$$\hat{u}_i = u_{\min} + (i-1)\Delta_u$$

and $\Delta_u = (u_{\max} - u_{\min}/L)$. As a consequence of input discretization, the Preisach plane is discretized into cells.

When restricted to inputs taking values in U_L, the Preisach operator becomes a weighted combination of a finite number of hysterons, where the weight of each hysteron equals the integral of the original Preisach density function over the corresponding grid (see Fig. 4 for illustration). Denote this discretized Preisach operator as Γ_L and its set of memory curves as Ψ_L. Note that an element of Ψ_L consists of L vertical or horizontal segments, each with length Δ_u.

Discretization of time is performed similarly. Given $N \geq 1$, the time interval $[0,T]$ is uniformly divided into N sub-intervals with consecutive end-points denoted as $\{t_j\}_{j=0}^{N}$, where $t_j = j\Delta_t$ with $\Delta_t \triangleq (T/N)$.

Let D_J^N denote the set of sequences of length $N+1$ taking values in J, i.e., $\forall\, s \in D_J^N$, $s[j] \in J$, for $j = 0, 1, \ldots, N$. For the discretized Preisach operator Γ_L, an approximate inversion problem can be formulated as follows: Given $\psi_{-1} \in \Psi_L$ and $s_y \in D_J^N$, find $s_u^* \in D_{U_L}^N$ (set of sequences taking values in U_L), such that

$$\|\Gamma_L[s_u^*, \psi_{-1}] - s_y\|_\infty = \min_{s_u \in D_{U_L}^N} \|\Gamma_L(s_u, \psi_{-1}) - s_y\|_\infty. \tag{30}$$

Since $\Gamma_L : D_{U_L}^N \rightarrow D_J^N$ is not "onto", only an approximate inverse s_u^* is sought in (30).

Dynamic programming can be used to solve (30) [24]. However, as N and L get large, this approach becomes prohibitive in terms of computational and storage costs. A sub-optimal scheme is to *sequentially* generate an input sequence s_u of length N so that at time j, $|\Gamma_L[s_u, \psi_{-1}][j] - s_y[j]|$ is minimized. This decomposes the original (approximate) inverse problem of length $N+1$ into $N+1$ successive problems of length 1. To be precise, at each time instant, given the current memory curve $\psi^{(0)}$ (from which the current input $u^{(0)}$ and output $y^{(0)}$ can be derived) and a desired output value $\hat{y}$, find $u^{\#} \in U_L$, such that

$$|\Gamma_L[u^{\#}, \psi^{(0)}] - \hat{y}| = \min_{u \in U_L} |\Gamma_L[u, \psi^{(0)}] - \hat{y}|. \tag{31}$$

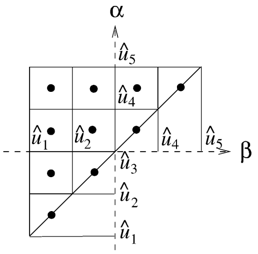

Fig. 4. Illustration of the discretization scheme ($L = 4$), where weighting masses are located at the centers of cells.

Also, the resulting memory curve $\psi^{\#}$ should be returned for use at the next time instant.

The following algorithm can be used to efficiently solve (31) (see Fig. 5 for an illustration). As the fixed-point algorithm, it is also based on the piecewise strictly increasing property of the Preisach operator, and it fully utilizes the discrete structure of the problem. Consider the case $y^{(0)} \leq \hat{y}$ (the other case $y^{(0)} > \hat{y}$ is dealt with analogously). Intuitively, in this algorithm we keep increasing the input by one level in each iteration, until either: a) the input reaches the maximum $\hat{u}_{L+1}$, or b) $y^{(n)}$ exceeds $\hat{y}$. For case a), take $u^{\#} = \hat{u}_{L+1}$; for case b), take $u^{\#}$ to be $u^{(n)}$ or $u^{(n-1)}$ whichever yields smaller output error. In both cases, $u^{\#}$ so obtained solves (31).

```
Closest-Match Algorithm
• Step 0. Set $n = 0$.
• Step 1. If $u^{(n)} = \hat{u}_{L+1}$, let $u^{\#} = u^{(n)}$, $\psi^{\#} = \psi^{(n)}$, go to Step 4; otherwise $u^{(n+1)} = u^{(n)} + \Delta_u$, $\tilde{\psi} = \psi^{(n)}$ [backup the memory curve], $n = n + 1$, go to Step 2;
• Step 2. Evaluate $y^{(n)} = \Gamma[u^{(n)}, \psi^{(n-1)}]$, and (at the same time) update the memory curve to $\psi^{(n)}$. Compare $y^{(n)}$ with $\hat{y}$: if $y^{(n)} = \hat{y}$, let $u^{\#} = u^{(n)}$, $\psi^{\#} = \psi^{(n)}$, go to Step 4; if $y^{(n)} < \hat{y}$, go to Step 1; otherwise go to Step 3;
• Step 3. If $|y^{(n)} - \hat{y}| \leq |y^{(n-1)} - \hat{y}|$, let $u^{\#} = u^{(n)}$, $\psi^{\#} = \psi^{(n)}$, go to Step 4; otherwise $u^{\#} = u^{(n-1)}$, $\psi^{\#} = \tilde{\psi}$ [restore the memory curve], go to Step 4;
• Step 4. Exit.
```

It is not hard to see that this algorithm yields $u^{\#}$ in at most L iterations.

B. Approximate Inversion Based on the Closest-Match Algorithm

An algorithm to approximately solve $\Gamma[u, \psi_{-1}] = y$ is proposed as follows: Pick $N \geq 1, L \geq 1$.

- Step 1): Construct $\tilde{\psi}_{-1} \in \Psi_L$ from $\psi_{-1} \in \Psi_0$ based on the input discretization rules (i.e., approximating the given $\psi_{-1} \in \Psi_0$ by an element in Ψ_L).
- Step 2): For $y \in C_J([0, T])$, construct $s_y \in D_J^N$ via $s_y[j] = y(j\Delta_t)$.
- Step 3): Obtain $s_u \in D_{U_L}^N$ by applying the closest-match algorithm described previously.

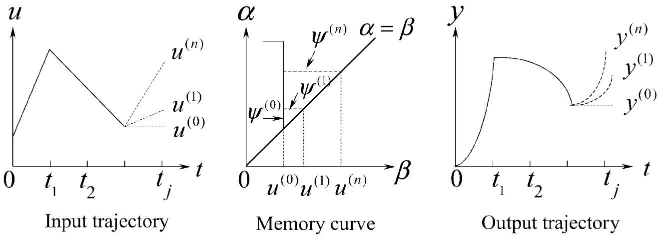

Fig. 5. Illustration of the convergence of the closest-match algorithm.

- Step 4): Construct $u_{N,L} \in C_I^{0,1}[0, T]$ using linear splines based on s_u, i.e.,

$$u_{N,L}(t) = \tau s_u[j] + (1 - \tau) s_u[j + 1]$$

if $t = (j + \tau)\Delta_t, j = 0, \ldots, N - 1$, and $0 \leq \tau \leq 1$.

Analogous to (25) and (26), denote $\bar{u}_{N,L}(t) = u_{N,L}(t) - u_{N,L}(0)$, $\bar{y}(t) = y(t) - y(0)$, and define

$$\begin{aligned} R_{N,L}[\cdot, \psi_{-1}] : C_{\bar{J}}[0, T] &\rightarrow C_{\bar{I}}^{0,1}[0, T] \\ \bar{y} &\mapsto \bar{u}_{N,L}. \end{aligned} \tag{32}$$

Similar to Proposition 4.1, for $y \in C_{m+,J}[0, T]$, we have the following convergence results for the closest-match algorithm-based inversion scheme:

Proposition 5.1: Assume that the density function of the Preisach operator Γ satisfies the conditions of Proposition 4.1. Let k_2 denote the Lipschitz constant for Γ. Then for any $\psi_{-1} \in \Psi_0, y \in C_{m+,J}[0, T]$

1) for any $N, L \geq 1$, $u_{N,L} \in C_{m^+,I}^{0,1}[0, T]$;

2) as $L, N \rightarrow \infty$

$$\|\Gamma(u_{N,L}, \psi_{-1}) - y\|_\infty \rightarrow 0; \tag{33}$$

3) as $N, L \rightarrow \infty$, we have $u_{N,L} \rightarrow u^*$ uniformly on $[0, T]$, where $u^* = \Gamma^{-1}[y, \psi_{-1}]$, and $u^* \in C_{m^+,I}[0, T]$.

Proof:

1) As $u_{N,L}$ is constructed using linear splines, it is clear that $u_{N,L} \in C_I^{0,1}[0, T]$. As y is monotone nondecreasing, $u_{N,L}$ is also monotone and nondecreasing by the nonnegativity condition on the density function.

2) Note that by the construction of Γ_L, it is also Lipschitz continuous with the same Lipschitz constant k_2 for Γ. Hence, if the input at any instant t is increased (or decreased) by Δ_u, the output of Γ_L at time t is increased (or decreased) by no more than $k_2\Delta_u$. From the closest-match algorithm

$$|\tilde{s}_y[j] - s_y[j]| < k_2\Delta_u, \qquad j = 0, \ldots, N \tag{34}$$

where $\tilde{s}_y = \Gamma_L[s_u, \tilde{\psi}_{-1}]$. By the construction of $\tilde{\psi}_{-1}$, it is within the Δ_u-neighborhood of ψ_{-1} (see [14, p. 113], for the definition of neighborhood of a memory curve), and hence by the Lipschitz continuity of Γ,

$$\|\Gamma[u_{N,L}, \psi_{-1}] - \Gamma[u_{N,L}, \tilde{\psi}_{-1}]\|_\infty \leq k_2\Delta_u. \tag{35}$$

Noting $\Gamma_L[s_u, \tilde{\psi}_{-1}][j] = \Gamma[u_{N,L}, \tilde{\psi}_{-1}](t_j)$, we get from (34) and (35), for $j = 0, \dots, N$

$$\begin{aligned}
&|\Gamma[u_{N,L}, \psi_{-1}](t_j) - s_y[j]| \\
&\quad \le |\Gamma[u_{N,L}, \psi_{-1}](t_j) - \Gamma[u_{N,L}, \tilde{\psi}_{-1}](t_j)| \\
&\qquad + |\Gamma[u_{N,L}, \tilde{\psi}_{-1}](t_j) - s_y[j]| \\
&\quad = |\Gamma[u_{N,L}, \psi_{-1}](t_j) - \Gamma[u_{N,L}, \tilde{\psi}_{-1}](t_j)| + |\tilde{s}_y[j] - s_y[j]| \\
&\quad \le k_2\Delta_u + k_2\Delta_u = 2k_2\Delta_u.
\end{aligned}$$

Since both y and $\Gamma[u_{N,L}, \psi_{-1}]$ are monotone, nondecreasing on $[t_j, t_{j+1}]$, for $t \in [t_j, t_{j+1}]$; $j = 0, \dots, N$, if $\Gamma[u_{N,L}, \psi_{-1}](t) \le y(t)$, one has

$$\begin{aligned}
&|\Gamma[u_{N,L}, \psi_{-1}](t) - y(t)| \\
&\quad \le |\Gamma[u_{N,L}, \psi_{-1}](t_j) - y(t_{j+1})| \\
&\quad \le |\Gamma[u_{N,L}, \psi_{-1}](t_j) - y(t_j)| + |y(t_j) - y(t_{j+1})| \\
&\quad \le 2k_2\Delta_u + \rho_y(\Delta_t) \qquad (36)
\end{aligned}$$

where $\rho_y(\cdot)$ is the continuity modulus of y. Same inequality can be obtained if $\Gamma[u_{N,L}, \psi_{-1}](t) \ge y(t)$. Therefore, for each t

$$\lim_{N,L \to \infty} |\Gamma[u_{N,L}, \psi_{-1}](t) - y(t)| = 0.$$

As $y \in C_J[0,T]$, y is uniformly continuous on $[0,T]$. Thus, the right hand side of (36) is independent of t. Therefore

$$\|\Gamma[u_{N,L}, \psi_{-1}] - y\|_\infty \le 2k_2\Delta_u + \rho_y(\Delta_t). \qquad (37)$$

Equation (33) follows, since $\rho_y(\Delta_t)) \to 0$ as $\Delta_t \to 0$.

3) Let $u^* = \Gamma^{-1}[y, \psi_{-1}]$. Then, $u^* \in C_I[0,T]$ as $\Gamma^{-1} : C_J[0,T] \to C_I[0,T]$. The function u^* is also monotone, by the nonnegativity condition on the density function and by $y \in C_{m+,J}[0,T]$.

From item 3 of Theorem 2.1, $\Gamma^{-1} : C_J[0,T] \to C_I[0,T]$ is continuous and, hence, we get from (33) that $\|u_{N,L} - u^*\|_\infty \to 0$ as $N, L \to \infty$. □

Again let $\bar{\Gamma}[\cdot, \psi_{-1}]$ and $\bar{\Gamma}^{-1}[\cdot, \psi_{-1}]$ be defined by (14) and (15), and $\bar{y}$ defined by (25). The following theorem shows a continuity property of R_{Δ_u,Δ_t} similar to that for the Fixed Point iteration method.

Theorem 5.1: Assume that the density function of the Preisach operator Γ satisfies the conditions of Proposition 4.1. Then

1) $\forall\, \bar{y} \in C_{\bar{J}}[0,T]$

$$\lim_{N,L \to \infty} \bar{\Gamma} \circ R_{\Delta_u,\Delta_t}[\bar{y}, \psi_{-1}] = \bar{y}; \qquad (38)$$

2) $\forall\, \phi \in L^1[0,T]$

$$\lim\nolimits_{N,L \to \infty} \left\langle R_{\Delta_u,\Delta_t}[\bar{y}, \psi_{-1}] - \bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}], \phi \right\rangle = 0 \qquad (39)$$

uniformly for $\bar{y}$ on bounded sets of $C^{0,1}_{\bar{J}}[0,T]$.

Proof: The first item follows by simply repeating the proof of Proposition 5.1. Other than the monotonicity of u^* (defined to be $\bar{\Gamma}^{-1}[\bar{y}, \psi_{-1}]$), the rest of the proof applies to this case.

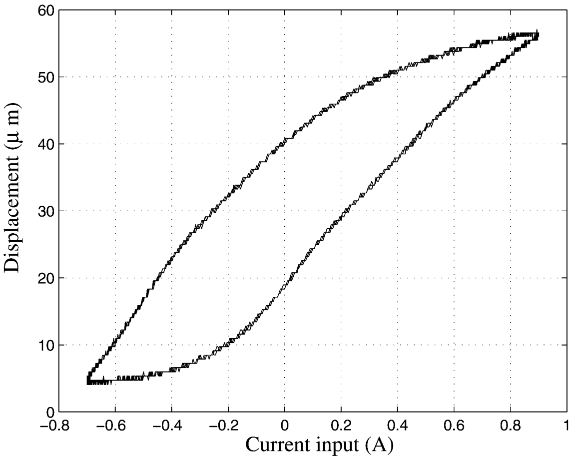

Fig. 6. Typical hysteresis curve of the Terfenol-D actuator.

The proof of the second statement is exactly analogous to that of Theorem 4.1, and utilizes the convergence in the $\|\cdot\|_\infty$ norm of the functions $\bar{u}_{N,L}$ to $\bar{u}^*$. □

C. Experimental Results on Tracking Control

The above inversion algorithm is applied to tracking control of a magnetostrictive actuator (made of Terfenol-D). Magnetostriction is the phenomenon of strong coupling between magnetic properties and mechanical properties of some ferromagnetic materials: strains are generated in response to an applied magnetic field, while conversely, mechanical stresses in the materials produce measurable changes in magnetization. By varying the current in the coil surrounding the Terfenol-D rod, one can vary the magnetic field inside the rod and thus control the displacement output of the actuator. The actuator used in this study is an AA-050H series Terfenol-D actuator manufactured by Etrema. The displacement of the actuator is measured with a LVDT sensor (Schaevitz 025MHR). Fig. 6 shows the hysteretic relationship between the current input and the displacement output.

When the input current is quasi-static, the hysteretic behavior of the magnetostrictive actuator can be modeled as [17]

$$\begin{cases} H = c_0 I \\ M = \Gamma[H, \psi_{-1}] \\ z = \frac{l_{\mathrm{rod}}\lambda_s}{M_s^2} M^2 \end{cases} \qquad (40)$$

where H and M are the magnetic field and the bulk magnetization along the rod direction, respectively, I is the current input, z is the displacement output, c_0 is the coil factor, l_{rod} is the rod length, λ_s is the saturation magnetostriction, and M_s is the saturation magnetization. In (40), the magnetostrictive hysteresis is essentially captured by the ferromagnetic hysteresis between M and H, which is modeled by the Preisach operator Γ.

For a discretization level of L, the weighting masses for Γ_L can be identified through a constrained least squares algorithm [7], [25]. Here L has been chosen to be 25, and $\Delta_t = (T/N) = 10$ ms. Fig. 7 shows the identified density function. As can be observed, the density function is nonzero along the $\beta = \alpha$ line,

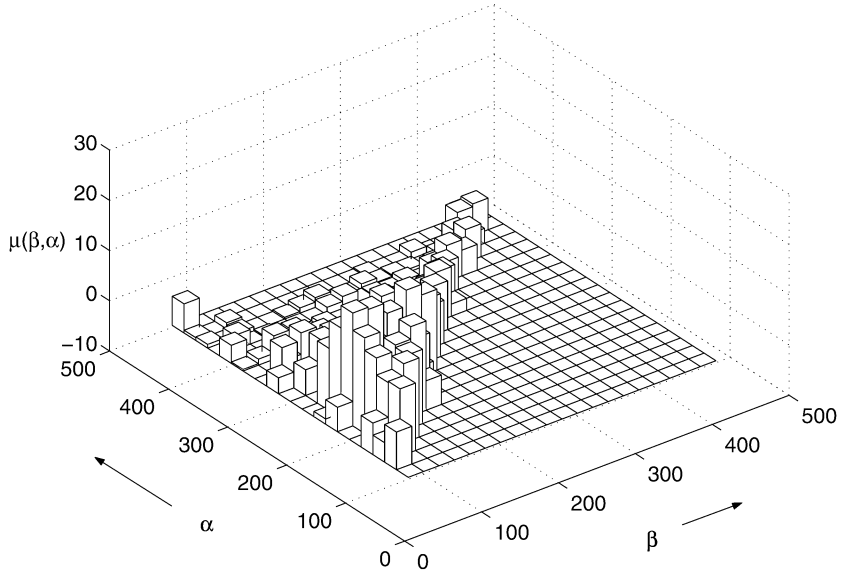

Fig. 7. Identified Preisach density function for a commercial magnetostrictive actuator.

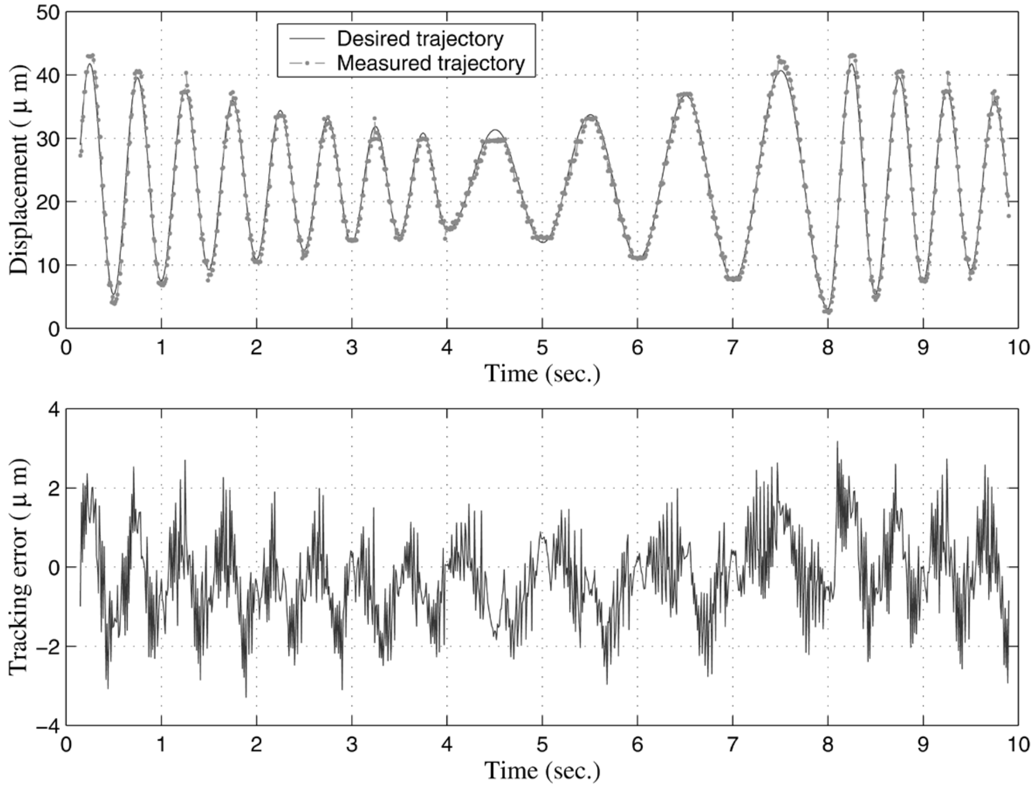

Fig. 8. Trajectory tracking of a magnetostrictive actuator based on the approximate inversion.

which is the same as the line $r = 0$ in (r, s) coordinates (recall that the variables (α, β) and (r, s) are related according to $r = (\alpha - \beta)/2$ and $s = (\alpha + \beta)/2$). Therefore, the key condition of Theorem 2.2 and Proposition 4.1 is satisfied, and both Theorems 4.1 and 5.1 can be applied to this actuator to find an approximate right-inverse.

An open-loop tracking experiment has been conducted based on the closest-match inversion algorithm. Fig. 8 shows the comparison of the desired trajectory and the actual trajectory, together with the tracking error. The desired trajectory is chosen to vary in both amplitude frequency. The tracking error is small (under 3 $\mu\mathrm{m}$), which shows that the inversion algorithm is effective. An extension of this approach to the closed-loop l^1 control of the magnetostrictive actuator over a 0–200 Hz range can be found in [26].

VI. CONCLUSION

The Preisach operator is a popular tool for hysteresis modeling in various smart materials. Inversion of the Preisach operator plays a fundamental role in effective control of these materials. This paper dealt with approximately inverting the Preisach operator, in such a way that the resulting functions have some regularity properties. We first presented a weak and easily verifiable condition that guarantees the weak-star continuity of the inverse operator. Motivated by this result, the notion of a regularization strategy was proposed for the inversion problem.

In practice, exact inversion of the Preisach operator is generally not possible due to numerical limitations. Two inversion schemes were developed in this paper, both of which fully utilized the piecewise strictly increasing property of the Preisach operator (under some mild conditions on the density function). Both algorithms yield Lipschitz continuous inputs. They were shown to satisfy the first condition for a regularization strategy. Both schemes also enjoy a continuity property that is similar to but weaker than that of a regularization strategy. An interesting direction for future work is to investigate whether the two schemes satisfy the second condition (18) for a regularization strategy.

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